Construction of a (pre-)abelian category from a pair of torsion pairs on a triangulated category

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Aim

Give a construction

triangulated category → abelian category

$\mathscr{C}$ ⇝ $\mathscr{A}$

$+\ {}^{\exists}$structure

(co-)torsion pair

Definition ((co-)torsion pair)

$\mathcal{U}, \mathcal{V} \subseteq \mathscr{C}$ full additive subcategories, closed under $\oplus, \cong$, direct summands

$(\mathcal{U}, \mathcal{V})$: *cotorsion pair*

$\overset{\text{def}}{\Longleftrightarrow}$

- $\mathrm{Ext}^1(\mathcal{U}, \mathcal{V}) = 0.$ i.e. $\mathscr{C}(U, V[1]) = 0 \quad ({}^\forall U \in \mathcal{U}, V \in \mathcal{V}).$
- $\mathscr{C} = \mathcal{U} * \mathcal{V}[1].$ i.e. ${}^\forall C \in \mathscr{C}$, ${}^\exists$distinguished $\triangle$

$$\begin{array}{ccccccc} {}^\exists V & \to & {}^\exists U & \to & C & \to & V[1] \\ \Cap & & \Cap & & & & \\ \mathcal{V} & & \mathcal{U} & & & & \end{array}$$

Remark

$(\mathcal{U}, \mathcal{V})$: cotorsion pair $\Longleftrightarrow$ $(\mathcal{U}, \mathcal{V}[1])$: torsion pair

Example

$(\mathcal{U}, \mathcal{V})$: cotorsion pair on $\mathscr{C}$.

1. *t-structure*

$$\boxed{\mathcal{U}[1] \subseteq \mathcal{U}} \iff (t^{\leq 0}, t^{\geq 0}) := (\mathcal{U}[-1], \mathcal{V}[1])$$

is a t-structure

2. *cluster tilting subcategory*

$$\boxed{\mathcal{U} = \mathcal{V}} \iff \mathcal{T} := \mathcal{U} (= \mathcal{V}) \subseteq \mathscr{C}$$

is a cluster tilting subcategory

3. *co-t-structure*

$$\cdots \quad \boxed{\mathcal{U}[-1] \subseteq \mathcal{U}} \quad (\iff \mathcal{V}[1] \subseteq \mathcal{V})$$

Example

1. *t-structure*

$$\boxed{\mathcal{U}[1] \subseteq \mathcal{U}}$$

$$(t^{\leq 0}, t^{\geq 0}) := (\mathcal{U}[-1], \mathcal{V}[1])$$

is a t-structure

heart

$$t^{\leq 0} \cap t^{\geq 0} \ : \ \text{abel}$$

([Beĭlinson-Bernstein-Deligne])

2. *cluster tilting subcategory*

$$\boxed{\mathcal{U} = \mathcal{V}}$$

$$\mathcal{T} := \mathcal{U}(= \mathcal{V}) \ \subseteq \mathscr{C}$$

is a cluster tilting subcategory

$$\mathscr{C}/\mathcal{T} \ : \ \text{abel}$$

([Keller-Reiten], [Koenig-Zhu])

Example

1. *t-structure*

heart ↓

⇒ $t^{\leq 0} \cap t^{\geq 0}$: abel

([Beĭlinson-Bernstein-Deligne])

2. *cluster tilting subcategory*

⇒ $\mathscr{C}/\mathcal{T}$: abel

([Keller-Reiten], [Koenig-Zhu])

Previous result

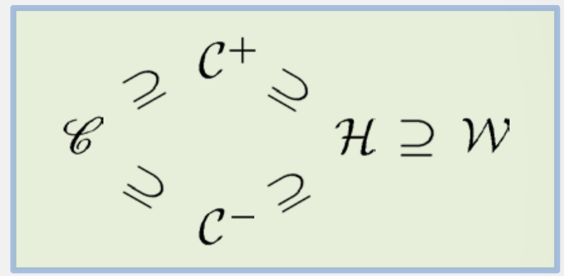
Generalizes these two constructions

Previous result

Theorem [N-]

$(\mathcal{U}, \mathcal{V})$: arbitrary cotorsion pair

$$\leadsto \begin{cases} \mathcal{W} := \mathcal{U} \cap \mathcal{V} \\ \mathcal{C}^+ := \mathcal{W} * \mathcal{V}[1] \\ \mathcal{C}^- := \mathcal{U}[-1] * \mathcal{W} \\ \mathcal{H} := \mathcal{C}^+ \cap \mathcal{C}^- \end{cases}$$



$\Longrightarrow$ the subfactor category

$\mathcal{H}/\mathcal{W}$ becomes abel.

$(\mathcal{U},\mathcal{V})$: arbitrary cotorsion pair

$\mathcal{W} := \mathcal{U}\cap\mathcal{V}$
$\mathcal{C}^+ := \mathcal{W} * \mathcal{V}[1]$
$\mathcal{C}^- := \mathcal{U}[-1] * \mathcal{W}$
$\mathcal{H} := \mathcal{C}^+ \cap \mathcal{C}^-$

$$\begin{array}{ccccc} & & \mathcal{C}^+ & & \\ & \supseteq & & \supseteq & \\ \mathscr{C} & & & & \mathcal{H} \supseteq \mathcal{W} \\ & \supseteq & & \supseteq & \\ & & \mathcal{C}^- & & \end{array}$$

$\Longrightarrow$ the subfactor category $\mathcal{H}/\mathcal{W}$ becomes abel.

In the above three cases $\cdots$

1. *t-structure*

$$\begin{array}{ccccc} & & \mathcal{C}^+ & & \\ & \supseteq & & \supseteq & \\ \mathscr{C} & & & & \mathcal{H} \supseteq \mathcal{W} \\ & \supseteq & & \supseteq & \\ & & \mathcal{C}^- & & \end{array} \quad = \quad \begin{array}{ccccc} & & t^{\geq 0} & & \\ & \supseteq & & \supseteq & \\ \mathscr{C} & & & & t^{\geq 0}\cap t^{\leq 0} \supseteq 0 \\ & \supseteq & & \supseteq & \\ & & t^{\leq 0} & & \end{array}$$

$$\mathcal{H}/\mathcal{W} = t^{\geq 0}\cap t^{\leq 0}$$ *heart*

2. *cluster tilting subcategory*

$$\begin{array}{ccccc} & & \mathcal{C}^+ & & \\ & \supseteq & & \supseteq & \\ \mathscr{C} & & & & \mathcal{H} \supseteq \mathcal{W} \\ & \supseteq & & \supseteq & \\ & & \mathcal{C}^- & & \end{array} \quad = \quad \begin{array}{ccccc} & & \mathscr{C} & & \\ & = & & = & \\ \mathscr{C} & & & & \mathscr{C} \supseteq \mathcal{T} \\ & = & & = & \\ & & \mathscr{C} & & \end{array}$$

$$\mathcal{H}/\mathcal{W} = \mathscr{C}/\mathcal{T}$$

3. *co-t-structure*

$$(\mathcal{U},\mathcal{V}) : \text{co-}t\text{-structure} \iff \mathcal{H}/\mathcal{W} = 0$$

Result by Buan & Marsh

$\cdots$ generalizes the cluster tilting case

$\mathscr{C}$: Hom-finite Krull-Schmidt, with a Serre functor

$T \in \mathscr{C}$: rigid object

$\mathcal{X}_T := (\mathrm{add}T)^{\perp} \quad (:= \mathrm{Ker}(\mathrm{add}T, -))$

$\Longrightarrow$ [Buan-Marsh] $\quad \mathscr{C}/\mathcal{X}_T$: integral preabelian category

$\Downarrow$ localization by regular (= mono & epi) morphisms [Rump]

$(\mathscr{C}/\mathcal{X}_T)_{\mathcal{R}}$: abelian category

Integral category

Definition

$\mathscr{A}$: additive category

(1) $\mathscr{A}$ is *preabelian*

$\overset{\text{def}}{\iff}$ $^\forall$kernel and $^\forall$cokernel exist

(2) $\mathscr{A}$ is *semi-abelian*

$\overset{\text{def}}{\iff}$ $\mathscr{A}$: preabelian

& pull-back of a cokernel morphism is epimorphic

& pushout of a kernel morphism is monomorphic

(3) $\mathscr{A}$ is *integral*

$\overset{\text{def}}{\iff}$ $\mathscr{A}$: preabelian

& epimorphisms are stable under pull-backs

& monomorphisms are stable under pushouts

Rump

abel $\Longrightarrow$ integral $\Longrightarrow$ semiabel $\Longrightarrow$ preabel cotorsion pair

$(\mathcal{U}, \mathcal{V})$

⇒ $\mathcal{H}/\mathcal{W}$: abel

Buan-Marsh's construction

$T \in \mathscr{C}$

⇒ $\mathscr{C}/\mathcal{X}_T$: integral

(⇒ $(\mathscr{C}/\mathcal{X}_T)_{\mathcal{R}}$: abel)

Question

Can we modify this construction
so that we can deal with this simultaneously?

Remark [Buan-Marsh]

- $(\mathrm{add}T[1], \mathcal{X}_T)$, $(\mathcal{X}_T, \mathcal{X}_T^{\perp}[-1])$ are cotorsion pairs

Setting

★ $(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})$: a pair of cotorsion pairs

satisfying $\mathrm{Ext}^1(\mathcal{S},\mathcal{V})=0$

cotorsion pair $(\mathcal{U},\mathcal{V})$

A pair ★

$(\mathcal{U},\mathcal{V}),(\mathcal{U},\mathcal{V})$

Buan-Marsh's triplet

$(\mathrm{add}T,\mathcal{X}_T,\mathcal{X}_T^{\perp})$

A pair ★

$(\mathrm{add}T[1],\mathcal{X}_T),(\mathcal{X}_T,\mathcal{X}_T^{\perp}[-1])$

Definition

$[\,(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})\,]$ is a ***twin cotorsion pair***

$\overset{\text{def}}{\Longleftrightarrow}$

- $(\mathcal{S},\mathcal{T})$: cotorsion pair
- $(\mathcal{U},\mathcal{V})$: cotorsion pair
- $\mathrm{Ext}^1(\mathcal{S},\mathcal{V})=0$

Example

1. *'single' cotorsion pair*

$$[\,(\mathcal{U},\mathcal{V}),(\mathcal{U},\mathcal{V})\,]$$

2. *Buan & Marsh's triplet*

$$[\,(\mathrm{add}T[1],\mathcal{X}_T),(\mathcal{X}_T,\mathcal{X}_T^{\perp}[-1])\,]$$

Theorem [N-]

1

$[(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})]$: (arbitrary) twin cotorsion pair

$$\begin{cases} \mathcal{W} := \mathcal{T} \cap \mathcal{U} \\ \mathcal{C}^{+} := \mathcal{W} * \mathcal{V}[1] \\ \mathcal{C}^{-} := \mathcal{S}[-1] * \mathcal{W} \\ \mathcal{H} := \mathcal{C}^{+} \cap \mathcal{C}^{-} \end{cases}$$

$$\mathscr{C} \supseteq \mathcal{C}^{+} \supseteq \mathcal{H} \supseteq \mathcal{W}, \qquad \mathscr{C} \supseteq \mathcal{C}^{-} \supseteq \mathcal{H}$$

$\Longrightarrow$ the subfactor category

$\mathcal{H}/\mathcal{W}$ becomes semi-abel.

Theorem [N-]

2

If $[(\mathcal{S},\mathcal{T}),(\mathcal{U},\mathcal{V})]$ satisfies

$$\mathcal{U}\subseteq\mathcal{S}*\mathcal{T} \quad \text{or} \quad \mathcal{T}\subseteq\mathcal{U}*\mathcal{V},$$

$\Longrightarrow$ the subfactor category

$\mathcal{H}/\mathcal{W}$ becomes integral.

Ex

Satisfied when

1. single cotorsion pair
2. the case $\mathcal{U}=\mathcal{T}$ (e.g. Buan-Marsh's triplet)
3. $(\mathcal{S},\mathcal{T})$ or $(\mathcal{U},\mathcal{V})$ is a co-t-structure