The moduli of absoltely thick representations

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§0. Motivation

(1) To study irreducible representations of discrete subgroups : $\mathrm{SL}(2,\mathbb{Z})$, $\mathrm{PSL}(2,\mathbb{Z})$, and so on.

(2) To classify several types of irreducible representations of several groups.

(3) Thickness and denseness are basic and essential.

§1. Preliminaries

G : a group
k : a field
V : a vector space over k with $\dim_k V = n$
$\rho : G \to \mathrm{GL}(V)$: a representation of G

Definition 1 (thick). $\rho : G \to \mathrm{GL}(V)$ is *m-thick* if for any subspaces V_1 and V_2 of V with $\dim V_1 = m$ and $\dim V_2 = n - m$, there exists $g \in G$ such that

$$(\rho(g)V_1) \oplus V_2 = V.$$

$\rho : G \to \mathrm{GL}(V)$ is *thick* if ρ is m-thick for each $0 < m < n$.

G : a group
k : a field
V : a vector space over k with $\dim_k V = n$
$\rho : G \to \mathrm{GL}(V)$: a representation of G

Definition 2 (dense)**.** $\rho : G \to \mathrm{GL}(V)$ is *m-dense* if the induced representation $(\wedge^m \rho) : G \to \mathrm{GL}(\wedge^m V)$ is irreducible.

$\rho : G \to \mathrm{GL}(V)$ is *dense* if ρ is m-dense for each $0 < m < n$.

Proposition 3. *For* $0 < m < n$*,*

$$m\text{-}dense \Longrightarrow m\text{-}thick \Longrightarrow irreducible$$

and

$$1\text{-}dense \Longleftrightarrow 1\text{-}thick \Longleftrightarrow irreducible.$$

In particular,

$$dense \Rightarrow thick \Rightarrow irreducible.$$

G : a group
k : a field
V : a vector space over k with $\dim_k V = n$
$\rho : G \to \mathrm{GL}(V)$: a representation of G

Definition 4 (absolutely thick and dense)**.** $\rho : G \to \mathrm{GL}(V)$ is *absolutely thick* if $\rho \otimes_k \overline{k} : G \to \mathrm{GL}(V \otimes_k \overline{k})$ is thick for an algebraic closure $\overline{k}$ of k.

$\rho : G \to \mathrm{GL}(V)$ is *absolutely dense* if $\rho \otimes_k \overline{k} : G \to \mathrm{GL}(V \otimes_k \overline{k})$ is dense.

Remark 5.

ρ : absolutely thick

$\iff \rho \otimes_k \Omega : G \to \mathrm{GL}(V \otimes_k \Omega)$ is thick
for any algebraically closed field Ω over k

$$\begin{array}{ccccc}
\text{abs. dense} & \Rightarrow & \text{abs. thick} & \Rightarrow & \text{abs. irreducible} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{dense} & \Rightarrow & \text{thick} & \Rightarrow & \text{irreducible}
\end{array}$$

§2. Moduli of absolutely irreducible representations

Definition 6.
(**Sch**) : category of schemes
(**Sets**) : category of sets

$\Gamma(X, \mathcal{O}_X)$: the ring of global functions on a scheme X.

Let us consider the contravariant functor:

$$\begin{array}{rccl} \mathrm{Rep}_n(G): & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & X & \mapsto & \{\rho : G \to \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X)) \text{ a representation }\}, \end{array}$$

$\rho : G \to \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X))$ is called an n-dimensional representation of G on X.

$$\begin{array}{rcl} \mathrm{Rep}_n(G) : \ (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ X & \mapsto & \{\rho : G \to \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X)) \text{ a representation } \}, \end{array}$$

Proposition 7. *The functor* $\mathrm{Rep}_n(G)$ *is representable by an affine scheme.*

Definition 8. The group scheme PGL_n over $\mathbb{Z}$ acts on $\mathrm{Rep}_n(G)$ by $\rho \mapsto P^{-1}\rho P$ for $\rho \in \mathrm{Rep}_n(G)$ and $P \in \mathrm{PGL}_n$.

Definition 9. For an n-dimensional representation $\rho : G \to \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X))$ on a scheme X, ρ is *absolutely irreducible* if

for each $x \in X$ the induced rerepsentation $\rho \otimes k(x) : G \to \mathrm{GL}_n(k(x))$ is absolutely irreducible, where $k(x) := \mathcal{O}_{X,x}/m_{X,x}$ is the residue field of x.

Remark 10. The scheme

$$\mathrm{Rep}_n(G)_{air} := \{\rho : \text{abs. irreducible}\}$$

is an open subscheme of $\mathrm{Rep}_n(G)$ which is invariant under PGL_n-action.

Theorem 11. *There exists a universal geometric quotient of* $\mathrm{Rep}_n(G)_{air}$ *by* PGL_n.

Definition 12. We call

$$\mathrm{Ch}_n(G)_{air} := \mathrm{Rep}_n(G)_{air}/\mathrm{PGL}_n$$

the moduli of n-dimensional absolutely irreducible representations of G.

$$F_m := \langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle : \text{ free group of rank } m$$

Proposition 13. *For $m > 1$, the moduli $\mathrm{Ch}_n(F_m)_{air}$ is a smooth irreducible scheme over $\mathbb{Z}$ of relative dimension $(m-1)n^2+1$.*

§3. Moduli of absolutely thick representations

Definition 14. For an n-dimensional representation $\rho : G \to \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X))$ on a scheme X, ρ is *absolutely thick* if

for each $x \in X$ the induced rerepsentation $\rho \otimes k(x) : G \to \mathrm{GL}_n(k(x))$ is absolutely thick, where $k(x) := \mathcal{O}_{X,x}/m_{X,x}$ is the residue field of x.

Definition 15. For an n-dimensional representation $\rho : G \to \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X))$ on a scheme X, ρ is *absolutely dense* if

for each $x \in X$ the induced rerepsentation $\rho \otimes k(x) : G \to \mathrm{GL}_n(k(x))$ is absolutely dense, where $k(x) := \mathcal{O}_{X,x}/m_{X,x}$ is the residue field of x.

Theorem 16.

$\mathrm{Rep}_n(G)_{thick} := \{\rho : \textit{abs. thick rep.}\}$
is an open subscheme of $\mathrm{Rep}_n(G)_{air}$ *which is invariant under* PGL_n*-action.*

Moreover, the universal geometric quotient

$\mathrm{Ch}_n(G)_{thick} := \mathrm{Rep}_n(G)_{thick}/\mathrm{PGL}_n.$
exists.

We call $\mathrm{Ch}_n(G)_{thick}$ the moduli of n-dimensional absolutely thick representations of G.

Key Point. "Thickness is open".

To show openness of absolutely thickness, we prove that

$$\mathrm{Rep}_n(G)_{air} \setminus \mathrm{Rep}_n(G)_{thick}$$

is closed.

Lemma 17. *For an n-dimensional representation $\rho : G \to \mathrm{GL}(V)$ of G over a field k, ρ is <u>not m-thick</u> if and only if*

there exist G-invariant "realizable" subspaces $W_1 \subseteq \wedge^m V$ and $W_2 \subseteq \wedge^{n-m} V$ such that $W_1 \wedge W_2 = 0$.

Here we say $W \subseteq \wedge^m V$ is realizable if there exist $v_1, \ldots, v_m \in V$ such that $0 \neq v_1 \wedge \cdots \wedge v_m \in W$.

Claim 1.

$$\begin{aligned}&\mathrm{Rep}_n(G)\times \mathrm{Gr}(d,\wedge^m\mathbb{A}^n)\times \mathrm{Gr}(d',\wedge^{n-m}\mathbb{A}^n)\\ &\supseteq Y(d,\wedge^m(n),\wedge^{n-m}(n))^{\perp}_{real}\\ &\quad := \left\{ (\rho,W_1,W_2) \,\middle|\, \begin{array}{l} W_1\subseteq \mathcal{O}^m:\ \text{rank } d \text{ realizable } G\text{-inv}\\ W_2\subseteq \mathcal{O}^m:\ \text{rank } d' \text{ realizable } G\text{-inv}\\ W_1\wedge W_2=0\end{array}\right\}\end{aligned}$$

is closed. Here $d' := \binom{n}{m} - d$.

Claim 2. Since

$$\begin{aligned} q_1 : \mathrm{Rep}_n(G)\times \mathrm{Gr}(d,\wedge^m\mathbb{A}^n)\times \mathrm{Gr}(d',\wedge^{n-m}\mathbb{A}^n)\\ \longrightarrow \mathrm{Rep}_n(G)\end{aligned}$$

is proper, $q_1(Y(d,\wedge^m(n),\wedge^{n-m}(n))^{\perp}_{real})$ is closed.

Remark 18. Similary, we can also construct the moduli of absolutely dense representations.

$$\begin{array}{ccccc}
\mathrm{Rep}_n(G)_{air} & \supseteq & \mathrm{Rep}_n(G)_{thick} & \supseteq & \mathrm{Rep}_n(G)_{dense} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ch}_n(G)_{air} & \supseteq & \mathrm{Ch}_n(G)_{thick} & \supseteq & \mathrm{Ch}_n(G)_{dense}
\end{array}$$

§4. Free groups case

$F_m := \langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle$: free group of rank m

Lemma 19. *The symmetric group* S_n *is generated by* (12) *and* $(123\cdots n)$.

Lemma 20. S_n *has a dense representation of dimension* $n-1$ *over* $\mathbb{C}$.

Theorem 21. $\mathrm{Ch}_n(F_2)_{dense} \neq \emptyset$. *Moreover,*

$$\mathrm{Ch}_n(F_m)_{dense} \neq \emptyset$$

and

$$\mathrm{Ch}_n(F_m)_{thick} \neq \emptyset$$

for $m \geq 2$.

Theorem 22. *The moduli schemes* $\mathrm{Ch}_n(F_m)_{dense}$ *and* $\mathrm{Ch}_n(F_m)_{thick}$ *are irreducible smooth scheme over* $\mathbb{Z}$ *of relative dimension* $(m-1)n^2+1$ *for* $m \geq 2$.

Theorem 23. *Let G be an arbitrary group. For $n \geq 4$ and $1 < m < n-1$, we have a morphism*

$$\begin{array}{rccc} \phi : & \mathrm{Ch}_n(G)_{dense} & \rightarrow & \mathrm{Ch}_{\binom{n}{m}}(G)_{air} \\ & [\rho] & \mapsto & [\wedge^m(\rho)]. \end{array}$$

Then $\mathrm{Im}\phi \subseteq \mathrm{Ch}_n(G)_{non-thick}$.

"Exterior produces No thick representations!"