A Gabriel-type theorem for cluster tilting

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Introduction

Throughout this talk,
K: algebraically closed field.
Λ: basic indecomposable finite dimensional K-algebra.
Q: finite connected acyclic quiver.
Q_0: set of vertices of Q.
We assume $\Lambda = KQ/(R)$, R: set of relations.

Motivation

Theorem (Gabriel)

Q :*Dynkin quiver*, q_Q :*Euler form*, X :*Indecomposable* KQ*-module.*

(1) $\underline{\mathrm{dim}}X$ *is a root of* q_Q.

(2) $\exists$ *bijection*

$$\{\text{indec.}KQ\text{-modules}\} \to \{\text{positive roots of } q_Q\}.$$

$$X \longmapsto \underline{\mathrm{dim}}X.$$

<u>Aim</u> Generalize Gabriel's Thm by higher dimensional AR theory.

(1) $\mathrm{gl.\,dim}\, KQ \leq 1 \longrightarrow \mathrm{gl.\,dim}\,\Lambda \leq n$.

(2) All modules $\longrightarrow$ Nice modules (Cluster-tilting modules).

n-representation-finite algebras

Definition

Λ : *n-representation-finite algebra* $\overset{\text{def}}{\Leftrightarrow}$
(1) $\operatorname{gl.dim}\Lambda \leq n$.
(2) $\exists$ *n-CT module M, i.e.*

$$\begin{aligned} \operatorname{add}M &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(M, X) = 0 \ \textit{for any} \ 0 < i < n\}, \\ &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, M) = 0 \ \textit{for any} \ 0 < i < n\}. \end{aligned}$$

- X : *Cluster indecomposable module* $\overset{\text{def}}{\Leftrightarrow}$
X is indecomposable and $X \in \operatorname{add}M$.

<u>Rem</u> Λ is 1-representation-finite.
$\Leftrightarrow$ $\operatorname{gl.dim}\Lambda \leq 1$ and Λ is representation-finite.
$\Leftrightarrow$ $\Lambda = KQ$ for a Dynkin quiver Q.

Q. How to obtain cluster-indecomposable modules?

Theorem (Iyama)

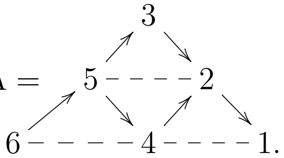
Λ : n-representation-finite algebra. Then basic n-CT module M is unique and given by

$$M \cong \bigoplus_{i \geq 0} \tau_n^{-i}(\Lambda) \cong \bigoplus_{i \geq 0} \tau_n^{i}(D\Lambda),$$

where $\tau_n := \tau\Omega^{n-1}$, $\tau_n^- := \tau^-\Omega^{-(n-1)}$ (n-AR translations).

<u>EX</u>.

$\Lambda =$



Then Λ is 2-representation-finite and the 2-CT module is given by
$M \cong I_1 \oplus I_2 \oplus I_3 \oplus I_4 \oplus I_5 \oplus I_6 \oplus \tau_2(I_4) \oplus \tau_2(I_5) \oplus \tau_2(I_6) \oplus \tau_2^2(I_6)$.

Euler form

Assume $\mathrm{gl.dim}\,\Lambda \leq n$. For $x \in \mathbb{Z}^{|Q_0|}$, define

$$q_\Lambda(x) := x^t (C_\Lambda^{-1})^t x. \quad (C_\Lambda : \text{Cartan matrix})$$

$$x_0 : \text{root} \overset{\text{def}}{\Leftrightarrow} q_\Lambda(x_0) = 1.$$

Rem For $X \in \mathrm{mod}\Lambda$, we have

$$q_\Lambda(\underline{\dim} X) = \sum (-1)^i \dim \mathrm{Ext}^i(X, X).$$

Theorem

Theorem

Λ :n-*representation-finite,* X :*Cluster-indec.module,* q_Λ :*Euler form.*

(1) $\underline{\dim} X$ *is a root of* q_Λ*.*

(2) $\exists$ *Injection*

$$\{\text{Cluster-indec.}\Lambda\text{-modules}\} \to \{\text{positive roots of } q_\Lambda\}.$$

$$X \longmapsto \underline{\dim} X.$$

Definition

$x \in \mathbb{Z}^{|Q_0|}$: *Cluster-root* $\overset{\text{def}}{\Leftrightarrow}$ $\exists$ *Cluster-indec.module* X *s.t.* $\underline{\dim} X = x$.

Key of proof

- Gabriel's theorem $\longrightarrow$
  BGP reflection functor, APR tilting modules.
- n-APR tilting modules [Iyama-Oppermann].
  A generalization of APR-tilting modules.

Key of proof

- Q : Dynkin quiver,
  $T_i := \tau^- P_i \oplus KQ/P_i$: APR tilting KQ-module,
  $KQ' := \mathrm{End}_{KQ}(T_i)$.
  $\exists$ equivalence

$$(\mathrm{mod} KQ)\backslash\{P_i\} \xrightarrow{\cong} (\mathrm{mod} KQ')\backslash\{I_i\}.$$

- Λ : n-representation-finite,
  $T_i := \tau_n^- P_i \oplus \Lambda/P_i$: n-APR tilting Λ-module.
  $\Rightarrow \Gamma := \mathrm{End}_\Lambda(T_i)$ is n-representation-finite and has n-CT Γ-module M'.
  $\Rightarrow \exists$ equivalence

$$\mathrm{add}(M/P_i) \xrightarrow{\cong} \mathrm{add}(M'/I_i).$$

A criterion of cluster-roots

Q. Is any positive root of q_Λ cluster-roots?
$\longrightarrow$ NO.
$\longrightarrow$ give a criterion of cluster-roots.

Φ-sign-coherent

Definition

Λ : n-representation-finite algebra, $\Phi := (-1)^n C_\Lambda^t C_\Lambda^{-1}$ Coxeter matrix of Λ and $x \in \mathbb{Z}^{|Q_0|}$.

- *x is Φ-sign-coherent $\overset{\text{def}}{\Leftrightarrow}$ $\Phi^m(x) \in \mathbb{Z}_{\geq 0}^{|Q_0|}$ or $\Phi^m(x) \in \mathbb{Z}_{\leq 0}^{|Q_0|}$ for $\forall m \in \mathbb{Z}$.*
- *x is Φ-positive $\overset{\text{def}}{\Leftrightarrow}$ $\Phi^m(x) \in \mathbb{Z}_{\geq 0}^{|Q_0|}$ for $\forall m \in \mathbb{Z}$.*

EX

(1) $\Phi = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\Phi} \cdots$

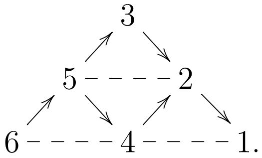
(2) $\Phi = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \cdots$

Theorem

Theorem

Cluster-roots are Φ-sign-coherent. If n is even, then cluster-roots are Φ-positive.

<u>EX</u>



2-representation-finite.

$$\underline{\dim} I_6 = \begin{pmatrix} 0\\0\\0\\0\\0\\1 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 0\\0\\0\\1\\0\\0 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 1\\0\\0\\0\\0\\0 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 1\\1\\1\\0\\0\\0 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 0\\0\\1\\0\\1\\1 \end{pmatrix} \xrightarrow{\Phi} \underline{\dim} I_6,$$

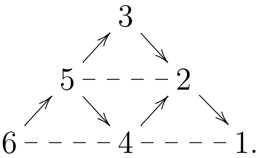
$$\underline{\dim} I_5 = \begin{pmatrix} 0\\1\\1\\1\\1\\0 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 0\\0\\0\\0\\1\\1 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 0\\1\\0\\1\\0\\0 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 0\\0\\0\\1\\1\\0 \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} 1\\1\\0\\0\\0\\0 \end{pmatrix} \xrightarrow{\Phi} \underline{\dim} I_5.$$

Conjecture

Λ : 2-representation-finite algebra. Then Φ-positive roots are cluster-roots.
Namely, $\exists$ bijection

$$\{\text{Cluster-indec.modules}\} \overset{1-1}{\longleftrightarrow} \{\Phi\text{-positive roots of } q_\Lambda\}.$$

<u>EX</u>



2-representation-finite.

$q_\Lambda(x) =$
$\sum_{i\in Q_0} x_i^2 - x_1x_2 - x_2x_3 - x_2x_4 - x_3x_5 - x_4x_5 - x_5x_6 + x_1x_4 + x_2x_5 + x_4x_6.$

All positive roots of q_Λ.

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix},$$

$$\begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\1\\0\end{pmatrix}.$$

All Φ-positive roots are given by

$$\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix}.$$

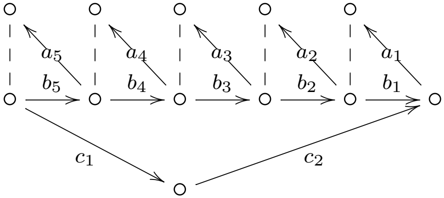
They correspond bijectively to cluster-roots.

Theorem

If Λ is iterated tiled, the conjecture is true.

EX 2-representation-finite algebra $\Lambda := KQ/(R)$, which is iterated tiled.

$Q =$



$$R = \{c_1 c_2 - b_5 b_4 b_3 b_2 b_1, b_i a_i\}.$$

All positive roots of q_Λ.

$$\begin{pmatrix}00000\\100000\\0\end{pmatrix},\begin{pmatrix}00000\\010000\\0\end{pmatrix},\begin{pmatrix}00000\\110000\\0\end{pmatrix},\begin{pmatrix}00000\\001000\\0\end{pmatrix},\begin{pmatrix}00000\\011000\\0\end{pmatrix},\begin{pmatrix}00000\\111000\\0\end{pmatrix},\begin{pmatrix}00000\\000100\\0\end{pmatrix}$$

$$\begin{pmatrix}00000\\011100\\0\end{pmatrix},\begin{pmatrix}00000\\111100\\0\end{pmatrix},\begin{pmatrix}00000\\000010\\0\end{pmatrix},\begin{pmatrix}00000\\000110\\0\end{pmatrix},\begin{pmatrix}00000\\001110\\0\end{pmatrix},\begin{pmatrix}00000\\011110\\0\end{pmatrix},\begin{pmatrix}00000\\111110\\0\end{pmatrix}$$

$$\begin{pmatrix}00000\\100000\\1\end{pmatrix},\begin{pmatrix}00000\\110000\\1\end{pmatrix},\begin{pmatrix}00000\\111000\\1\end{pmatrix},\begin{pmatrix}00000\\111100\\1\end{pmatrix},\begin{pmatrix}00000\\111110\\1\end{pmatrix},\begin{pmatrix}00000\\000001\\0\end{pmatrix},\begin{pmatrix}00000\\000011\\0\end{pmatrix}$$

$$\begin{pmatrix}00000\\001111\\0\end{pmatrix},\begin{pmatrix}00000\\011111\\0\end{pmatrix},\begin{pmatrix}00000\\000001\\1\end{pmatrix},\begin{pmatrix}00000\\000011\\1\end{pmatrix},\begin{pmatrix}00000\\000111\\1\end{pmatrix},\begin{pmatrix}00000\\001111\\1\end{pmatrix},\begin{pmatrix}00000\\011111\\1\end{pmatrix}$$

$$\begin{pmatrix}10000\\000000\\0\end{pmatrix},\begin{pmatrix}10000\\010000\\0\end{pmatrix},\begin{pmatrix}10000\\011000\\0\end{pmatrix},\begin{pmatrix}10000\\011100\\0\end{pmatrix},\begin{pmatrix}10000\\011110\\0\end{pmatrix},\begin{pmatrix}10000\\011111\\0\end{pmatrix},\begin{pmatrix}10000\\011111\\1\end{pmatrix}$$

$$\begin{pmatrix}01000\\001000\\0\end{pmatrix},\begin{pmatrix}01000\\001100\\0\end{pmatrix},\begin{pmatrix}01000\\001110\\0\end{pmatrix},\begin{pmatrix}01000\\001111\\0\end{pmatrix},\begin{pmatrix}01000\\001111\\1\end{pmatrix},\begin{pmatrix}00100\\000000\\0\end{pmatrix},\begin{pmatrix}00100\\000100\\0\end{pmatrix}$$

$$\begin{pmatrix}00100\\000111\\0\end{pmatrix},\begin{pmatrix}00100\\000111\\1\end{pmatrix},\begin{pmatrix}00010\\000000\\0\end{pmatrix},\begin{pmatrix}00010\\000010\\0\end{pmatrix},\begin{pmatrix}00010\\000011\\0\end{pmatrix},\begin{pmatrix}00010\\000011\\1\end{pmatrix},\begin{pmatrix}00001\\000000\\0\end{pmatrix}$$

$$\begin{pmatrix}00001\\000001\\1\end{pmatrix}.$$

All Φ-positive roots

$$\begin{pmatrix} 00001 \\ 000000 \\ 0 \end{pmatrix}, \begin{pmatrix} 00010 \\ 000000 \\ 0 \end{pmatrix}, \begin{pmatrix} 00100 \\ 000000 \\ 0 \end{pmatrix}, \begin{pmatrix} 01000 \\ 000000 \\ 0 \end{pmatrix}, \begin{pmatrix} 10000 \\ 000000 \\ 0 \end{pmatrix}, \begin{pmatrix} 00001 \\ 000001 \\ 0 \end{pmatrix}, \begin{pmatrix} 00001 \\ 000001 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 00100 \\ 000111 \\ 0 \end{pmatrix}, \begin{pmatrix} 01000 \\ 001111 \\ 0 \end{pmatrix}, \begin{pmatrix} 10000 \\ 011111 \\ 0 \end{pmatrix}, \begin{pmatrix} 00000 \\ 111111 \\ 1 \end{pmatrix}, \begin{pmatrix} 00010 \\ 000010 \\ 0 \end{pmatrix}, \begin{pmatrix} 00100 \\ 000100 \\ 0 \end{pmatrix}, \begin{pmatrix} 01000 \\ 001000 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 00000 \\ 100000 \\ 1 \end{pmatrix}, \begin{pmatrix} 00000 \\ 111110 \\ 0 \end{pmatrix}, \begin{pmatrix} 00000 \\ 111100 \\ 0 \end{pmatrix}, \begin{pmatrix} 00000 \\ 111000 \\ 0 \end{pmatrix}, \begin{pmatrix} 00000 \\ 110000 \\ 0 \end{pmatrix}, \begin{pmatrix} 00000 \\ 100000 \\ 0 \end{pmatrix}.$$

They correspond bijectively to cluster-indecomposable Λ-modules.

Thank you for your attention!