The p-regular subspaces of symmetric Nakayama algebras and algebras of dihedral and semidihedral type

Aditya Purwa Santika and Intan Muchtadi-Alamsyah
Algebra Research Group
Institut Teknologi Bandung, Indonesia

- Let p be a prime and G be a finite group with p divides the order of group $|G|$ and F be a ring with characteristic $p > 0$.

- For $h \in G$, the conjugation class generated by h is:

$$C_h = \{g \in G | \exists x \in G : xhx^{-1} = g\}$$

- The centre of the group algebra FG, denoted by $Z(FG)$ has basis

$$\{C_h^+ = \sum_{g \in C_h} g | h \in G\}.$$

- The p-*regular subspace* of $Z(FG)$

$$Z_{p'}(FG) = span\{C_h^+ : p \nmid o(h)\}$$

is the subspace of $Z(FG)$ spanned by class sum of p'-element. We see that this definition depends only on the order of elements in G.

- While $Z(FG)$ is always a ring, this is not the case for $Z_{p'}(FG)$ (see [Meyer], [Meyer2], [FanKulshammer], [EnsslenKulshammer]).

- For any algebra A we define $[A,A]$ as the commutator subspace of A.

- Given a symmetric K-algebra A with symmetrising form $\langle\ ,\ \rangle$, then for any K-basis $\mathcal{B}$ of $Z(A)$ we get a K-basis $\mathcal{B}^*$ of $A/[A,A]$ by the condition

$$\langle b, c^* \rangle = \begin{cases} 1 & \text{if } b = c \\ 0 & \text{otherwise} \end{cases}$$

- Hence we get an identification

(1)
$$\begin{array}{rcl} Z(A) & \stackrel{\delta}{\to} & A/[A,A] \\ \sum_{b \in B} \lambda_b b & \mapsto & \sum_{b' \in B'} \lambda_b b' \end{array}$$

This map depends on the choice of B and the choice of the symmetrizing form $\langle\ ,\ \rangle$.

- In particular $Z(FG)$ can be identified with $FG/[FG, FG]$. This isomorphism will give a way to identify $Z_{p'}FG$ without depending on the order of elements in G.

- Define the p-power map

$$\mu_p : FG/[FG, FG] \to FG/[FG, FG]$$

with $\mu_p(a) = a^p$ for every $a \in FG/[FG, FG]$. This map is well-defined as shown by Kulshammer [Kulshammer].

Lemma 1 (SantikaZimmermann). *Let F be a field of characteristic $p > 0$, let G be a finite group and let $h \in G$. Then*

$h + [FG, FG] \in \bigcap_{t \in \mathbb{N}} im(\mu_p^t)$ *if and only if* $p \nmid o(h)$.

- The above lemma says that an element C_h^+ is in the basis of $Z_{p'}FG$ if and only if $h + [FG, FG] \in \bigcap_{t \in \mathbb{N}} im(\mu_p^t)$.

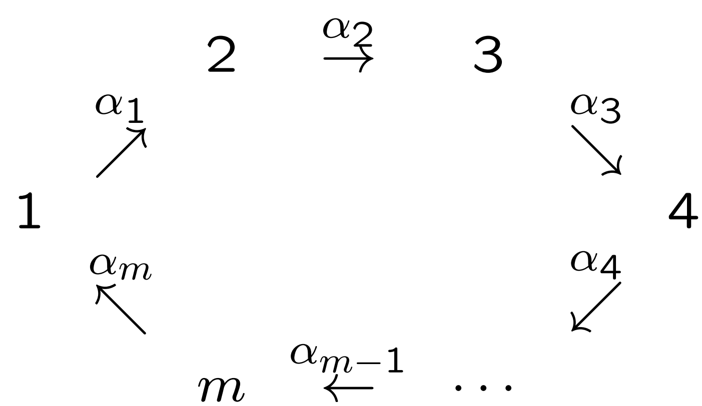
- This condition on the above lemma does not depend on the order of elements of G, so we apply this condition to define $Z_{p'}A$ for any symmetric algebra A.

Definition 1. *Let A be an algebra and let*

$$\mu_p : A/[A, A] \to A/[A, A]$$

with $\mu_p(a) = a^p$ for every $a \in A/[A, A]$.
The p-regular subspace *$Z_{p'}A$ of ZA is defined as the pre-image of $\bigcap_{t \in \mathbb{N}} im(\mu_p^t)$ via the identification δ (1).*

Let A be the symmetric Nakayama algebra N_m^n, where m divides n. The algebra A is the path algebra of the quiver



modulo the relations : $(\alpha_i \alpha_{i+1} \cdots \alpha_{i-2} \alpha_{i-1})^{(n/m)} \alpha_i = 0$ for all $1 \leq i \leq m$.

- A basis of ZA is given by union of $\mathbb{B}_1 \cup \mathbb{B}_2 \cup \mathbb{B}_3$, where $\mathbb{B}_1 = \{e_1 + \cdots + e_m\}$,

$$\mathbb{B}_2 = \{\sum_{i=1}^{m} (\alpha_i \alpha_{i+1} \cdots \alpha_{i-2}\alpha_{i-1})^k, 1 \leq k \leq n/m - 1\},$$

$$\mathbb{B}_3 = \{(\alpha_i \alpha_{i+1} \cdots \alpha_{i-2}\alpha_{i-1})^{n/m}, 1 \leq i \leq m\}.$$

- A basis of $A/[A, A]$ is given by

$$\{e_1, \cdots, e_m, (\alpha_1 \alpha_2 \cdots \alpha_m)^k, 1 \leq k \leq n/m\}.$$

- The first step to find $Z_{p'}A$ is to compute p^t-power of every basis elements of $A/[A, A]$. Since e_i are idempotents, then $e_i^{p^t} = e_i$ for all $1 \leq i \leq m$, for all $t \in \mathbb{N}$.

- For other basis elements, we get that

$$((\alpha_1\alpha_2\cdots\alpha_m)^k)^{(n/m)+1} = 0$$

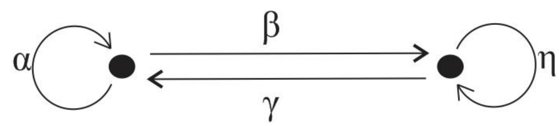
for all $1 \leq k \leq n/m$. Therefore only $e_1, \cdots, e_m$ that are in the image of the p^t-power map μ_p^t for every $t \in \mathbb{N}$.

- In $\mathcal{Z}$ these elements correspond to $\mathbb{B}_3$. Therefore the p-regular subspace of $A = N_m^n$ is $Z_{p'}A = span\mathbb{B}_3$.

- An algebra A (over an algebraically closed field) is *of dihedral type* if it is a symmetric and indecomposable algebra, its Cartan matrix is non-singular, and the stable AR-quiver consists of 1-tubes, at most two 3-tubes, and a non-periodic components of tree class $\mathbb{A}_{\infty}^{\infty}$ or $\mathbb{A}_{1,2}$. [Erdman]

- Any algebra of dihedral type with two simple modules is derived equivalent to the basic algebra $A_c^{k,s} = D(2\mathfrak{B})^{k,s}(c)$, where $k, s \geq 1$ are integers and $c \in \{0, 1\}$ (see [Holm]).

- These algebras are defined by path algebra of quiver



subject to relations

$$\beta\eta = 0, \eta\gamma = 0, \gamma\beta = 0, \alpha^2 = c(\alpha\beta\gamma)^k, (\alpha\beta\gamma)^k = (\beta\gamma\alpha)^k, \eta^s = (\gamma\alpha\beta)^k.$$

- Let us assume that the characteristic of F is 2. Every block with a dihedral defect group of order 2^n and two simple modules is Morita equivalent to $A_c^{1,s}$ (see [Erdman]).

- In [HolmZimmermann], it is shown that a basis of $Z(A_c^{k,s})$ is given by

$$\mathcal{Z} = \{1, (\alpha\beta\gamma)^i + (\beta\gamma\alpha)^i + (\gamma\alpha\beta)^i,$$

$$(\beta\gamma\alpha)^{k-1}\beta\gamma, (\alpha\beta\gamma)^k, \eta^j | 1 \leq i \leq k-1, 1 \leq j \leq s\},$$

and a basis of $A_c^{k,s}/[A_c^{k,s}, A_c^{k,s}]$ is given by

$$\mathbb{B} = \{e_1, e_2, \alpha, \alpha\beta\gamma, \ldots, (\alpha\beta\gamma)^{k-1}, \eta, \ldots, \eta^s\}.$$

- The first step to find $Z_{2'}A_c^{k,s}$ is to compute 2^n-power of every basis elements of $A_c^{k,s}/[A_c^{k,s}, A_c^{k,s}]$.

- Since e_1 and e_2 are idempotents, then $e_1^{2^n} = e_1$ and $e_2^{2^n} = e_2$.

- For other basis elements, we get that

$$
\begin{aligned}
((\alpha\beta\gamma)^i)^{2^k} &= \underbrace{\alpha\beta\gamma\ldots\alpha\beta\gamma}_{k+1}(\alpha\beta\gamma)^{i2^k-(k+1)} \\
&= \alpha(\underbrace{\beta\gamma\alpha\ldots\beta\gamma\alpha}_{k})\beta\gamma(\alpha\beta\gamma)^{i2^k-(k+1)} \\
&= \alpha(\beta\gamma\alpha)^k\beta\gamma(\alpha\beta\gamma)^{i2^k-(k+1)} \\
&= \alpha(\alpha\beta\gamma)^k\beta\gamma(\alpha\beta\gamma)^{i2^k-(k+1)} \\
&= \alpha(\alpha\beta\gamma)^{k-1}\alpha\beta\underbrace{\gamma\beta}_{=0}\gamma(\alpha\beta\gamma)^{i2^k-(k+1)} \\
&= 0
\end{aligned}
$$

for every $i = 1, \ldots, k$.

- For $c = 0$, then $\alpha^2 = 0$.

- If $c = 1$, then

$$\begin{aligned}\alpha^4 &= (\alpha\beta\gamma)^{2k} = (\alpha\beta\gamma)^{k+1}(\alpha\beta\gamma)^{k-1}\\ &= \underbrace{\alpha\beta\gamma\ldots\alpha\beta\gamma}_{k+1}(\alpha\beta\gamma)^{k-1}\\ &= \alpha(\underbrace{\beta\gamma\alpha\ldots\beta\gamma\alpha}_{k})\beta\gamma(\alpha\beta\gamma)^{k-1}\\ &= \alpha(\beta\gamma\alpha)^k\beta\gamma(\alpha\beta\gamma)^{k-1}\\ &= \alpha(\alpha\beta\gamma)^k\beta\gamma(\alpha\beta\gamma)^{k-1} = 0\end{aligned}$$

by using the relation $\gamma\beta = 0$.

- Also, for every $i = 1, 2, \ldots, s$,

$$(\eta^i)^{2^{s+1}} = \eta^{2s}\eta^{i2^{s+1}-2s} = (\gamma\alpha\beta)^{2k}\eta^{i2^{s+1}-2s}$$

$$= \gamma(\alpha\beta\gamma)^k(\gamma\alpha\beta)^{k-1}\eta^{i2^{s+1}-2s}$$

which also equals to 0 by the same relation.

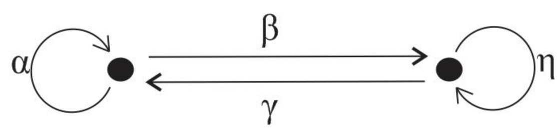
- These show that only e_1 and e_2 that are in the image of the 2^n-power map μ_2^n for every $n \in \mathbb{N}$.

- These elements correspond to $\{(\alpha\beta\gamma)^k, \eta^s\} \subseteq \mathcal{Z}$. Therefore, $Z_{2'}A_c^{k,s} = span\{(\alpha\beta\gamma)^k, \eta^s\}$.

- An algebra A (over an algebraically closed field) is said to be *of semidihedral type* if A is symmetric and indecomposable, the Cartan matrix of A is non-singular, and the stable Auslander-Reiten quiver of A has tube of rank at most 3, at most one 3-tube, and non-periodic components isomorphic to $\mathbb{Z}\mathbb{A}_{\infty}^{\infty}$ and $\mathbb{Z}\mathbb{D}_{\infty}$. [Erdman]

- Any algebra of semidihedral type with two simple modules is derived equivalent to one of the two following classes.

- The first class is algebra $A_c^{k,t} := SD(2\mathfrak{B})_1^{k,t}(c)$, where integers $k \geq 1, t \geq 2$ and $c \in \{0,1\}$.

- These algebras are defined by path algebra of quiver 

  subject to relations

$$\beta\eta = 0, \eta\gamma = 0, \gamma\beta = 0, \alpha^2 = (\beta\gamma\alpha)^{k-1}\beta\gamma + c(\alpha\beta\gamma)^k, (\alpha\beta\gamma)^k = (\beta\gamma\alpha)^k, \eta^t = (\gamma\alpha\beta)^k.$$

- The only difference between $A_c^{k,t}$ and $A_c^{k,s}$ in algebra of dihedral type is the relation $\alpha^2 = (\beta\gamma\alpha)^{k-1}\beta\gamma + c(\alpha\beta\gamma)^k$.

- Because our main concern is in $A_c^{k,t}/[A_c^{k,t}, A_c^{k,t}]$ and

$$(\beta\gamma\alpha)^{k-1}\beta\gamma = [(\beta\gamma\alpha)^{k-1}\beta, \gamma] \text{ (from the relation } \gamma\beta = 0)$$

then we can apply our result in dihedral case to $A_c^{k,t}$.

- The 2-regular subspace of $A_c^{k,t}$ is

$$Z_{2'}A_c^{k,t} = span\{(\alpha\beta\gamma)^k, \eta^t\}.$$

- The second class is algebra $B_c^{k,t} := SD(2\mathfrak{B})_2^{k,t}(c)$, with $k \geq 1, t \geq 2$ such that $k+t \geq 4$ and $c \in \{0,1\}$.

- This algebra defined by the same quiver as above, but subject to the relations

$$\beta\eta = (\alpha\beta\gamma)^{k-1}\alpha\beta, \gamma\beta = \eta^{t-1}, \eta\gamma = (\gamma\alpha\beta)^{k-1}\gamma\alpha, \beta\eta^2 = 0, \eta^2\gamma = 0, \alpha^2 = c(\alpha\beta\gamma)^k.$$

- The basis for $B_c^{k,t}/[B_c^{k,t}, B_c^{k,t}]$ (as shown in [HolmZimmermann]) is

$$\mathbb{B} = \{e_1, e_2, \alpha, \alpha\beta\gamma, \ldots, (\alpha\beta\gamma)^{k-1}, \eta, \ldots, \eta^t\}.$$

- Idempotents e_1 and e_2 again are in the image of μ_2^n for every natural number n.

- For other basis elements, we get

$$(\eta^i)^{2^t} = \eta^{t-1}\eta^{i2^t-(t-1)} = \gamma\beta\eta^{i2^t-(t-1)} = 0$$

for every $i = 1, \ldots, t$.

$$((\alpha\beta\gamma)^i)^{2^{k+1}} = (\alpha\beta\gamma)^k(\alpha\beta\gamma)^k(\alpha\beta\gamma)^{i2^{k+1}-2k}$$

$$= (\alpha\beta\gamma)^{k-1}\alpha\beta\gamma(\alpha\beta\gamma)^k(\alpha\beta\gamma)^{i2^{k+1}-2k}$$

$$= (\alpha\beta\gamma)^{k-1}\alpha\beta(\gamma(\alpha\beta\gamma)^{k-2}\alpha\beta)\gamma\alpha\beta\gamma(\alpha\beta\gamma)^{i2^{k+1}-2k}$$

$$= ((\alpha\beta\gamma)^{k-1}\alpha\beta)((\gamma\alpha\beta)^{k-1}\gamma\alpha)\beta\gamma(\alpha\beta\gamma)^{i2^{k+1}-2k}$$

$$= (\beta\eta)(\eta\gamma)\beta\gamma(\alpha\beta\gamma)^{i2^{k+1}-2k} = 0$$

for every $i = 1, \ldots, k$.

- By the same method, we see that

$$\alpha^8 = (\alpha\beta\gamma)^{4k} = (\alpha\beta\gamma)^k(\alpha\beta\gamma)^k(\alpha\beta\gamma)^{2k} = 0.$$

- Hence only e_1 and e_2 are in the image of the 2^n-power map μ_2^n for every $n \in \mathbb{N}$.

- These elements correspond to $\{(\alpha\beta\gamma)^k, \eta^t\}$ in the centre of $B_c^{k,t}$. Therefore, $Z_{2'}B_c^{k,t} = span\{(\alpha\beta\gamma)^k, \eta^t\}$.

EnsslenKulshammer S. Ensslen and B. Külshammer, A note on blocks with dihedral defect groups, Arch. Math. (Basel) 91 (2008), 205211.

Erdman Blocks of tame representation type and related algebras, Lecture Notes in Mathematics, 1428, Berlin, New York: Springer-Verlag.

FanKulshammer Y. Fan and B. Külshammer, A note on blocks with abelian defect groups, J. Algebra 317 (2007), 250-259.

Holm T. Holm, Block of Tame Representation Type and Related Algebras: Derived Equivalences and Hochschild Cohomology, Habilitation thesis, Otto-von-Guericke-Universitt Magdeburg, 2001.

HolmZimmermann T. Holm and A. Zimmermann, Generalized Reynolds Ideals and Derived Equivalences for Algebras of Dihedral and Semidihedral Type, Journal of Algebra 320 (2008) 3425-3437.

Kulshammer B. Külshammer, Group-theoretical descriptions of ring theoretical invariants of group algebras, Progress in Mathematics, 95 (1991), 425–441.

Meyer H. Meyer, On a subalgebra of the center of a group ring, J.Algebra 295 (2006), 293-302.

Meyer2 H. Meyer, On a subalgebra of the center of a group ring II, Archiv der Mathematik Vol 90 Number 2 (2008), 112-122.

SantikaZimmermann A.P.Santika and A.Zimmermann, Derived equivalences and substructures of the centre related to p-power maps, manuscript.