Hochschild cohomology of quiver algebras defined by two cycles and a quantum-like relation

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1

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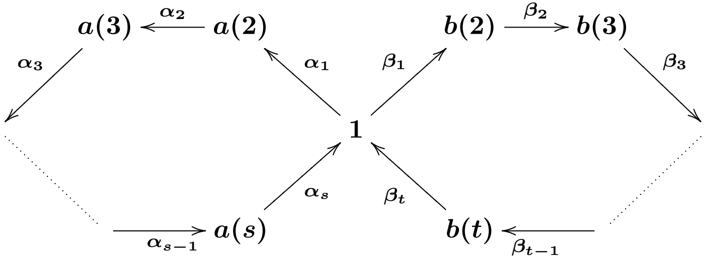
Quiver algebras defined by two cycles and a quantum-like relation

- k : field.

We consider the quiver algebra $A_q = kQ/I_q$.

- Q : quiver with $s+t-1$ vertices and $s+t$ arrows as follows:



  for integers $s, t \geq 1$.

- $I_q = \langle X^{sa}, X^sY^t - qY^tX^s, Y^{tb} \rangle$ for $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$, $Y := \beta_1 + \beta_2 + \cdots + \beta_t$, integers $a, b \geq 2$ and $q (\neq 0) \in k$.

Quantum complete intersection

In the case $s = t = 1$, $A_q = k\langle x, y\rangle / \langle x^a, xy - qyx, y^b\rangle$ is a quantum complete intersection. This algebra is self-injective algebra. And in the case $a = b = 2$, this algebra is Koszul algebra.

- In the case $a = b = 2$, $\mathrm{HH}^*(A_q)$ was determined by [BGMS(2005)] for any element q in k.
- In the case $a, b \geq 2$ and q is not a root of unity, $\mathrm{HH}^*(A_q)$ was determined by [BE(2008)].
- In the case $a, b \geq 2$, Bergh and Oppermann showed that A_q holds the finiteness conditions if and only if q is a root of unity in [BO(2008)].

Finiteness conditions [EHSST(2004)]

(Fg1) There is a graded subalgebra H of $\mathrm{HH}^*(A)$ such that H is a commutative Noetherian ring and $H^0 = \mathrm{HH}^0(A)$.

(Fg2) $\mathrm{Ext}^*_A(A/\mathrm{rad}\,A, A/\mathrm{rad}\,A)$ is a finitely generated H-module.

Hochschild cohomology ring of A_q

	$q = 1$	$q = -1$	q: an r-th root of unity ($r \geq 3$)	q: not a root of unity
$s = t = 1$, $a = b = 2$	[BGMS(2005)]	[BGMS(2005)]	[BGMS(2005)]	[BGMS(2005)]
$s = t = 1$, $a, b \geq 2$	[BO(2008)]			[BE(2008)]
$s \geq 2$ or $t \geq 2$, $a, b \geq 2$	[D.Obara(2012)]	[D.Obara(preprint)]		

Hochschild cohomology ring of A_q

Step 1: We determine the minimal projective bimodule resolution $\mathbb{P}$ of A_q using the same method as [BE(2008)].

Step 2: Using the resolution $\mathbb{P}$, we determine a basis of $\mathrm{HH}^n(A_q)$ for $n \geq 0$.

Step 3: We give liftings of basis elements of $\mathrm{HH}^n(A_q)$ for $n \geq 0$.

Step 4: As a main result, we determine the ring structure of $\mathrm{HH}^*(A_q)$ by means of generators and Yoneda product in all characteristics.

Main result

Let $\mathcal{N}$ be the ideal of $\mathrm{HH}^*(A_q)$ generated by all homogeneous nilpotent elements.

Theorem 1

In the case where q is a root of unity, $\mathrm{HH}^*(A_q)/\mathcal{N}$ is isomorphic to the polynomial ring of two variables.

Theorem 2

In the case where q is not a root of unity, $\mathrm{HH}^*(A_q)/\mathcal{N} \cong k$.

Theorem 3

A_q satisfies the finiteness conditions if and only if q is a root of unity.

2
Support variety

The support variety of M

Definition 1 [[SnSo(2004)], Defininsion 3.3]

The support variety of A-module M is given by

$$V(M) = \{m \in \operatorname{MaxSpec} \operatorname{HH}^*(A)/\mathcal{N} | \operatorname{AnnExt}^*_A(M, M) \subseteq m'\}$$

where $\operatorname{AnnExt}^*_A(M, M)$ is the annihilator of $\operatorname{Ext}^*_A(M, M)$ and m' is the preimage in $\operatorname{HH}^*(A)$ of the ideal m in $\operatorname{HH}^*(A)/\mathcal{N}$.

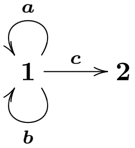
Question[Sn(2008)]

Whether we can give necessary and sufficient conditions on a finite dimensional algebra for the Hochschild cohomology ring modulo nilpotence to be finitely generated as an algebra?

With respect to sufficient conditions, it is shown that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as an algebra for various classes of algebras by many authors as follows:

- **Any block of a group ring of a finite group (See [Ev(1961)], [V(1959)])**
- **Any block of a finite dimensional cocommutative Hopf algebra (See [FSu(1997)])**
- **Finite dimensional algebras of finite global dimension (See [Ha(1989)])**
- **Finite dimensional self-injective algebras of finite representation type over an algebraically closed field (See [GSnSo(2003)])**
- **Finite dimensional monomial algebras (See [GSnSo(2006)])**
- **A class of special biserial algebras (See [SnT(2010)])**
- **A Hecke algebra (See [ScSn(2011)])**

- **Koenig and Nagase produce many examples of finite dimensional algebras A with a stratifying ideal for which $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as an algebra. (See [KN(2009)])**
- **There is a finite dimensional algebra A for which $\mathrm{HH}^*(A)/\mathcal{N}$ is not a finitely generated algebra.(See [Sn(2009)], [Xu(2008)]) Let $A = kQ/I$ where Q is the quiver**



and $I = \langle a^2, b^2, ab - ba, ac \rangle$. Snashall showed the following Theorem.

[Sn(2009), Theorem 4.5]

1. $\mathrm{HH}^*(A)/\mathcal{N} \cong \begin{cases} k \oplus k[a,b]b & \text{if } \mathrm{char}\, k = 2, \\ k \oplus k[a^2, b^2]b^2 & \text{if } \mathrm{char}\, k \neq 2. \end{cases}$
2. $\mathrm{HH}^*(A)/\mathcal{N}$ **is not finitely generated as an algebra.**

Main result

Theorem 1

In the case where q is a root of unity, $\mathrm{HH}^*(A_q)/\mathcal{N}$ is isomorphic to the polynomial ring of two variables.

Theorem 2

In the case where q is not a root of unity, $\mathrm{HH}^*(A_q)/\mathcal{N} \cong k$.

Theorem 3

A_q satisfies the finiteness conditions if and only if q is a root of unity.

By Theorem 1 and 2, $\mathrm{HH}^*(A_q)/\mathcal{N}$ is finitely generated as an algebra. This algebras A_q is new example of a class of algebra for which the Hochschild cohomology ring modulo nilpotence is finitely generated as an algebra.

3

Example

Example

In the case $s = 2$, $t = 1$ and $a = b = 2$, Q is the quiver:

$$a(2) \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} 1 \circlearrowright \beta_1$$

and I_q is the ideal of kQ generated by

$$X^4, X^2Y - qYX^2, Y^2$$

for $X = \alpha_1 + \alpha_2$, $Y = \beta_1$. Then indecomposable projective and injective modules are given by the following.

Example

$$P(1) = A_q e_1 = K^4 \underset{\begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&1&0&0 \end{pmatrix}}{\overset{1}{\leftrightarrows}} K^4 \circlearrowright \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&q^{-1}&0&0 \end{pmatrix},$$

$$P(2) = A_q e_{a(2)} = K^4 \underset{\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&q&0 \end{pmatrix}}{\overset{\begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}}{\leftrightarrows}} K^4 \circlearrowright \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix},$$

$$I(1) = D(e_1 A_q) = K^4 \underset{1}{\overset{\begin{pmatrix} 0&1&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix}}{\leftrightarrows}} K^4 \circlearrowright \begin{pmatrix} 0&0&1&0\\ 0&0&0&q\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix},$$

Then $I(1) \ncong P(i)$ for $i = 1, 2$. Therefore A_q is not self-injective.

Example

Stratifying ideal

Let e be an idempotent of A. If the two sided ideal AeA satisfies the following conditions, this ideal is called a stratifying ideal.

- **The multiplication map $Ae \otimes_{eAe} eA \to AeA$ is an isomorphism.**
- **For all $n \geq 1$, $\mathrm{Tor}_n^{eAe}(Ae, eA) = 0$.**

In the case $s = 2$, $t = 1$ and $a = b = 2$, we consider the multiiplicasion maps

$$\mathcal{M}_1 : A_q e_1 \otimes_{e_1 A e_1} e_1 A_q \to A_q e_1 A_q,$$
$$\mathcal{M}_{a(2)} : A_q e_{a(2)} \otimes_{e_{a(2)} A e_{a(2)}} e_{a(2)} A_q \to A_q e_{a(2)} A_q.$$

Then, we have $\mathcal{M}_1(\alpha_2\alpha_1\alpha_2 \otimes \alpha_1) = 0$, $\mathcal{M}_{a(2)}(\alpha_1\alpha_2\alpha_1 \otimes \alpha_2) = 0$. So multiplication maps $\mathcal{M}_1$ and $\mathcal{M}_{a(2)}$ are not isomorphisms and A_q have no stratifying ideal.

Example

In the case $s = 2$, $t = 1$ and $a = b = 2$, the following sequence $\mathbb{P}$ is a minimal projective resolution of the A_q-module $A_q/\operatorname{rad} A_q$:

$$\mathbb{P} : \cdots \to P_n \xrightarrow{d_n} \cdots \to P_1 \xrightarrow{d_1} A_q \xrightarrow{\pi} A_q/\operatorname{rad} A_q \to 0.$$

where $P_n = e_1 A_q \oplus e_1 A_q \oplus e_{a(2)} A_q$, π: $A_q \to A_q/\operatorname{rad} A_q$ is the natural epimorphism, and we define right A_q-homomorphisms d_n by

$$d_1 : \begin{cases} e_{1,1} \mapsto \alpha_2, \\ e_{1,2} \mapsto \beta_1, \\ e_{a(2)} \mapsto \alpha_1, \end{cases} \qquad d_{2m+2} : \begin{cases} e_{1,1} \mapsto e_{a(2)}\alpha_2\alpha_1\alpha_2, \\ e_{1,2} \mapsto e_{1,2}\beta_1, \\ e_{a(2)} \mapsto e_{1,1}\alpha_1\alpha_2\alpha_1, \end{cases}$$

$$d_{2m+3} : \begin{cases} e_{1,1} \mapsto e_{a(2)}\alpha_2, \\ e_{1,2} \mapsto e_{1,2}\beta_1, \\ e_{a(2)} \mapsto e_{1,1}\alpha_1, \end{cases}$$

for $m \geq 0$. By d_{2m+2}, P_n is not generated in degree n for $n \geq 2$. So A_q is not Koszul algebra.

Let A be a graded k-algebra.

Graded module is generated in degree i

A graded A-module M is generated in degree i if $M_j = (0)$ for $j < i$ and $M_{i+j} = A_j M_i$ for all $j \geq 0$.
f: $M \to N$ is an A-homomorphism of degree i if $f(M_j) = M_{i+j}$ for all j.

Koszul algebra

A is a Koszul algebra if the minimal projective resolution of A-module $A/\operatorname{rad} A$

$$\cdots \to P_n \xrightarrow{d_n} \cdots \to P_1 \xrightarrow{d_1} P_0 \to A/\operatorname{rad} A \to 0$$

is linear, that is, P_i is generated in degree i and d_i is A-homomorphism of degree 0 for $i \geq 0$.

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