Super-decomposable pure-injective modules over strongly simply connected algebras of non-polynomial growth

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(joint work with Stanisław Kasjan)

Basic definitions

Let R be a ring with an identity. By a **module** we always mean a left module.

- A module $M \in Mod(R)$ is called **super-decomposable** if it has no indecomposable direct summand. Thus, it cannot be finitely-generated.

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- A module $M \in Mod(R)$ is called **pure-injective** if it is injective with respect to the pure monomorphisms.
- A module $M \in Mod(R)$ is called **super-decomposable pure-injective** if it is both super-decomposable and pure-injectective.

Motivation

K-an algebraically closed field

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What is the connection between representation type of a finite-dimensional algebra A and existence of super-decomposable pure-injective A-module?

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Question

May complexity of the category $mod(A)$ be measured by existence of super-decomposable pure-injective A-module?

Known results

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Every **string algebra of non-polynomial growth** over a **countable** field possesses a super-decomposable pure-injective module.

Theorem (Harland, 2011)

Every **tubular algebra** over a **countable** field possesses a super-decomposable pure-injective module.

Aim of my talk

To sketch the proof of the fact that there exist SPI modules over **strongly simply connected algebras of non-polynomial growth**, if the base field K is countable and $char(K) \neq 2$.

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Definition (Skowronski, 1993)

Assume that A is a finite dimensional algebra without oriented cycles in its Gabriel quiver. Then A is called **strongly simply connected** if the first Hochschild cohomology group $HH^1(C, C)$ vanishes for any convex subcategory C of A.

Plan of my talk

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2. The existence of SPI module over the incidence algebra of a garland of length 3 and over the diamond algebra.
3. Some consequences of 2. and an idea of the proof of the main result.

1. Why we assume that the field is countable?

There is only one general criterion for existence of SPI modules:

Theorem (Ziegler, 1984)

If the "width of the lattice of all pp-formulae in one free variable" over a **countable** ring R (in particular a countable K-algebra) is infinite, then there exists a super-decomposable pure-injective R-module.

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Remark

If the field K is countable, then every bound-quiver K-algebra is countable. Thus in this case we may use the above theorem.

The notions

- A **pp-formula** $\varphi(v)$ in one free variable v is a formula of the following form:

$$\exists_{x_1,\dots,x_m} \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{t1} & \dots & a_{tm} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_t \end{bmatrix} \cdot v$$

for some $n, m \in \mathbb{N}$ and $a_{ij}, b_k \in R$. Thus it expresses an existence of a solution of some system of R-linear equations.

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  for some $n, m \in \mathbb{N}$ and $a_{ij}, b_k \in R$. Thus it expresses an existence of a solution of some system of R-linear equations.
- The set of all pp-formulae forms a modular lattice L in which:
  - the meet (the infimum) of φ and ψ is given by the conjunction $\varphi \wedge \psi$,
  - the join (the supremum) of φ and ψ is given by $(\varphi + \psi)(x) := \exists_y(\varphi(y) \wedge \psi(x - y))$.

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- The width of the lattice L expresses the "level of its complexity".

pp-formulae and pointed modules

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Answer

- Every pp-formula $\varphi(v)$ induces a **pointed module** (M_φ, m_φ), where M_φ is a finitely-presented R-module and $m_\varphi \in M_\varphi$.

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Answer

- Every pp-formula $\varphi(v)$ induces a **pointed module** (M_φ, m_φ), where M_φ is a finitely-presented R-module and $m_\varphi \in M_\varphi$.
- Pointed modules provide us with a practical method of applying the theorem of Ziegler - the existence of **some special family of pointed modules** implies that the width of the lattice of all pp-formulae is infinite.

Two concepts of Puninski

Definition 1 (a dense chain of pointed R-modules)

A **dense chain of pointed R-modules** is an indexed family $(M_q, m_q)_{q \in \mathbb{Q}}$ of pointed R-modules such that:

1. modules M_q are indecomposable and $m_q \neq 0$ for all $q \in \mathbb{Q}$,
2. there exist pointed homomorphisms $\mu_{q,q'} : (M_q, m_q) \to (M_{q'}, m_{q'})$ for all $q < q'$,
3. pointed modules (M_q, m_q) and $(M_{q'}, m_{q'})$ are not isomorphic.

Two concepts of Puninski

Definition 2 (an independent pair of dense chains of pointed R-modules)

An **independent pair of dense chains of pointed R-modules** is a pair $((M_q, m_q)_{q \in \mathbb{Q}^+}, (N_t, n_t)_{t \in \mathbb{Q}^-})$ of dense chains of pointed R-modules such that:

1. there is no pointed homomorhism from (M_q, m_q) to (N_t, n_t) nor from (N_t, n_t) to (M_q, m_q) for all $t < 0 < q$,
2. the pointed pushout $(M_q, m_q) * (N_t, n_t)$ of (M_q, m_q) and (N_t, n_t) is indecomposable for all $t < 0 < q$,
3. $(M_q, m_q) * (N_t, n_t) \ncong (M_{q'}, m_{q'}) * (N_t, n_t)$ and $(M_q, m_q) * (N_t, n_t) \ncong (M_q, m_q) * (N_{t'}, n_{t'})$ for all $t < t' < 0 < q < q'$.

The criterion

Theorem (Puninski, 2008)

If there exists an independent pair of dense chains of pointed modules over the ring R, then the width of the lattice of all pp-formulae over R is infinite.

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Corollary (Puninski, Ziegler)

Assume that R be a countable ring. If there exists an independent pair of dense chains of pointed modules over the ring R, then there is a super-decomposable pure-injective R-module.

The criterion

In our considerations, we will also use the following:

Theorem (Prest-Puninski, 1999)

Assume that R, S are rings, $\mathcal{A}$ is a full subcategory of $mod(R)$ and $F : \mathcal{A} \to mod(S)$ is full, faithful and exact functor. If there exists an independent pair of dense chains of pointed R-modules in $\mathcal{A}$, then there is a super-decomposable pure-injective S-module.

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Remark

The above theorem is not at all an obvious corollary. Its proof uses rather advanced techniques of model theory of modules.

2. The garland and the diamond

A **garland** $\mathcal{G}_3$ of length 3 is a bound-quiver K-algebra of the form KQ/I, where

$$Q := \begin{tikzcd}
11 \arrow[d, "\alpha_1"'] \arrow[dr, "\beta_1" near start] & 12 \arrow[dl, "\beta_2"' near start] \arrow[d, "\alpha_2"] \\
21 \arrow[d, "\delta_1"'] \arrow[dr, "\gamma_1" near start] & 22 \arrow[dl, "\gamma_2"' near start] \arrow[d, "\delta_2"] \\
31 & 32
\end{tikzcd}$$

and $I = \langle \gamma_1\alpha_1 - \delta_2\beta_1, \delta_1\alpha_1 - \gamma_2\beta_1, \gamma_2\alpha_2 - \delta_1\beta_2, \delta_2\alpha_2 - \gamma_1\beta_2 \rangle$.

A **diamond** $\mathcal{D}$ is a bound-quiver K-algebra of the form KQ'/I', where

$$
Q' = \begin{tikzcd}
 & & 1 \arrow[dll, "\upsilon_1"'] \arrow[dl, "\upsilon_2"] \arrow[dr, "\upsilon_3"'] \arrow[drr, "\upsilon_4"] & & \\
2 \arrow[drr, "\omega_1"'] & 3 \arrow[dr, "\omega_2"] & & 4 \arrow[dl, "\omega_3"'] & 5 \arrow[dll, "\omega_4"] \\
 & & 6 & &
\end{tikzcd}
$$

and $I' = \langle \upsilon_1\omega_1 + \upsilon_3\omega_3 + \upsilon_4\omega_4, \upsilon_2\omega_2 + \upsilon_3\omega_3 + \upsilon_4\omega_4 \rangle$.

Existence of SPIs over garland and diamond: the idea of the proof

- Assume $char(K) \neq 2$. Then there is a Galois covering $F : Q \to Q''$ of the string algebra $S := KQ''/I''$ of non-polynomial growth by $\mathcal{G}_3$, where

$$Q'' := \begin{tikzcd} x_1 \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] \\ x_2 \arrow[d, shift right, "\delta"'] \arrow[d, shift left, "\gamma"] \\ x_3 \end{tikzcd}$$

and $I'' = \langle \delta\alpha, \gamma\beta \rangle$.

The idea

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- It turns out there are many of such independent pairs, so choose some special independent pair $\mathcal{P}$ such that "$F_\bullet(\mathcal{P})$" is an independent pair of dense chains of pointed modules in $mod(\mathcal{G}_3)$, where $F_\bullet : mod(S) \to mod(\mathcal{G}_3)$ is the pull-up functor associated to F (quite hard and technically complicated).

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- It turns out there are many of such independent pairs, so choose some special independent pair $\mathcal{P}$ such that "$F_\bullet(\mathcal{P})$" is an independent pair of dense chains of pointed modules in $mod(\mathcal{G}_3)$, where $F_\bullet : mod(S) \to mod(\mathcal{G}_3)$ is the pull-up functor associated to F (quite hard and technically complicated).
- Use the independent pair constructed in $mod(\mathcal{G}_3)$ to build an independent pair in $mod(\mathcal{D})$ (rather easy).

Theorem (Kasjan-P., 2011)

1. There is an independent pair of dense chains of pointed modules in $mod(\mathcal{G}_3)$. Thus there exists a super-decomposable pure-injective $\mathcal{G}_3$-module.
2. There is an independent pair of dense chains of pointed modules in $mod(\mathcal{D})$. Thus there exists a super-decomposable pure-injective $\mathcal{D}$-module.

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Technical refinement

1. The independent pair of dense chains of pointed modules in $mod(\mathcal{G}_3)$ is entirely contained in $prin(\mathcal{G}_3)$.
2. The independent pair of dense chains of pointed modules in $mod(\mathcal{D})$ is entirely contained in $prin(\mathcal{D})$.

3. Applications to strongly simply connected algebras

Remark

Noerenberg and Skowronski defined a class of so called **polynomial-growth critical algebras** (pg-critical) algebras such that:

- They are certain extensions of representation-infinite tilted algebras of type $\widetilde{\mathbb{D}}_n$, naturally dived into two classes: $B[M]$ and $B[N, t]$.

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- Every such an algebra is tame of non-polynomial growth.

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- They are certain extensions of representation-infinite tilted algebras of type $\widetilde{\mathbb{D}}_n$, naturally dived into two classes: $B[M]$ and $B[N, t]$.
- Every such an algebra is tame of non-polynomial growth.
- Every proper convex subcategory of such an algebra is of polynomial growth.

Theorem (Noerenberg-Skowronski, 1997)

Assume that A is a tame strongly simply connected algebra. Then A is of non-polynomial growth if and only if A contains a pg-critical convex subcategory.

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$$\Downarrow$$

Corollary

If every pg-critical algebra possesses a super-decomposable pure-injective module, then every tame strongly simply connected algebra possesses such module.

Theorem (Simson, 1995)

1. There exists fully faithful exact functor $F_\Gamma : prin(\mathcal{G}_3) \to mod(\Gamma)$ for any pg-critical algebra Γ of the form $B[N, t]$.
2. There exists fully faithful exact functor $G_\Lambda : prin(\mathcal{D}) \to mod(\Lambda)$ for any pg-critical algebra Λ of the form $B[M]$.

Summary

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- Existence of fully faithful exact functor $F : \mathcal{A} \to mod(S)$ from the category $\mathcal{A}$ which possesses an independent dense chains of pointed modules implies the existence of SPI module over S.

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- Existence of fully faithful exact functor $F : \mathcal{A} \to mod(S)$ from the category $\mathcal{A}$ which possesses an independent dense chains of pointed modules implies the existence of SPI module over S.

Corollary

There are super-decomposable pure-injective modules over any pg-critical algebra.

Theorem

Assume that A is a tame strongly simply connected algebra of non-polynomial growth. Then there exists a super-decomposable pure-injective A-module.

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THANK YOU FOR YOUR ATTENTION