

The strong no loop conjecture

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joint work with K. Igusa and S. Liu

No loop conjecture

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Let Λ be an artin algebra and S be a simple Λ -module with $\text{Ext}_{\Lambda}^1(S, S) \neq 0$. Then the global dimension of Λ is infinite.

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- [Igusa, Lenzing] This conjecture holds true for finite dimensional elementary k -algebras where k is a field.
- From this, the no loop conjecture holds true for finite dimensional algebras over an algebraically closed field.

Strong no loop conjecture

Conjecture (Strong no loop conjecture (SNLC))

Let Λ be an artin algebra and S be a simple Λ -module with $\text{Ext}_{\Lambda}^1(S, S) \neq 0$. Then the projective dimension (and injective dimension) of S is infinite.

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 - [Skorodumov] Mild algebras (hence representation finite algebras)
 - Some other cases have also been obtained

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- Definition of the localized trace function

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Trace function of Hattori and Stallings

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- We define $\mathrm{tr}(\varphi) = \sum a_{ii} + [\Lambda, \Lambda] \in \mathrm{HH}_0(\Lambda)$.

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- We define $\mathrm{tr}(\varphi) = \sum a_{ii} + [\Lambda, \Lambda] \in \mathrm{HH}_0(\Lambda)$.
- This definition is independent of the decomposition of P .

Trace function of Hattori and Stallings

Proposition (Hattori-Stallings)

Let P, P' be projective modules in $\text{mod}(\Lambda)$.

(1) If $\varphi, \psi \in \text{End}_\Lambda(P)$, then $\text{tr}(\varphi + \psi) = \text{tr}(\varphi) + \text{tr}(\psi)$.

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- (3) If $\varphi : \Lambda \rightarrow \Lambda$ is the left multiplication by $a \in \Lambda$, then $\text{tr}(\varphi) = a + [\Lambda, \Lambda]$.*

Lenzing's trace function

Let M be a Λ -module of finite projective dimension and $\varphi : M \rightarrow M$.
Then we have a finite projective resolution:

$$\bullet \quad \mathcal{P}_M \quad \cdots \longrightarrow P_i \xrightarrow{d_i} P_{i-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$$

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commutes.

- We set $\text{tr}(\varphi) = \sum_{i=0}^{\infty} (-1)^i \text{tr}(\varphi_i)$.

The e -trace function

For the rest of the talk, we let e denote an idempotent in Λ and r a positive integer.

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- If $\varphi : P \rightarrow P$ is an endomorphism of a projective Λ -module, then we set

$$\mathrm{tr}_e(\varphi) = H_e(\mathrm{tr}(\varphi)) \in \mathrm{HH}_0(\Lambda_e).$$

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Proposition

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The (e, r) -trace function

Definition

- ① Let M be a module and

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a projective resolution of M . We say that $\mathcal{P}_M$ is **(e, r) -bounded** if $\text{top}(P_r) \cdot e = 0$.

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- 2 Let $\varphi \in \text{End}_\Lambda(M)$ with a lifting $\{\varphi_i\}_{i \geq 0}$ to $\mathcal{P}_M$. We define

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Lemma

The (e, r) -trace is well defined for endomorphisms of modules having an (e, r) -bounded projective resolution.

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Proposition

Consider a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L & \xrightarrow{u} & M & \xrightarrow{v} & N \longrightarrow 0 \\
 & & \downarrow \varphi_L & & \downarrow \varphi_M & & \downarrow \varphi_N \\
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 \end{array}$$

in $\text{mod } \Lambda$ with exact rows. If L, N have (e, r) -bounded projective resolutions, then so does M and $\text{tr}_{(e,r)}(\varphi_M) = \text{tr}_{(e,r)}(\varphi_L) + \text{tr}_{(e,r)}(\varphi_N)$.

Hochschild homology and injective dimensions

Theorem

If $\text{Ext}^r(a^j \Lambda / a^{j+1} \Lambda, S_e) = 0$ for every $a \in eJe$ and $j \geq 0$, then $\text{rad}(\Lambda_e) \subseteq [\Lambda_e, \Lambda_e]$.

PROOF (SKETCH).

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- Consider $a^i\Lambda, i = 0, 1, \dots, m$
- $\varphi_i : a^i\Lambda \rightarrow a^i\Lambda$ induced by $\varphi_0 := a \cdot$

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PROOF (TO BE CONTINUED).

- We have commutative diagrams

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- Hence,

$$0 = \mathrm{tr}_{(e,r)}(\varphi_m) = \cdots = \mathrm{tr}_{(e,r)}(\varphi_0) = \bar{a} + [\Lambda_e, \Lambda_e] \in \mathrm{HH}_0(\Lambda_e) \quad \square$$

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Lemma

If e is primitive, then $B(e) := (e + J^2)(\Lambda/J^2)(e + J^2)$ is a commutative algebra and $\mathrm{HH}_0(B(e)) = B(e)$.

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- For $i \geq 0$, denote by $\Omega_i(S_e)$ and $\Omega^i(S_e)$ the i -th syzygy and i -th co-syzygy of S_e , respectively.

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Theorem

Suppose that e is primitive. If $\text{Ext}^1(S_e, S_e) \neq 0$, then $\Omega_i(S_e)e \neq 0$ and $\Omega^i(S_e)e \neq 0$ for all $i \geq 0$. In particular, $\text{pd}(S_e) = \text{id}(S_e) = \infty$.

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- Suffices to show that $eJe/eJ^2e = 0$
- For $a \in eJe$ and $j \geq 0$, we have $\text{Hom}(a^j\Lambda/a^{j+1}\Lambda, \Omega^r(S_e)) = 0$.
Hence, $\text{Ext}^r(a^j\Lambda/a^{j+1}\Lambda, S_e) = 0$.

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Hence, $\text{Ext}^r(a^j\Lambda/a^{j+1}\Lambda, S_e) = 0$.
- Therefore, we know that $\text{rad}(\Lambda_e) \subseteq [\Lambda_e, \Lambda_e]$

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- $a + eJ^2e = f(a + \Lambda(1-e)\Lambda)$ lies in the commutator group of $B(e)$
- $a + eJ^2e = 0$ $\square$

Strong no loop conjecture

Theorem

Let S be a simple Λ -module with $\text{Ext}^1(S, S) \neq 0$. Then S is a composition factor of all syzygies and co-syzygies of S . In particular, S has infinite projective and injective dimensions.

Some generalizations

- Let Q be a finite quiver and I an admissible ideal in kQ .

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- The **idempotent supporting σ** is the “smallest” idempotent e such that $e\sigma_i = \sigma_i$ for all i

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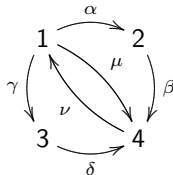
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Theorem

Let $\Lambda = kQ/I$ with Q a finite quiver and I an admissible ideal in kQ , and let σ be an oriented cycle in Q with supporting idempotent $e \in \Lambda$. If σ is cyclically free in Λ , then S_e has infinite projective and injective dimensions.

An example

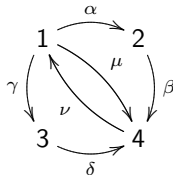
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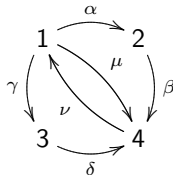


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and I is the ideal in kQ generated by $\alpha\beta - \gamma\delta, \beta\nu, \nu\mu\nu$.

- The oriented cycle $\mu\nu$ is cyclically free in Λ
- By the last theorem, one of the simple modules S_1, S_4 has infinite projective dimension, and one has infinite injective dimension.

Stronger version of the SNLC

Conjecture (Extension conjecture)

Let S be a simple module over an artin algebra. If $\text{Ext}^1(S, S)$ is non-zero, then $\text{Ext}^i(S, S)$ is non-zero for infinitely many integers i .

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- It remains open except for monomial algebras and special biserial algebras.

THANK YOU

Questions ?