The strong no loop conjecture

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joint work with K. Igusa and S. Liu

No loop conjecture

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- [Igusa, Lenzing] This conjecture holds true for finite dimensional elementary k-algebras where k is a field.
- From this, the no loop conjecture holds true for finite dimensional algebras over an algebraically closed field.

Strong no loop conjecture

Conjecture (Strong no loop conjecture (SNLC))

Let Λ be an artin algebra and S be a simple Λ-module with $\mathrm{Ext}^1_\Lambda(S, S) \neq 0$. Then the projective dimension (and injective dimension) of S is infinite.

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  - [Skorodumov] Mild algebras (hence representation finite algebras)
  - Some other cases have also been obtained

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Trace function of Hattori and Stallings

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- This definition is independent of the decomposition of P.

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Proposition (Hattori-Stallings)

Let P, P' be projective modules in $\mathrm{mod}(\Lambda)$.

(1) *If $\varphi, \psi \in \mathrm{End}_\Lambda(P)$, then $\mathrm{tr}(\varphi + \psi) = \mathrm{tr}(\varphi) + \mathrm{tr}(\psi)$.*

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(3) *If $\varphi : \Lambda \to \Lambda$ is the left multiplication by $a \in \Lambda$, then $\mathrm{tr}(\varphi) = a + [\Lambda, \Lambda]$.*

Lenzing's trace function

Let M be a Λ-module of finite projective dimension and $\varphi : M \to M$. Then we have a finite projective resolution:

- $\mathcal{P}_M \qquad \cdots \longrightarrow P_i \xrightarrow{d_i} P_{i-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$

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  commutes.
- We set $\mathrm{tr}(\varphi) = \sum_{i=0}^{\infty} (-1)^i \mathrm{tr}(\varphi_i)$.

The e-trace function

For the rest of the talk, we let e denote an idempotent in Λ and r a positive integer.

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$$H_e : \mathrm{HH}_0(\Lambda) \to \mathrm{HH}_0(\Lambda_e).$$
- If $\varphi : P \to P$ is an endomorphism of a projective Λ-module, then we set
$$\mathrm{tr}_e(\varphi) = H_e(\mathrm{tr}(\varphi)) \in \mathrm{HH}_0(\Lambda_e).$$

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The (e, r)-trace function

Definition

1. Let M be a module and

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a projective resolution of M. We say that $\mathcal{P}_M$ is (e, r)-bounded if $\mathrm{top}(P_r) \cdot e = 0$.

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2. Let $\varphi \in \mathrm{End}_\Lambda(M)$ with a lifting $\{\varphi_i\}_{i\geq 0}$ to $\mathcal{P}_M$. We define

$$\mathrm{tr}_{(e,r)}(\varphi) = \sum_{i=0}^{r-1} (-1)^i \,\mathrm{tr}_e(\varphi_i) \in \mathrm{HH}_0(\Lambda_e).$$

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Lemma

The (e, r)-trace is well defined for endomorphisms of modules having an (e, r)-bounded projective resolution.

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Proposition

Consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{u} & M & \xrightarrow{v} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_L} & & \downarrow{\scriptstyle \varphi_M} & & \downarrow{\scriptstyle \varphi_N} & & \\ 0 & \longrightarrow & L & \xrightarrow{u} & M & \xrightarrow{v} & N & \longrightarrow & 0 \end{array}$$

in $\mathrm{mod}\Lambda$ *with exact rows. If* L, N *have* (e, r)*-bounded projective resolutions, then so does* M *and* $\mathrm{tr}_{(e,r)}(\varphi_M) = \mathrm{tr}_{(e,r)}(\varphi_L) + \mathrm{tr}_{(e,r)}(\varphi_N)$.

Hochschild homology and injective dimensions

Theorem

If $\mathrm{Ext}^r(a^j\Lambda/a^{j+1}\Lambda, S_e) = 0$ *for every* $a \in eJe$ *and* $j \geq 0$, *then* $\mathrm{rad}(\Lambda_e) \subseteq [\Lambda_e, \Lambda_e]$.

PROOF (SKETCH).

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- $\varphi_i : a^i\Lambda \to a^i\Lambda$ induced by $\varphi_0 := a\cdot$

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PROOF (TO BE CONTINUED).

- We have commutative diagrams

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- Hence,
$0 = \mathrm{tr}_{(e,r)}(\varphi_m) = \cdots = \mathrm{tr}_{(e,r)}(\varphi_0) = \bar{a} + [\Lambda_e, \Lambda_e] \in \mathrm{HH}_0(\Lambda_e)$ $\square$

Strong no loop conjecture

Lemma

If e is primitive, then $B(e) := (e + J^2)(\Lambda/J^2)(e + J^2)$ *is a commutative algebra and* $\mathrm{HH}_0(B(e)) = B(e)$.

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- For $i \geq 0$, denote by $\Omega_i(S_e)$ and $\Omega^i(S_e)$ the i-th syzygy and i-th co-syzygy of S_e, respectively.

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Suppose that e is primitive. If $\mathrm{Ext}^1(S_e, S_e) \neq 0$*, then* $\Omega_i(S_e)e \neq 0$ *and* $\Omega^i(S_e)e \neq 0$ *for all* $i \geq 0$*. In particular,* $\mathrm{pd}(S_e) = \mathrm{id}(S_e) = \infty$.

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- For $a \in eJe$ and $j \geq 0$, we have $\mathrm{Hom}(a^j\Lambda/a^{j+1}\Lambda, \Omega^r(S_e)) = 0$. Hence, $\mathrm{Ext}^r(a^j\Lambda/a^{j+1}\Lambda, S_e) = 0$.

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- Therefore, we know that $\mathrm{rad}(\Lambda_e) \subseteq [\Lambda_e, \Lambda_e]$

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- $a + eJ^2e = 0$ $\square$

Strong no loop conjecture

Theorem

Let S be a simple Λ-module with $\mathrm{Ext}^1(S,S) \neq 0$. Then S is a composition factor of all syzygies and co-syzygies of S. In particular, S has infinite projective and injective dimensions.

Some generalizations

- Let Q be a finite quiver and I an admissible ideal in kQ.

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- The idempotent supporting σ is the "smallest" idempotent e such that $e\sigma_i = \sigma_i$ for all i

Some generalizations

- We say that σ is cyclically free in Λ if none of the σ_i with $1 \leq i \leq r$ is a summand of a minimal relation.

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Theorem

Let $\Lambda = kQ/I$ with Q a finite quiver and I an admissible ideal in kQ, and let σ be an oriented cycle in Q with supporting idempotent $e \in \Lambda$. If σ is cyclically free in Λ, then S_e has infinite projective and injective dimensions.

An example

Let $\Lambda = kQ/I$, where Q is the following quiver

$$\begin{tikzcd}
1 \arrow[r, bend left, "\alpha"] \arrow[d, bend right, "\gamma"'] \arrow[dr, bend left, "\mu"] & 2 \arrow[d, bend left, "\beta"] \\
3 \arrow[r, bend right, "\delta"'] & 4 \arrow[ul, bend left, "\nu"]
\end{tikzcd}$$

and I is the ideal in kQ generated by $\alpha\beta - \gamma\delta, \beta\nu, \nu\mu\nu$.

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and I is the ideal in kQ generated by $\alpha\beta - \gamma\delta, \beta\nu, \nu\mu\nu$.

- The oriented cycle $\mu\nu$ is cyclically free in Λ
- By the last theorem, one of the simple modules S_1, S_4 has infinite projective dimension, and one has infinite injective dimension.

Stronger version of the SNLC

Conjecture (Extension conjecture)

Let S be a simple module over an artin algebra. If $\mathrm{Ext}^1(S,S)$ is non-zero, then $\mathrm{Ext}^i(S,S)$ is non-zero for infinitely many integers i.

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- It remains open except for monomial algebras and special biserial algebras.

THANK YOU

Questions ?