On the classification of irreducible representations of special class

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This talk is based on joint work with Kazunori Nakamoto (University of Yamanashi).

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- Definition of representations of special class which we call thick representations
- Properties and some characterization of thick representations
- Application: the classification of thick representations of simple Lie groups

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for any subspaces V_1, V_2 of V with $\dim V_1 = 1$ and $\dim V_2 = \dim V - 1$ there exists $g \in G$ such that $\rho(g)V_1 \oplus V_2 = V$.

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Example1

Suppose $G = \mathrm{SL}_2(\mathbb{C})$ and $V = V_d$ is the vector space of binary forms of degree d on which G acts naturally. Then this representation is thick.

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Definition

For a subspace $W \subseteq \Lambda^d V$, we say that W is *realizable* if W contains a non-zero vector $\Lambda^d V'$ obtained by a d-dimensional subspace V' of V.

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Proposition

V is not m-thick if and only if there exist G-invariant realizable subspaces $W_1 \subseteq \Lambda^m V$ and $W_2 \subseteq \Lambda^{n-m} V$ such that $W_1^{\perp} = W_2$.

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Example1

The vector space of binary forms of degree d is thick but not dense if $d \geq 3$.

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For any finite dimensional representations of a group G, the following implications hold for $0 < m < \dim V$:

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Corollary

For any representation of a group G, the following implications hold:

$$\textit{dense} \Rightarrow \textit{thick} \Rightarrow \textit{irreducible}.$$

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- We can regard $W(V)$ as a poset with respect to the usual root order and call it a weight poset.
  ($\mu > \gamma$ for $\mu, \gamma \in W(V)$ if and only if $\mu - \gamma$ is a nonzero sum of simple roots with nonnegative coefficients.)

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  ($\mu > \gamma$ for $\mu, \gamma \in W(V)$ if and only if $\mu - \gamma$ is a nonzero sum of simple roots with nonnegative coefficients.)
- We say that V is weight multiplicity-free (WMF) if all weight spaces in V are one dimensional.

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R. Howe classified weight multiplicity-free representations of simple Lie groups in "Perspectives on Invariant Theory(1995)".

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The classification of thick representations

Theorem

The list of thick representations of simple Lie groups is following.

$$(A_n, \omega_1), (A_n, \omega_n), (A_1, m\omega_1), (B_n, \omega_1), (C_n, \omega_1), (G_2, \omega_1).$$

In particular the list of representations of simple Lie groups which is thick but not dense is following.

$$(A_1, m\omega_1)(m \geq 3), (C_n, \omega_1), (G_2, \omega_1).$$

Thank you!