McKay Type Correspondence for AS-regular Algebras

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Classical McKay Correspondence

Setup: A finite subgroup $G \leq \mathrm{GL}(n,k)$ acts on $S = k\langle x_1, \dots, x_n\rangle / I$.

- $S^G := \{f \in S \mid \sigma(f) = f, \forall \sigma \in G\}$: **the fixed subalgebra**

- $\mathrm{End}_{S^G} S$: **the endomorphism algebra**

- $S * G := \bigoplus_{\sigma \in G} S * \sigma$: **the skew group algebra** $(f * \sigma)(g * \tau) := f\sigma(g) * \sigma\tau$.

- $\Pi\widetilde{Q_G}$: **the preprojective algebra of the reduced McKay quiver** $\widetilde{Q_G}$.

$Q = (Q_0, Q_1)$: **quiver** $:\Leftrightarrow$
$Q_0 = \{\text{vertices}\}, Q_1 = \{\text{arrows}\}$

$\overline{Q}$: **the double of** Q $:\Leftrightarrow$
$\overline{Q}_0 = Q_0$, $\overline{Q}_1 = Q_1 \cup \{\alpha^* : j \to i \mid \alpha : i \to j \in Q_1\}$.

$\Pi Q := k\overline{Q}/(\sum_{\alpha \in Q_1}(\alpha\alpha^* - \alpha^*\alpha))$: **the preprojective algebra of** Q.

Setup:

- $a_1, \dots, a_n \in \mathbb{N}^+, \gcd(a_1, \dots, a_n) = 1$
- $\omega \in \mathbb{C}$: **primitive ℓ-th root of unity**
- $G = \langle \mathrm{diag}(\omega^{a_1}, \dots, \omega^{a_n}) \rangle \leq \mathrm{GL}(n, k)$

Q_G: **McKay quiver of G :⇔**
$(Q_G)_0 := \mathbb{Z}_\ell = \{0, 1, \dots, \ell - 1\}$
$(Q_G)_1 := \{\alpha : i \to i + a_j (\mathrm{mod}\, \ell) \mid i \in \mathbb{Z}_\ell, 1 \leq j \leq n\}$

$\widetilde{Q_G}$: **reduced McKay quiver of G :⇔**
$(\widetilde{Q_G})_0 := (Q_G)_0$
$(\widetilde{Q_G})_1 := \{\alpha : i \to j \in (Q_G)_1 \mid i < j\}$

Theorem (classical McKay correspondence)

A finite subgroup $G \leq \mathrm{SL}(2,k)$ acts on $S = k[x,y]$.

1. $\mathrm{mod}\, \Pi\widetilde{Q_G} \cong \mathrm{mod}(S * G) \cong \mathrm{mod}\, \mathrm{End}_{S^G} S$
2. $\mathcal{D}^b(\mathrm{mod}\, \Pi\widetilde{Q_G}) \cong \mathcal{D}^b(\mathrm{coh}\, \widetilde{\mathrm{Spec}\, S^G})$ where $\widetilde{\mathrm{Spec}\, S^G}$ is the minimal resolution of the quotient singularity $\mathrm{Spec}\, S^G \cong \mathbb{A}^2/G$.
3. $\widetilde{Q_G}$ is extended Dynkin.

McKay Type Correspondence for AS-regular Algebras

Definition (Artin-Schelter)

A graded algebra $S = k\langle x_1, \dots, x_n\rangle / I_S$ is AS-regular of dimension d and of Gorenstein parameter ℓ if

- $\deg x_i = a_i \in \mathbb{N}^+$ **(connected graded),**
- S **is graded right coherent,**
- $\operatorname{gldim} S = d < \infty$, **and**
- $\operatorname{Ext}^i_S(k, S)_{-j} \cong \begin{cases} k & \text{if } i = d, j = \ell \\ 0 & \text{otherwise.} \end{cases}$

Example

Commutative AS-regular algebras are exactly weighted polynomial algebras $S = k[x_1, \dots, x_n]$ where $\deg x_i = a_i \in \mathbb{N}^+$. Gorenstein parameter of S is $\ell = a_1 + \cdots + a_n$.

Example

A weighted skew polynomial algebra $S = k\langle x_1, \dots, x_n\rangle/(x_jx_i - \alpha_{ij}x_ix_j)$ where $\alpha_{ii} = \alpha_{ij}\alpha_{ji} = 1$ and $\deg x_i = a_i \in \mathbb{N}^+$ is an AS-regular algebra of dimension n and of Gorenstein parameter $\ell = a_1 + \cdots + a_n$.

$R = kQ/I$, $I \lhd kQ$: a path algebra with relations such that $\operatorname{gldim} R = d < \infty$.

The preprojective algebra of a quiver with relations (Q, I) is defined by

$$\Pi(Q, I) := T_R(\operatorname{Ext}^d_R(DR, R))$$

where $DR = \operatorname{Hom}_k(R, k)$.

Fact: $\Pi(Q, 0) \cong \Pi Q$.

Definition (Artin-Zhang)

The noncommutative projective scheme associated to a graded right coherent algebra S is defined by

$$\operatorname{tails} S := \operatorname{grmod} S/\sim$$

where $M \sim N \iff M_{\geq j} \cong N_{\geq j},\ j \gg 0$.

If $S = k[x_1, \dots, x_n]/I$ is commutative and $\deg x_i = a_i = 1$ for all i, then

$$\operatorname{tails} S \cong \operatorname{coh} \operatorname{Proj} S.$$

Setup:

- $S = k\langle x_1, \dots, x_n\rangle / I_S$ **is an AS-regular algebra of dimension** $d \geq 2$ **and of Gorenstein parameter** ℓ.
- $\deg x_i = a_i \in \mathbb{N}^+$ **such that** $\gcd(a_1, \dots, a_n) = 1$ **and** $\gcd(a_i, \ell) = 1$ **for all** $i = 1, \dots, n$.
- $G = \langle \mathrm{diag}(\omega^{a_1}, \dots, \omega^{a_n})\rangle \leq \mathrm{GL}(n, k)$ **where** $\omega \in \mathbb{C}$ **is a primitive** ℓ**-th root of unity.**

Theorem (McKay type correspondence for AS-regular algebras)

1. $\operatorname{grmod} \Pi(\widetilde{Q_G}, \widetilde{I_S}) \cong \operatorname{grmod}(S * G)$.
2. If S is a weighted skew polynomial algebra or $a_i = 1$ for all i, then $\operatorname{grmod}(S * G) \cong \operatorname{grmod} \operatorname{End}_{S^G} S$.
3. If S is a noetherian domain, then $\mathcal{D}^b(\operatorname{tails} \Pi(\widetilde{Q_G}, \widetilde{I_S})) \cong \mathcal{D}^b(\operatorname{coh} \widetilde{\operatorname{Proj}_{nc} S^G})$ (If S is commutative, then $\operatorname{Proj}_{nc} S^G \cong [\mathbb{A}^n \setminus \{0\}/G_m]$.)
4. $\operatorname{gldim} S = 2 \iff \widetilde{I_S} = 0 \iff Q_G = \overline{\widetilde{Q_G}}$.

Example

$$S = k[x, y], \deg x = 1, \deg y = 3$$

$\Rightarrow$ S is an AS-regular algebra of dimension 2 and of Gorenstein parameter 4

$$\Rightarrow \quad G = \left\langle \begin{pmatrix} \omega & 0 \\ 0 & \omega^3 \end{pmatrix} \right\rangle \leq \mathrm{SL}(2, k)$$

ω: primitive 4-th root of unity.

$$\widetilde{Q_G} = \begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 3 & \longleftarrow & 2 \end{array} = \widetilde{A}_3, \qquad Q_G = \begin{array}{ccc} 0 & \rightleftarrows & 1 \\ \uparrow\downarrow & & \uparrow\downarrow \\ 3 & \rightleftarrows & 2 \end{array} = \overline{\widetilde{Q_G}}$$

Example

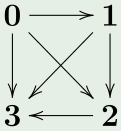
$$S = k\langle x, y, z\rangle/(xz + y^2 + zx)$$

$$\deg x = 1, \deg y = 2, \deg z = 3$$

$\Rightarrow$ S **is an AS-regular algebra of dimension 2 and of Gorenstein parameter 4**

$$\Rightarrow \quad G = \left\langle \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^3 \end{pmatrix} \right\rangle \leq \mathrm{GL}(3, k)$$

ω: **primitive 4-th root of unity.**

$\widetilde{Q_G} =$ 

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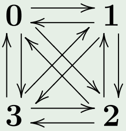
$Q_G =$ 

$= \overline{\widetilde{Q_G}}$

Example

$$S = k\langle x, y\rangle/(x^2y - yx^2, xy^2 - y^2x)$$

$$\deg x = \deg y = 1$$

$\Rightarrow$ S **is an AS-regular algebra of dimension 3 and of Gorenstein parameter 4**

$$\Rightarrow \quad G = \left\langle \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} \right\rangle \leq \mathrm{GL}(2, k)$$

ω: **primitive 4-th root of unity.**

$$\widetilde{Q_G} = \begin{array}{ccc} 0 & \rightrightarrows & 1 \\ & & \downdownarrows \\ 3 & \leftleftarrows & 2 \end{array}, \qquad Q_G = \begin{array}{ccc} 0 & \rightrightarrows & 1 \\ \upuparrows & & \downdownarrows \\ 3 & \leftleftarrows & 2 \end{array} \neq \overline{\widetilde{Q_G}}$$