Parametrizatons for integral slope homogeneous modules over tubular canonical algebras

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Conventions

- Λ – a finite-dimensional algebra over a field k,
- $\operatorname{mod}\Lambda$ = finite-dimensional right Λ-modules.

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If $\Lambda = kQ/I$, for a bound quiver (Q, I), then

$$\operatorname{mod}\Lambda \equiv \operatorname{rep}_k(Q, I) \ni M = (\{M_i\}_{i\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1}),$$

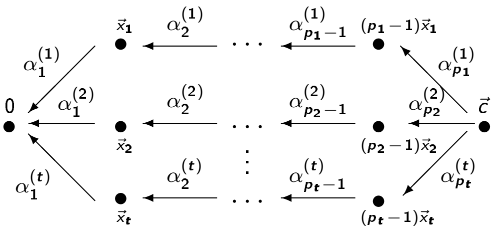
- M_i - vector space over k ($M_i = k^{n_i}$),
- $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ - k-map ($M_\alpha \in \mathbb{M}_{n_{t(\alpha)}\times n_{s(\alpha)}}(k)$)

($\operatorname{mod}^{\mathrm{mat}}\Lambda$ = *matrix modules*).

Canonical algebras

Since now $k = \overline{k}$.

Definition. *Canonical algebra* = k-algebra $\Lambda(\mathbf{p}, \boldsymbol{\lambda}) = kQ_{\mathbf{p}}/I_{\mathbf{p},\boldsymbol{\lambda}}$,
with fixed $\mathbf{p} = (p_1, p_2, \ldots, p_t) \in \mathbb{N}^t$, $t \geq 2$,
$\boldsymbol{\lambda} = (\lambda_3, \ldots, \lambda_t) \in (k^*)^{t-2}$,
$Q_{\mathbf{p}}$:



and $I_{\mathbf{p},\boldsymbol{\lambda}} \lhd kQ_{\mathbf{p}}$ is an ideal generated by all
$\alpha_{p_i}^{(i)} \cdots \alpha_2^{(i)}\alpha_1^{(i)} - \alpha_{p_1}^{(1)} \cdots \alpha_2^{(1)}\alpha_1^{(1)} - \lambda_i \alpha_{p_2}^{(2)} \cdots \alpha_2^{(2)}\alpha_1^{(2)}$, $3 \leq i \leq t$.
(We can assume that $\lambda_3 = 1$).

Tubular canonical algebras

Definition. *Tubular canonical algebra* = canonical algebra $\Lambda = \Lambda(\mathbf{p}, \boldsymbol{\lambda})$, where $\mathbf{p}$ equals:
$(2,2,2,2), \ (3,3,3), \ (2,4,4), \ \text{or} \ (2,3,6)$.

- In case $\mathbf{p} = (2,2,2,2)$, Λ depends on one parameter $\lambda = \lambda_4$.
- Set $p := \mathrm{lcm}(\mathbf{p})$ $(= \max(\mathbf{p})$ in tubular case).

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Tubular canonical algebras are tame of polynomial (linear) growth representation type (but non-domestic).

Tubular canonical algebras - module invariants

Λ = a fixed tubular canonical algebra.

For $M \in \operatorname{mod}\Lambda$ we define its *rank* and *degree*:

$$\operatorname{rk}(M) := \dim_k M_0 - \dim_k M_{\vec{c}}\,,$$

$$\deg(M) := \sum_{i=1}^{t} \frac{p}{p_i}\left(\sum_{j=1}^{p_i-1} \dim_k M_{j\vec{x}_i}\right) - p \cdot \dim_k M_{\vec{c}}$$

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and (if M is indecomposable) its *slope*: $\mu(M) := \frac{\deg(M)}{\operatorname{rk}(M)} \in \bar{\mathbb{Q}}$,
where $\bar{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$.

Tubular canonical algebras - the module category

The structure of the Auslander-Reiten quiver Γ_Λ:

$$\Gamma_\Lambda = \mathcal{P} \sqcup \Big(\coprod_{q \in \bar{\mathbb{Q}}'} \widetilde{\mathcal{T}}_q \Big) \sqcup \mathcal{Q}$$

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- $\mathcal{P}$ is a preprojective component;
- each $\widetilde{\mathcal{T}}_q$, $q \in \bar{\mathbb{Q}}'$, is a $\mathbb{P}^1(k)$*-family of tubes of type* $\mathbf{p}$, almost all are stable except two cases:
  - $\widetilde{\mathcal{T}}_p$ contains tube obtained by coray deletion from a stable tube of rank p ($\Rightarrow$ contains projective),
  - $\widetilde{\mathcal{T}}_0$ contains tube obtained by ray deletion from a stable tube of rank p ($\Rightarrow$ contains injective);
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• $\mu(M) = q$, for all $M \in \widetilde{\mathcal{T}}_q$ and $q \in \bar{\mathbb{Q}}'$.
• $\widetilde{\mathcal{T}}_q \supseteq \widetilde{\mathcal{T}}_q^{\mathrm{h}}$ = homogeneous tubes. Non-homogeneous tubes in each $\widetilde{\mathcal{T}}_q$ have "parameters": $0, 1, \infty \in \mathbb{P}^1(k)$ (and λ in case $(2, 2, 2, 2)$).

Tubular canonical algebras - the module category

- One describes the correspondence:

slope $\bar{\mathbb{Q}}' = \bar{\mathbb{Q}} \setminus (0, p)$	$\leftrightarrow$	Ringel's index $\mathbb{Q}_{\geq 0} \cup \{\infty\}$
p	$\mapsto$	0
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Goal:

to describe the modules for the remaining (homogeneous) c's...

Bimodules - generalities

- Λ – a finite-dimensional k-algebra,
- R – a commutative k-algebra,
- $\operatorname{mod} R \otimes_k \Lambda$ = R-Λ-bimodules fin. gen. as R-modules,
- $\operatorname{mod} R \otimes_k \Lambda \supseteq \operatorname{mod}_R \Lambda$ = R-free.

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$$\mathrm{mod}_R \Lambda \;\equiv\; \mathrm{rep}_R(Q, I) \;\ni\; B = (\{B_i\}_{i \in Q_0}, \{B_\alpha\}_{\alpha \in Q_1}),$$

- B_i - fin. gen. free R-module ($B_i = R^{n_i}$),
- $B_\alpha : B_{s(\alpha)} \to B_{t(\alpha)}$ - R-map ($B_\alpha \in \mathbb{M}_{n_{t(\alpha)} \times n_{s(\alpha)}}(R)$)

($\mathrm{mod}_R^{\mathrm{mat}} \Lambda \;=\;$ *matrix bimodules*).

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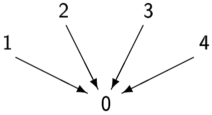
In other words, the specializations

$$B(c,\ell) := R/(T-c)^\ell \otimes_R B,$$

for $(c,\ell) \in (k \setminus \mathcal{Z}(f)) \times \mathbb{N}_+$, form 1-parameter family $\widetilde{\mathcal{T}}^{\mathrm{h}} = \{\mathcal{T}_c\}_{c \in k \setminus \mathcal{Z}(f)}$ of homogeneous tubes in $\mathrm{mod}\,\Lambda$ consisting of modules with dimension vectors in $\mathbb{N} \cdot \underline{\mathrm{rank}}(B)$.

(We say B is a *parametrizing bimodule for the family* $\widetilde{\mathcal{T}}^{\mathrm{h}}$).

Bimodules - generalities

Example. $\Lambda = kQ$, where $Q =$ 

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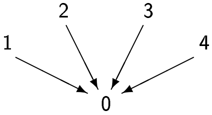
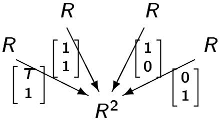


B := 

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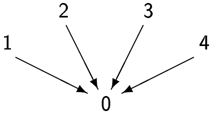

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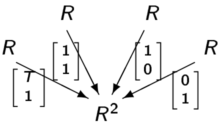

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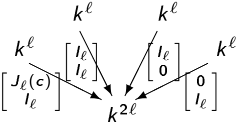

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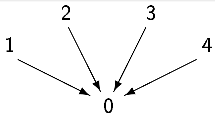
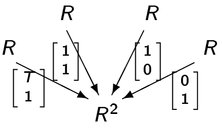
Example. $\Lambda = kQ$, where $Q =$ 

B := 

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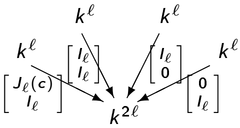
$B(c, \ell) \;\cong$ 

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Note: there are three tubes of rank 2 (with "parameters" $0, 1, \infty$).
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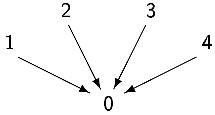

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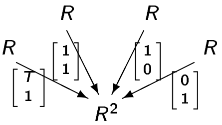

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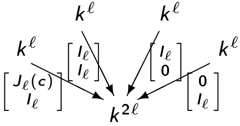

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Remark. For a tubular canonical algebra Λ one can expect parametrizing bimodules indexed by $\bar{\mathbb{Q}}'$.

The main result - bimodules for integral slopes

$$k[T] \ni f_{\mathbf{p}} := \begin{cases} T(T-1)(T-\lambda), & \text{if} \quad \mathbf{p} = (2,2,2,2), \\ T(T-1), & \text{if} \quad \mathbf{p} \neq (2,2,2,2). \end{cases}$$

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Theorem (Dowbor-Meltzer-M.)

Let $\Lambda = \Lambda(\mathbf{p}, \boldsymbol{\lambda})$ be a tubular canonical algebra. Then for any $n \in \mathbb{Z}_{\geq p}$ the matrix bimodule

$$(B^{(n)})_{f_{\mathbf{p}}}$$

is a parametrizing bimodule for $\widetilde{\mathcal{T}}_n^{\mathrm{h}}$, where

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- the remaining (non-integer) slopes – **Piotr Dowbor's talk.**

The main result - bimodules for integral slopes

Notation: For any $m \in \mathbb{N}$, we denote by $X = X_m$ and $Y = Y_m$ the matrices

$$X = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \in \mathbb{M}_{(m+1)\times m}(k).$$

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We set

- $Z^\lambda = Z_m^\lambda := X_m + \lambda Y_m$, for $\lambda \in k$,
- we write $Z = Z_m^1$,
- $I = I_m$ – the identity matrix,
- $\dot{X} := X_{m+1}$, $\dot{Y} := Y_{m+1}$, $\dot{Z} := Z_{m+1}$, $\dot{Z}^\lambda = Z_{m+1}^\lambda$,
- $\dot{I} := I_{m+1}$, $\ddot{I} := I_{m+2}$.

[We also consider the empty matrices, i.e. for $m = 0$.]

Bimodules for integral slopes for $(2,2,2,2;\lambda)$

$B^{(n)}$, $n = 2 + 2m$:

$$\begin{bmatrix} & 1+2m & \\ & 1+2m & \\ & (m+1)+m & \\ 2(m+1) & m+(m+1) & 2m \end{bmatrix}$$

$$\begin{bmatrix}
\left[\begin{array}{c|c|c} 0 & & \\ \vdots & X & \\ 0 & & \\ T & & \\ \hline 0 & & \\ \vdots & & X \\ 0 & & \\ 1 & & \end{array}\right] & \left[\begin{array}{c|c} 0\cdots 0 & 0\cdots 0 \\ \hline I & \\ \hline & I \end{array}\right] \\
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\left[\begin{array}{c|c} \dot{I} & \\ \hline & Z \end{array}\right] & \left[\begin{array}{c|c} Z & \\ \hline & I \end{array}\right] \\
\left[\begin{array}{c|c} Z^{\lambda} & \\ \hline & \dot{I} \end{array}\right] & \left[\begin{array}{c|c} I & \\ \hline & Z^{\lambda} \end{array}\right]
\end{bmatrix}$$

$B^{(n)}$, $n = 2 + 2m + 1$:

$$\begin{bmatrix} & 1+m+(m+1) & \\ & 1+m+(m+1) & \\ & 2(m+1) & \\ (m+1)+(m+2) & 2(m+1) & m+(m+1) \end{bmatrix}$$

$$\begin{bmatrix}
\left[\begin{array}{c|c|c} 0 & & \\ \vdots & X & \\ 0 & & \\ T & & \\ \hline 0 & & \\ \vdots & & \dot{X} \\ 0 & & \\ 1 & & \\ \lambda & & \end{array}\right] & \left[\begin{array}{c|c} 0\cdots 0 & 0\cdots 0 \\ \hline I & \\ \hline & \dot{I} \end{array}\right] \\
\left[\begin{array}{c|c|c} 1 & & \\ 0 & Y & \\ \vdots & & \\ 0 & & \\ \hline 1 & & \\ \lambda & & \\ 0 & & \dot{Y} \\ \vdots & & \\ 0 & & \end{array}\right] & \left[\begin{array}{c|c} 0\cdots 0 & 0\cdots 0 \\ \hline I & \\ \hline & \dot{I} \end{array}\right] \\
\left[\begin{array}{c|c} \dot{I} & \\ \hline & \dot{Z} \end{array}\right] & \left[\begin{array}{c|c} Z & \\ \hline & \dot{I} \end{array}\right] \\
\left[\begin{array}{c|c} \dot{I} & \\ \hline & \dot{Z}^{\lambda} \end{array}\right] & \left[\begin{array}{c|c} Z^{\lambda} & \\ \hline & \dot{I} \end{array}\right]
\end{bmatrix}$$

$(3,3,3)$, bimodules $B^{(n)}$, $n = 3 + 3m$

$$
\left[\begin{array}{ccc}
\left[\begin{array}{cc|c|c|c}
0 & 0 & X & & \\
\vdots & \vdots & & & \\
0 & 0 & & & \\
1 & T & & & \\ \hline
0 & 0 & & X & \\
\vdots & \vdots & & & \\
0 & 0 & & & \\
1 & 1 & & & \\ \hline
0 & 0 & & & X \\
\vdots & \vdots & & & \\
0 & 0 & & & \\
1 & 0 & & &
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
1 & 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 \\
0 & 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 \\ \hline
0 & I & & \\
\vdots & & & \\
0 & & & \\ \hline
0 & & I & \\
\vdots & & & \\
0 & & & \\ \hline
0 & & & I \\
\vdots & & & \\
0 & & &
\end{array}\right]
&
\left[\begin{array}{c|c|c}
0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 \\ \hline
I & & \\ \hline
 & I & \\ \hline
 & & I
\end{array}\right]
\\
\left[\begin{array}{c|c|c|c}
\dot{I} & 0 & & \\
 & \vdots & & \\
 & 0 & & \\ \hline
 & 1 & Y & \\
 & 0 & & \\
 & \vdots & & \\
 & 0 & & \\ \hline
 & 0 & & Y \\
 & \vdots & & \\
 & 0 & &
\end{array}\right]
&
\left[\begin{array}{c|c|c}
\dot{I} & & \\ \hline
0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 \\ \hline
 & I & \\ \hline
 & & I
\end{array}\right]
&
\left[\begin{array}{c|c|c}
Y & & \\ \hline
 & I & \\ \hline
 & & I
\end{array}\right]
\\
\left[\begin{array}{c|c|c}
Z & & \\ \hline
 & \dot{I} & \\ \hline
 & & \dot{I}
\end{array}\right]
&
\left[\begin{array}{c|c|c}
I & & \\ \hline
 & Z & \\ \hline
 & & \dot{I}
\end{array}\right]
&
\left[\begin{array}{c|c|c}
I & & \\ \hline
 & I & \\ \hline
 & & Z
\end{array}\right]
\end{array}\right]
$$

$(3,3,3)$, bimodules $B^{(n)}$, $n = 3 + 3m + 1$

$$
\left[\begin{array}{ccc}
\left[\begin{array}{cc|c|c|c}
0 & 0 & & & \\
\vdots & \vdots & X & & \\
0 & 0 & & & \\
{\color{red}1} & {\color{red}T} & & & \\ \hline
0 & 0 & & & \\
\vdots & \vdots & & X & \\
0 & 0 & & & \\
{\color{red}1} & {\color{red}1} & & & \\ \hline
0 & 0 & & & \\
\vdots & \vdots & & & \dot{X} \\
0 & 0 & & & \\
{\color{red}1} & 0 & & & \\
{\color{red}1} & 0 & & &
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
{\color{red}1} & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 \\
0 & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 \\ \hline
0 & & & \\
\vdots & I & & \\
0 & & & \\ \hline
0 & & & \\
\vdots & & I & \\
0 & & & \\ \hline
0 & & & \\
\vdots & & & \dot{I} \\
0 & & &
\end{array}\right]
&
\left[\begin{array}{c|c|c}
0\cdots 0 & 0\cdots 0 & 0\cdots 0 \\ \hline
I & & \\ \hline
 & I & \\ \hline
 & & \dot{I}
\end{array}\right]
\\
\left[\begin{array}{c|c|c|c}
 & 0 & & \\
\dot{I} & \vdots & & \\
 & 0 & & \\ \hline
 & {\color{red}1} & & \\
 & 0 & Y & \\
 & \vdots & & \\
 & 0 & & \\ \hline
 & 0 & & \\
 & \vdots & & \dot{Y} \\
 & 0 & &
\end{array}\right]
&
\left[\begin{array}{c|c|c}
\dot{I} & & \\ \hline
0\cdots 0 & 0\cdots 0 & 0\cdots 0 \\ \hline
 & I & \\ \hline
 & & \dot{I}
\end{array}\right]
&
\left[\begin{array}{c|c|c}
Y & & \\ \hline
 & I & \\ \hline
 & & \dot{I}
\end{array}\right]
\\
\left[\begin{array}{c|c|c}
\dot{I} & & \\ \hline
 & I & \\ \hline
 & & \dot{Z}
\end{array}\right]
&
\left[\begin{array}{c|c|c}
Z & & \\ \hline
 & \dot{I} & \\ \hline
 & & \dot{I}
\end{array}\right]
&
\left[\begin{array}{c|c|c}
I & & \\ \hline
 & Z & \\ \hline
 & & \dot{I}
\end{array}\right]
\end{array}\right]
$$

$(3,3,3)$, bimodules $B^{(n)}$, $n = 3 + 3m + 2$

$$
\left[\begin{array}{ccc}
\left[\begin{array}{cc|c|c|c}
0 & 0 & X & & \\
\vdots & \vdots & & & \\
0 & 0 & & & \\
1 & T & & & \\
\hline
0 & 0 & & \dot{X} & \\
\vdots & \vdots & & & \\
0 & 0 & & & \\
1 & 1 & & & \\
1 & 1 & & & \\
\hline
0 & 0 & & & \dot{X} \\
\vdots & \vdots & & & \\
0 & 0 & & & \\
1 & 0 & & & \\
1 & 0 & & &
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
1 & 0\cdots0 & 0\cdots0 & 0\cdots0 \\
0 & 0\cdots0 & 0\cdots0 & 0\cdots0 \\
\hline
0 & I & & \\
\vdots & & & \\
0 & & & \\
\hline
0 & & \dot{I} & \\
\vdots & & & \\
0 & & & \\
\hline
0 & & & \dot{I} \\
\vdots & & & \\
0 & & &
\end{array}\right]
&
\left[\begin{array}{c|c|c}
0\cdots0 & 0\cdots0 & 0\cdots0 \\
\hline
I & & \\
\hline
& \dot{I} & \\
\hline
& & \dot{I}
\end{array}\right]
\\
\left[\begin{array}{c|c|c|c}
\dot{I} & 0 & & \\
& \vdots & & \\
& 0 & & \\
\hline
& 1 & \dot{Y} & \\
& 1 & & \\
& 0 & & \\
& \vdots & & \\
& 0 & & \\
\hline
& 0 & & \dot{Y} \\
& \vdots & & \\
& 0 & &
\end{array}\right]
&
\left[\begin{array}{c|c|c}
\dot{I} & & \\
\hline
0\cdots0 & 0\cdots0 & 0\cdots0 \\
\hline
& \dot{I} & \\
\hline
& & \dot{I}
\end{array}\right]
&
\left[\begin{array}{c|c|c}
Y & & \\
\hline
& \dot{I} & \\
\hline
& & \dot{I}
\end{array}\right]
\\
\left[\begin{array}{c|c|c}
\dot{I} & & \\
\hline
& \dot{Z} & \\
\hline
& & \ddot{I}
\end{array}\right]
&
\left[\begin{array}{c|c|c}
\dot{I} & & \\
\hline
& \dot{I} & \\
\hline
& & \dot{Z}
\end{array}\right]
&
\left[\begin{array}{c|c|c}
Z & & \\
\hline
& \dot{I} & \\
\hline
& & \dot{I}
\end{array}\right]
\end{array}\right]
$$

$(2,4,4)$, bimodules $B^{(n)}$, $n = 4 + 4m$

$$
\left[\begin{array}{cccc}
\left[\begin{array}{cc|c|c|c|c}
\begin{smallmatrix}0\\ \vdots\\ 0\\ T\end{smallmatrix} & \begin{smallmatrix}0\\ \vdots\\ 0\\ 1\end{smallmatrix} & X & & & \\ \hline
\begin{smallmatrix}0\\ \vdots\\ 0\\ 1\end{smallmatrix} & \begin{smallmatrix}0\\ \vdots\\ 0\\ 0\end{smallmatrix} & & X & & \\ \hline
\begin{smallmatrix}0\\ \vdots\\ 0\\ 1\end{smallmatrix} & \begin{smallmatrix}0\\ \vdots\\ 0\\ 1\end{smallmatrix} & & & X & \\ \hline
\begin{smallmatrix}0\\ \vdots\\ 0\\ 0\end{smallmatrix} & \begin{smallmatrix}0\\ \vdots\\ 0\\ 1\end{smallmatrix} & & & & X
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
\begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} & \begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} & \begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} & \begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} \\ \hline
I & & & \\ \hline
 & I & & \\ \hline
 & & I & \\ \hline
 & & & I
\end{array}\right]
& & \\
\left[\begin{array}{c|cc|c|c|c}
\dot{I} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & & & \\ \hline
 & \begin{smallmatrix}1\\ 0\\ \vdots\\ 0\end{smallmatrix} & \begin{smallmatrix}0\\ 0\\ \vdots\\ 0\end{smallmatrix} & Y & & \\ \hline
 & \begin{smallmatrix}0\\ 0\\ \vdots\\ 0\end{smallmatrix} & \begin{smallmatrix}1\\ 0\\ \vdots\\ 0\end{smallmatrix} & & Y & \\ \hline
 & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & & & Y
\end{array}\right]
&
\left[\begin{array}{c|c|c|c|c}
\dot{I} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & & & \\ \hline
\begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} & \begin{smallmatrix}1\\ 0\end{smallmatrix} & \begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} & \begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} & \begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} \\ \hline
 & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & I & & \\ \hline
 & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & & I & \\ \hline
 & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & & & I
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
\dot{I} & & & \\ \hline
0\cdots 0 & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 \\ \hline
 & I & & \\ \hline
 & & I & \\ \hline
 & & & I
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
Y & & & \\ \hline
 & I & & \\ \hline
 & & I & \\ \hline
 & & & I
\end{array}\right] \\
\left[\begin{array}{c|c|c|c}
Z & & & \\ \hline
 & \dot{I} & & \\ \hline
 & & \dot{I} & \\ \hline
 & & & \dot{I}
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
I & & & \\ \hline
 & Z & & \\ \hline
 & & \dot{I} & \\ \hline
 & & & \dot{I}
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
I & & & \\ \hline
 & I & & \\ \hline
 & & Z & \\ \hline
 & & & \dot{I}
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
I & & & \\ \hline
 & I & & \\ \hline
 & & I & \\ \hline
 & & & Z
\end{array}\right]
\end{array}\right]
$$

$(2,4,4)$, bimodules $B^{(n)}$, $n = 4+4m+1$

$$
\left[\begin{array}{cccc}
\left[\begin{array}{cc|c|c|c|c}
\begin{smallmatrix}0\\ \vdots\\ 0\\ T\end{smallmatrix} & \begin{smallmatrix}0\\ \vdots\\ 0\\ 1\end{smallmatrix} & X & & & \\ \hline
\begin{smallmatrix}0\\ \vdots\\ 0\\ 1\end{smallmatrix} & \begin{smallmatrix}0\\ \vdots\\ 0\\ 0\end{smallmatrix} & & X & & \\ \hline
\begin{smallmatrix}0\\ \vdots\\ 0\\ 1\end{smallmatrix} & \begin{smallmatrix}0\\ \vdots\\ 0\\ 1\end{smallmatrix} & & & X & \\ \hline
\begin{smallmatrix}0\\ \vdots\\ 0\\ 0\\ 0\end{smallmatrix} & \begin{smallmatrix}0\\ \vdots\\ 0\\ 1\\ 1\end{smallmatrix} & & & & \dot X
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
\begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} & \begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} & \begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} & \begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} \\ \hline
I & & & \\ \hline
 & I & & \\ \hline
 & & I & \\ \hline
 & & & \dot I
\end{array}\right]
& & \\
\left[\begin{array}{c|cc|c|c|c}
\dot I & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & & & \\ \hline
 & \begin{smallmatrix}1\\ 0\\ \vdots\\ 0\end{smallmatrix} & \begin{smallmatrix}0\\ 0\\ \vdots\\ 0\end{smallmatrix} & Y & & \\ \hline
 & \begin{smallmatrix}0\\ 0\\ \vdots\\ 0\end{smallmatrix} & \begin{smallmatrix}1\\ 0\\ \vdots\\ 0\end{smallmatrix} & & Y & \\ \hline
 & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & & & \dot Y
\end{array}\right]
&
\left[\begin{array}{c|c|c|c|c}
\dot I & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & & & \\ \hline
\begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} & \begin{smallmatrix}1\\ 0\end{smallmatrix} & \begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} & \begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} & \begin{smallmatrix}0\cdots 0\\ 0\cdots 0\end{smallmatrix} \\ \hline
 & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & I & & \\ \hline
 & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & & I & \\ \hline
 & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} & & & \dot I
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
\dot I & & & \\ \hline
0\cdots 0 & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 \\ \hline
 & I & & \\ \hline
 & & I & \\ \hline
 & & & \dot I
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
Y & & & \\ \hline
 & I & & \\ \hline
 & & I & \\ \hline
 & & & \dot I
\end{array}\right]
\\
\left[\begin{array}{c|c|c|c}
\dot I & & & \\ \hline
 & \dot I & & \\ \hline
 & & \dot I & \\ \hline
 & & & \dot Z
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
Z & & & \\ \hline
 & \dot I & & \\ \hline
 & & \dot I & \\ \hline
 & & & \dot I
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
I & & & \\ \hline
 & Z & & \\ \hline
 & & \dot I & \\ \hline
 & & & \dot I
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
I & & & \\ \hline
 & I & & \\ \hline
 & & Z & \\ \hline
 & & & \dot I
\end{array}\right]
\end{array}\right]
$$

$(2,4,4)$, bimodules $B^{(n)}$, $n = 4 + 4m + 2$

$$
\left[\begin{array}{cccc}
\left[\begin{array}{c|c|c|c|c}
\begin{smallmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ T & 1 \end{smallmatrix} & X & & & \\ \hline
\begin{smallmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 1 & 0 \end{smallmatrix} & & X & & \\ \hline
\begin{smallmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix} & & & \dot{X} & \\ \hline
\begin{smallmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{smallmatrix} & & & & \dot{X}
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
\begin{smallmatrix} 0 \cdots 0 \\ 0 \cdots 0 \end{smallmatrix} & \begin{smallmatrix} 0 \cdots 0 \\ 0 \cdots 0 \end{smallmatrix} & \begin{smallmatrix} 0 \cdots 0 \\ 0 \cdots 0 \end{smallmatrix} & \begin{smallmatrix} 0 \cdots 0 \\ 0 \cdots 0 \end{smallmatrix} \\ \hline
I & & & \\ \hline
 & I & & \\ \hline
 & & \dot{I} & \\ \hline
 & & & \dot{I}
\end{array}\right]
& & \\
\left[\begin{array}{c|c|c|c|c}
\dot{I} & \begin{smallmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{smallmatrix} & & & \\ \hline
 & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{smallmatrix} & Y & & \\ \hline
 & \begin{smallmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{smallmatrix} & & \dot{Y} & \\ \hline
 & \begin{smallmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{smallmatrix} & & & \dot{Y}
\end{array}\right]
&
\left[\begin{array}{c|c|c|c|c}
\dot{I} & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} & & & \\ \hline
\begin{smallmatrix} 0 \cdots 0 \\ 0 \cdots 0 \end{smallmatrix} & \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} & \begin{smallmatrix} 0 \cdots 0 \\ 0 \cdots 0 \end{smallmatrix} & \begin{smallmatrix} 0 \cdots 0 \\ 0 \cdots 0 \end{smallmatrix} & \begin{smallmatrix} 0 \cdots 0 \\ 0 \cdots 0 \end{smallmatrix} \\ \hline
 & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} & I & & \\ \hline
 & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} & & \dot{I} & \\ \hline
 & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} & & & \dot{I}
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
\dot{I} & & & \\ \hline
\begin{smallmatrix} 0 \cdots 0 \end{smallmatrix} & \begin{smallmatrix} 0 \cdots 0 \end{smallmatrix} & \begin{smallmatrix} 0 \cdots 0 \end{smallmatrix} & \begin{smallmatrix} 0 \cdots 0 \end{smallmatrix} \\ \hline
 & I & & \\ \hline
 & & \dot{I} & \\ \hline
 & & & \dot{I}
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
Y & & & \\ \hline
 & I & & \\ \hline
 & & \dot{I} & \\ \hline
 & & & \dot{I}
\end{array}\right] \\
\left[\begin{array}{c|c|c|c}
\dot{I} & & & \\ \hline
 & \dot{I} & & \\ \hline
 & & \dot{Z} & \\ \hline
 & & & \ddot{I}
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
\dot{I} & & & \\ \hline
 & \dot{I} & & \\ \hline
 & & \dot{I} & \\ \hline
 & & & \dot{Z}
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
Z & & & \\ \hline
 & \dot{I} & & \\ \hline
 & & \dot{I} & \\ \hline
 & & & \dot{I}
\end{array}\right]
&
\left[\begin{array}{c|c|c|c}
I & & & \\ \hline
 & Z & & \\ \hline
 & & \dot{I} & \\ \hline
 & & & \dot{I}
\end{array}\right]
\end{array}\right]
$$

$(2,4,4)$, bimodules $B^{(n)}$, $n = 4 + 4m + 3$

$$
\left[\begin{array}{cc|c|c|c|c}
0 & 0 & X & & & \\
\vdots & \vdots & & & & \\
0 & 0 & & & & \\
{\color{red}T} & {\color{red}1} & & & & \\
\hline
0 & 0 & & \dot{X} & & \\
\vdots & \vdots & & & & \\
0 & 0 & & & & \\
{\color{red}1} & 0 & & & & \\
{\color{red}1} & 0 & & & & \\
\hline
0 & 0 & & & \dot{X} & \\
\vdots & \vdots & & & & \\
0 & 0 & & & & \\
{\color{red}1} & {\color{red}1} & & & & \\
{\color{red}1} & {\color{red}1} & & & & \\
\hline
0 & 0 & & & & \dot{X} \\
\vdots & \vdots & & & & \\
0 & 0 & & & & \\
0 & {\color{red}1} & & & & \\
0 & {\color{red}1} & & & &
\end{array}\right]
\qquad
\left[\begin{array}{c|c|c|c}
0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 \\
0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 \\
\hline
I & & & \\
\hline
 & \dot{I} & & \\
\hline
 & & \dot{I} & \\
\hline
 & & & \dot{I}
\end{array}\right]
$$

$$
\left[\begin{array}{c|cc|c|c|c}
\dot{I} & 0 & 0 & & & \\
 & \vdots & \vdots & & & \\
 & 0 & 0 & & & \\
\hline
 & {\color{red}1} & 0 & \dot{Y} & & \\
 & {\color{red}1} & 0 & & & \\
 & 0 & 0 & & & \\
 & \vdots & \vdots & & & \\
 & 0 & 0 & & & \\
\hline
 & 0 & {\color{red}1} & & \dot{Y} & \\
 & 0 & {\color{red}1} & & & \\
 & 0 & 0 & & & \\
 & \vdots & \vdots & & & \\
 & 0 & 0 & & & \\
\hline
 & 0 & 0 & & & \dot{Y} \\
 & \vdots & \vdots & & & \\
 & 0 & 0 & & &
\end{array}\right]
\quad
\left[\begin{array}{c|c|c|c|c}
\dot{I} & 0 & & & \\
 & \vdots & & & \\
 & 0 & & & \\
\hline
0 \cdots 0 & {\color{red}1} & 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 \\
0 \cdots 0 & 0 & 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 \\
\hline
 & 0 & \dot{I} & & \\
 & \vdots & & & \\
 & 0 & & & \\
\hline
 & 0 & & \dot{I} & \\
 & \vdots & & & \\
 & 0 & & & \\
\hline
 & 0 & & & \dot{I} \\
 & \vdots & & & \\
 & 0 & & &
\end{array}\right]
\quad
\left[\begin{array}{c|c|c|c}
\dot{I} & & & \\
\hline
0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 \\
\hline
 & \dot{I} & & \\
\hline
 & & \dot{I} & \\
\hline
 & & & \dot{I}
\end{array}\right]
\quad
\left[\begin{array}{c|c|c|c}
Y & & & \\
\hline
 & \dot{I} & & \\
\hline
 & & \dot{I} & \\
\hline
 & & & \dot{I}
\end{array}\right]
$$

$$
\left[\begin{array}{c|c|c|c}
\dot{I} & & & \\
\hline
 & \dot{Z} & & \\
\hline
 & & \ddot{I} & \\
\hline
 & & & \cdots
\end{array}\right]
\quad
\left[\begin{array}{c|c|c|c}
\dot{I} & & & \\
\hline
 & \dot{I} & & \\
\hline
 & & \dot{Z} & \\
\hline
 & & & \cdots
\end{array}\right]
\quad
\left[\begin{array}{c|c|c|c}
\dot{I} & & & \\
\hline
 & \dot{I} & & \\
\hline
 & & \dot{I} & \\
\hline
 & & & \cdots
\end{array}\right]
\quad
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Z & & & \\
\hline
 & \dot{I} & & \\
\hline
 & & \dot{I} & \\
\hline
 & & & \cdots
\end{array}\right]
$$

$(2,3,6)$, bimodules $B^{(n)}$

Six series: for

$$n = 6 + 6m + i,$$

$i = 0, 1, 2, 3, 4, 5.$

Remarks

The fact that those bimodules:

- appear in a finite number of series,
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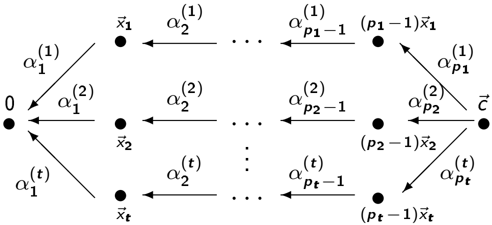
We have obtained such appearance of bimodules thanks to:

- a periodical nature of (functorial) techniques we applied,
- certain not completely trivial computer algebra facts concerning matrix calculus over $k[T]$ we used.

About the construction

Recall that $Q = Q_{\mathbf{p}}$:



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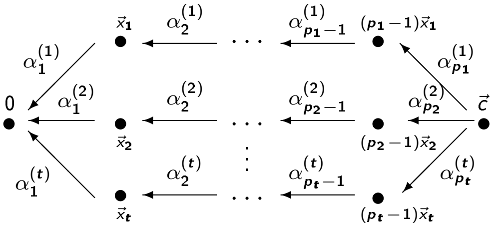

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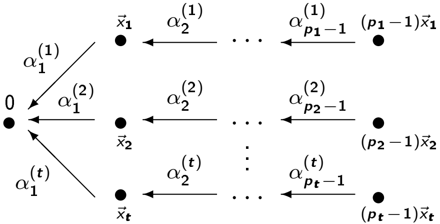

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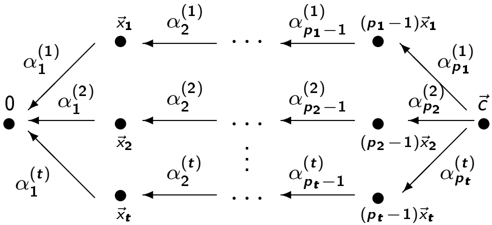

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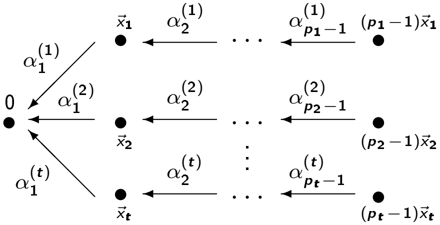

Λ	Λ_0
(2,2,2,2)	$\tilde{\mathbb{D}}_4$
(3,3,3)	$\tilde{\mathbb{E}}_6$
(2,4,4)	$\tilde{\mathbb{E}}_7$
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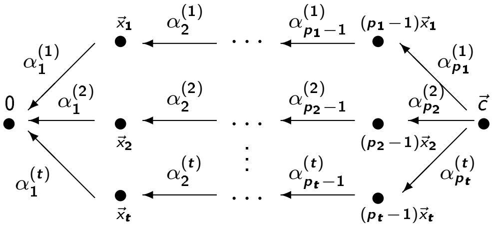

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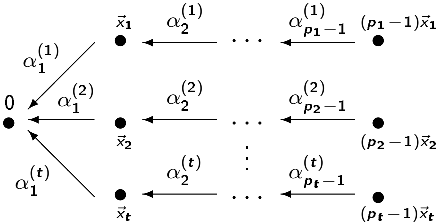

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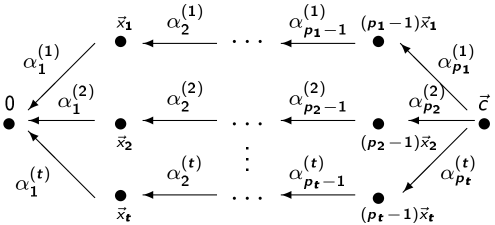

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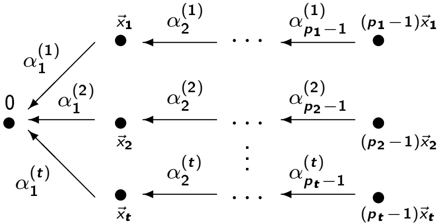

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$$\operatorname{mod}\Lambda_0 \hookrightarrow \operatorname{mod}\Lambda$$

$$\operatorname{reg}\Lambda_0 \hookrightarrow \widetilde{\mathcal{T}}_p$$

About the construction - main steps

Step 1: construct a proj. pres. $\forall$ homogeneous mod. $\mathcal{H}^{(c,\ell)} \in \widetilde{\mathcal{T}}_p$:

$$0 \longrightarrow P_1^{(c,\ell)} \xrightarrow[F^{(c,\ell)}]{} P_0^{(c,\ell)} \longrightarrow \mathcal{H}^{(c,\ell)} \longrightarrow 0.$$

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It can be done s.t.:

$$B := \mathrm{Coker}\left(k[T] \otimes_k P_1^{(1)} \underset{F^{(T,1)}}{\longrightarrow} k[T] \otimes_k P_0^{(1)} \right)$$

is a parametrizing bimodule for $\widetilde{\mathcal{T}}_p^{\mathrm{h}}$, i.e. $B(c,\ell) = \mathcal{H}^{(c,\ell)} \ \ \forall_{c,\ell}$.

About the construction - main steps

Step 2: precisely describe how the functor $\Delta_{(n)}$, for $n \in \mathbb{N}$:

$$\begin{array}{ccc} \mathcal{D}^b(\mathrm{coh}\,\mathbb{X}) & \xrightarrow{\mathcal{S}^n} & \mathcal{D}^b(\mathrm{coh}\,\mathbb{X}) \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{D}^b(\mathrm{mod}\,\Lambda) & \xrightarrow{\Delta_{(n)}} & \mathcal{D}^b(\mathrm{mod}\,\Lambda) \end{array}$$

behave on $\mathrm{mod}^{\mathrm{mat}}\,\Lambda$, where $\mathcal{S}$ is induced by the shift functor

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In particular

$$\Delta_{(n)_|} : \widetilde{\mathcal{T}}_p \to \widetilde{\mathcal{T}}_{p+n}$$

is an equivalence on homogeneous modules.

About the construction - main steps

Step 3: compute

$$k[T] \otimes_k \Delta_{(n)}(P_1^{(1)}) \underset{\Delta_{(n)}(F^{(T,1)})}{\longrightarrow} k[T] \otimes_k \Delta_{(n)}(P_0^{(1)})$$

for all $n \in \mathbb{N}$.

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Step 4: prove that $B^{(p+n)} := \mathrm{Coker}(\Delta_{(n)}(F^{(T,1)}))$ belong to $\mathrm{mod}_{k[T]}\Lambda$ and is a parametrizing bimodule for $\widetilde{\mathcal{T}}^{\mathrm{h}}_{p+n}$.

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Step 5: compute the explicit matrices of $B^{(p+n)}$ (Smith forms of T-matrices...).

Final remark

We checked that the specializations

$$B^{(n)}(c,\ell) = k[T]/(T-c)^{\ell} \otimes_{k[T]} B^{(n)},$$

for $(c,\ell) \in (k \setminus \mathcal{Z}(f_{\mathbf{p}})) \times \mathbb{N}$ indeed:

Final remark

We checked that the specializations

$$B^{(n)}(c,\ell) = k[T]/(T-c)^{\ell} \otimes_{k[T]} B^{(n)},$$

for $(c,\ell) \in (k \setminus \mathcal{Z}(f_{\mathbf{p}})) \times \mathbb{N}$ indeed:

- have appropriate dimension vectors,
- are indecomposable (their endomorphism algebras turned out to be local),
- are Hom-orthogonal (i.e. for fixed slope n,

  $$\mathrm{Hom}_{\Lambda}(B^{(n)}(c,\ell), B^{(n)}(c',\ell')) = 0$$

  for any $c, c' \in k \setminus \mathcal{Z}(f_{\mathbf{p}})$, $c \neq c'$ and $\ell, \ell' \geq 1$).