Super-decomposable pure-injective modules over strongly simply connected algebras of non-polynomial growth

Grzegorz Pastuszak
(joint work with Stanisław Kasjan)
Basic definitions

Let $R$ be a ring with an identity. By a module we always mean a left module.

- A module $M \in \text{Mod}(R)$ is called super-decomposable if it has no indecomposable direct summand. Thus, it cannot be finitely-generated.
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▶ A module $M \in \text{Mod}(R)$ is called **super-decomposable** if it has no indecomposable direct summand. Thus, it cannot be finitely-generated.

▶ A monomorphism $f : X \rightarrow Y$ is called **pure** if the map $\text{id}_M \otimes f : M \otimes X \rightarrow M \otimes Y$ is a monomorphism for every $M \in \text{Mod}(R)$. 

▶ A module $M \in \text{Mod}(R)$ is called **pure-injective** if it is injective with respect to the pure monomorphisms.

▶ A module $M \in \text{Mod}(R)$ is called **super-decomposable pure-injective** if it is both super-decomposable and pure-injective.
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Motivation

Let $K$ be an algebraically closed field.

**Problem (Prest, 1988)**

What is the connection between representation type of a finite-dimensional algebra $A$ and existence of super-decomposable pure-injective $A$-module?
Motivation

\( K \)-an algebraically closed field

**Problem (Prest, 1988)**
What is the connection between representation type of a finite-dimensional algebra \( A \) and existence of super-decomposable pure-injective \( A \)-module?

**Question**
May complexity of the category \( mod(A) \) be measured by existence of super-decomposable pure-injective \( A \)-module?
Known results

Theorem (Prest, 1988)
Every strictly wild algebra over a countable field possesses a super-decomposable pure-injective module.

Theorem (Puninski, 2003)
Every string algebra of non-polynomial growth over a countable field possesses a super-decomposable pure-injective module.

Theorem (Harland, 2011)
Every tubular algebra over a countable field possesses a super-decomposable pure-injective module.
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Aim of my talk

To sketch the proof of the fact that there exist SPI modules over strongly simply connected algebras of non-polynomial growth, if the base field $K$ is countable and $\text{char}(K) \neq 2$. 
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Definition (Skowronski, 1993)

Assume that \( A \) is a finite dimensional algebra without oriented cycles in its Gabriel quiver. Then \( A \) is called strongly simply connected if the first Hochschild cohomology group \( HH^1(C, C) \) vanishes for any convex subcategory \( C \) of \( A \).
1. The role of countability of the base field $K$ in our result.
Plan of my talk

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2. The existence of SPI module over the incidence algebra of a garland of length 3 and over the diamond algebra.
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1. The role of countability of the base field $K$ in our result.
2. The existence of SPI module over the incidence algebra of a garland of length 3 and over the diamond algebra.
3. Some consequences of 2. and an idea of the proof of the main result.
1. Why we assume that the field is countable?

There is only one general criterion for existence of SPI modules:

**Theorem (Ziegler, 1984)**

If the ”width of the lattice of all pp-formulae in one free variable” over a **countable** ring $R$ (in particular a countable $K$-algebra) is infinite, then there exists a super-decomposable pure-injective $R$-module.
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**Remark**

If the field $K$ is countable, then every bound-quiver $K$-algebra is countable. Thus in this case we may use the above theorem.
The notions

- A **pp-formula** $\varphi(\nu)$ in one free variable $\nu$ is a formula of the following form:

$$\exists x_1, \ldots, x_m \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{t1} & \cdots & a_{tm} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_t \end{bmatrix} \cdot \nu$$

for some $n, m \in \mathbb{N}$ and $a_{ij}, b_k \in R$. Thus it expresses an existence of a solution of some system of $R$-linear equations.
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for some \( n, m \in \mathbb{N} \) and \( a_{ij}, b_k \in R \). Thus it expresses an existence of a solution of some system of \( R \)-linear equations.

▶ The set of all pp-formulae forms a modular lattice \( L \) in which:

▶ the meet (the infimum) of \( \varphi \) and \( \psi \) is given by the conjunction \( \varphi \land \psi \),

▶ the join (the supremum) of \( \varphi \) and \( \psi \) is given by \( (\varphi + \psi)(x) := \exists y(\varphi(y) \land \psi(x - y)) \).
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- The set of all pp-formulae forms a modular lattice \( L \) in which:
  - the meet (the infimum) of \( \varphi \) and \( \psi \) is given by the conjunction \( \varphi \wedge \psi \),
  - the join (the supremum) of \( \varphi \) and \( \psi \) is given by \((\varphi + \psi)(x) := \exists y(\varphi(y) \wedge \psi(x - y))\).

- The width of the lattice \( L \) expresses the “level of its complexity”.
Question

How to apply the Ziegler criterion in a practical way?
pp-formulae and pointed modules

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Answer

- Every pp-formula $\varphi(v)$ induces a **pointed module** $(M_\varphi, m_\varphi)$, where $M_\varphi$ is a finitely-presented $R$-module and $m_\varphi \in M_\varphi$. 

Pointed modules provide us with a practical method of applying the theorem of Ziegler - the existence of some special family of pointed modules implies that the width of the lattice of all pp-formulae is infinite.
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Definition 1 (a dense chain of pointed R-modules)

A dense chain of pointed R-modules is an indexed family 
\((M_q, m_q)_{q \in \mathbb{Q}}\) of pointed R-modules such that:
1. modules \(M_q\) are indecomposable and \(m_q \neq 0\) for all \(q \in \mathbb{Q}\),
2. there exist pointed homomorphisms
   \(\mu_{q,q'} : (M_q, m_q) \rightarrow (M_{q'}, m_{q'})\) for all \(q < q'\),
3. pointed modules \((M_q, m_q)\) and \((M_{q'}, m_{q'})\) are not isomorphic.
Two concepts of Puninski

Definition 2 (an independent pair of dense chains of pointed \(R\)-modules)

An **independent pair of dense chains of pointed \(R\)-modules** is a pair \(\left( (M_q, m_q)_{q \in \mathbb{Q}^+}, (N_t, n_t)_{t \in \mathbb{Q}^-} \right)\) of dense chains of pointed \(R\)-modules such that:

1. there is no pointed homomorphism from \((M_q, m_q)\) to \((N_t, n_t)\) nor from \((N_t, n_t)\) to \((M_q, m_q)\) for all \(t < 0 < q\),
2. the pointed pushout \(\left( M_q, m_q \right) \ast \left( N_t, n_t \right)\) of \((M_q, m_q)\) and \((N_t, n_t)\) is indecomposable for all \(t < 0 < q\),
3. \(\left( M_q, m_q \right) \ast \left( N_t, n_t \right) \not\cong \left( M_{q'}, m_{q'} \right) \ast \left( N_t, n_t \right)\) and \(\left( M_q, m_q \right) \ast \left( N_t, n_t \right) \not\cong \left( M_q, m_q \right) \ast \left( N_{t'}, n_{t'} \right)\) for all \(t < t' < 0 < q < q'\).
The criterion

Theorem (Puninski, 2008)
If there exists an independent pair of dense chains of pointed modules over the ring $R$, then the width of the lattice of all pp-formulae over $R$ is infinite.

Corollary (Puninski, Ziegler)
Assume that $R$ be a countable ring. If there exists an independent pair of dense chains of pointed modules over the ring $R$, then there is a super-decomposable pure-injective $R$-module.
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The criterion

In our considerations, we will also use the following:

**Theorem (Prest-Puninski, 1999)**

Assume that $R, S$ are rings, $\mathcal{A}$ is a full subcategory of $\text{mod}(R)$ and $F: \mathcal{A} \to \text{mod}(S)$ is full, faithful and exact functor. If there exists an independent pair of dense chains of pointed $R$-modules in $\mathcal{A}$, then there is a super-decomposable pure-injective $S$-module.
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In our considerations, we will also use the following:

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**Remark**

The above theorem is not at all an obvious corollary. Its proof uses rather advanced techniques of model theory of modules.
2. The garland and the diamond

A garland $G_3$ of length 3 is a bound-quiver $K$-algebra of the form $KQ/I$, where

$$Q := \begin{array}{ccc}
11 & 12 \\
\alpha_1 & \beta_2 & \alpha_2 \\
\beta_1 & & \\
21 & 22 \\
\delta_1 & \gamma_2 & \delta_2 \\
31 & 32 \\
\delta_1 & \gamma_1 & \delta_2
\end{array}$$

and $I = \langle \gamma_1\alpha_1 - \delta_2\beta_1, \delta_1\alpha_1 - \gamma_2\beta_1, \gamma_2\alpha_2 - \delta_1\beta_2, \delta_2\alpha_2 - \gamma_1\beta_2 \rangle$. 
A **diamond** $D$ is a bound-quiver $K$-algebra of the form $KQ'/I'$, where

$$Q' = \begin{array}{cccccc}
2 & 3 & 4 & 5 \\
\omega_1 & v_1 & v_2 & v_3 & v_4 \\
\omega_2 & & & & \\
\omega_3 & & & & \\
\omega_4 & & & & \\
6 & & & &
\end{array}$$

and $I' = \langle v_1\omega_1 + v_3\omega_3 + v_4\omega_4, v_2\omega_2 + v_3\omega_3 + v_4\omega_4 \rangle$. 
Existence of SPIs over garland and diamond: the idea of the proof

Assume $\text{char}(K) \neq 2$. Then there is a Galois covering $F: Q \to Q''$ of the string algebra $S := KQ''/I''$ of non-polynomial growth by $G_3$, where

\[
\begin{align*}
Q'' := \begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\delta
\end{array},
\end{align*}
\]

and $I'' = \langle \delta \alpha, \gamma \beta \rangle$. 
The idea

Since $S$ is a string algebra of non-polynomial growth, by theorem of Puninski it possesses an independent pair of dense chains of pointed modules.
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- Since $S$ is a string algebra of non-polynomial growth, by theorem of Puninski it possesses an independent pair of dense chains of pointed modules.

- It turns out there are many of such independent pairs, so choose some special independent pair $\mathcal{P}$ such that “$F_\bullet(\mathcal{P})$” is an independent pair of dense chains of pointed modules in $\text{mod}(G_3)$, where $F_\bullet : \text{mod}(S) \rightarrow \text{mod}(G_3)$ is the pull-up functor associated to $F$ (quite hard and technically complicated).
The idea

▶ Since $S$ is a string algebra of non-polynomial growth, by theorem of Puninski it possesses an independent pair of dense chains of pointed modules.

▶ It turns out there are many of such independent pairs, so choose some special independent pair $\mathcal{P}$ such that \( F_\bullet(\mathcal{P}) \) is an independent pair of dense chains of pointed modules in $\text{mod}(\mathcal{G}_3)$, where $F_\bullet : \text{mod}(S) \rightarrow \text{mod}(\mathcal{G}_3)$ is the pull-up functor associated to $F$ (quite hard and technically complicated).

▶ Use the independent pair constructed in $\text{mod}(\mathcal{G}_3)$ to build an independent pair in $\text{mod}(\mathcal{D})$ (rather easy).
Theorem (Kasjan-P., 2011)

1. There is an independent pair of dense chains of pointed modules in $mod(\mathcal{G}_3)$. Thus there exists a super-decomposable pure-injective $\mathcal{G}_3$-module.

2. There is an independent pair of dense chains of pointed modules in $mod(\mathcal{D})$. Thus there exists a super-decomposable pure-injective $\mathcal{D}$-module.
Theorem (Kasjan-P., 2011)

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2. There is an independent pair of dense chains of pointed modules in $mod(D)$. Thus there exists a super-decomposable pure-injective $D$-module.

Technical refinement

1. The independent pair of dense chains of pointed modules in $mod(G_3)$ is entirely contained in $prin(G_3)$.

2. The independent pair of dense chains of pointed modules in $mod(D)$ is entirely contained in $prin(D)$. 
Remark
Noerenberg and Skowronski defined a class of so called polynomial-growth critical algebras (pg-critical) algebras such that:

- They are certain extensions of representation-infinite tilted algebras of type $\widetilde{D}_n$, naturally divided into two classes: $B[M]$ and $B[N, t]$. 
Remark
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▶ They are certain extensions of representation-infinite tilted algebras of type $\widehat{\mathbb{D}}_n$, naturally divided into two classes: $B[M]$ and $B[N, t]$.
▶ Every such an algebra is tame of non-polynomial growth.
3. Applications to strongly simply connected algebras

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- They are certain extensions of representation-infinite tilted algebras of type $\widetilde{D}_n$, naturally divided into two classes: $B[M]$ and $B[N, t]$.
- Every such an algebra is tame of non-polynomial growth.
- Every proper convex subcategory of such an algebra is of polynomial growth.
Theorem (Noerenberg-Skowronski, 1997)
Assume that $A$ is a tame strongly simply connected algebra. Then $A$ is of non-polynomial growth if and only if $A$ contains a pg-critical convex subcategory.

Corollary
If every pg-critical algebra possesses a super-decomposable pure-injective module, then every tame strongly simply connected algebra possesses such module.
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Corollary
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Theorem (Simson, 1995)

1. There exists fully faithful exact functor $F_\Gamma : \text{prin}(G_3) \to \text{mod}(\Gamma)$ for any pg-critical algebra $\Gamma$ of the form $B[N, t]$.

2. There exists fully faithful exact functor $G_\Lambda : \text{prin}(D) \to \text{mod}(\Lambda)$ for any pg-critical algebra $\Lambda$ of the form $B[M]$.
Summary

- There are independent dense chains of pointed modules in $\text{prin}(G_3)$ and $\text{prin}(\mathcal{D})$.

Corollary

There are super-decomposable pure-injective modules over any pg-critical algebra.
Summary

- There are independent dense chains of pointed modules in \( \text{prin}(\mathcal{G}_3) \) and \( \text{prin}(\mathcal{D}) \).
- There are fully faithful exact functors from \( \text{prin}(\mathcal{G}_3) \) or \( \text{prin}(\mathcal{D}) \) to categories of modules over any pg-critical algebra.
- Existence of fully faithful exact functor \( F : A \to \text{mod}(S) \) from the category \( A \) which possesses an independent dense chains of pointed modules implies the existence of SPI module over \( S \).
- Corollary: There are super-decomposable pure-injective modules over any pg-critical algebra.
Summary

- There are independent dense chains of pointed modules in $\text{prin}(G_3)$ and $\text{prin}(D)$.
- There are fully faithful exact functors from $\text{prin}(G_3)$ or $\text{prin}(D)$ to categories of modules over any pg-critical algebra.
- Existence of fully faithful exact functor $F : \mathcal{A} \rightarrow \text{mod}(S)$ from the category $\mathcal{A}$ which possesses an independent dense chains of pointed modules implies the existence of SPI module over $S$.

Corollary: There are super-decomposable pure-injective modules over any pg-critical algebra.
Summary

- There are independent dense chains of pointed modules in prin($G_3$) and prin($D$).
- There are fully faithful exact functors from prin($G_3$) or prin($D$) to categories of modules over any pg-critical algebra.
- Existence of fully faithful exact functor $F : \mathcal{A} \to \text{mod}(S)$ from the category $\mathcal{A}$ which possesses an independent dense chains of pointed modules implies the existence of SPI module over $S$.

Corollary

There are super-decomposable pure-injective modules over any pg-critical algebra.
Theorem
Assume that $A$ is a tame strongly simply connected algebra of non-polynomial growth. Then there exists a super-decomposable pure-injective $A$-module.
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THANK YOU FOR YOUR ATTENTION