Elementary subalgebras of modular Lie algebras and vector bundles on projective varieties

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joint work with E. Friedlander and J. Carlson

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Examples

(1). G - finite group $\sim$ *constant* finite group scheme $\rightsquigarrow$ kG - group algebra
(2). $\mathfrak{g}$ - restricted Lie algebra $\rightsquigarrow$ $\mathfrak{u}(\mathfrak{g})$ restricted enveloping algebra
(3). $\mathcal{G}$ - algebraic group $\rightsquigarrow$ $\mathcal{G}_{(r)}$, the r^{th} Frobenius kernel of $\mathcal{G}$, an *infinitesimal* finite group scheme

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- *Study invariants* (cohomology, support varieties, local behaviour, coherent sheaves associated to representations)
- *Classify coarser structures* (thick subcategories)
- *Study special classes of representations* (e.g., modules of constant Jordan type)

Cohomology

kG - Hopf algebra $\quad\Rightarrow\quad \mathrm{H}^*(G,k) := \mathrm{H}^*(kG,k)$ is graded commutative.

$$\mathrm{H}^\bullet(G,k) = \begin{cases} \mathrm{H}^{ev}(G,k) & p > 2 \\ \mathrm{H}^*(G,k) & p = 2 \end{cases}$$

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Theorem (Friedlander-Suslin, '95)

For any finite group scheme G over k, $\mathrm{H}^*(G,k)$ is a finitely generated k-algebra. For a finite-dimensional G-module M, $\mathrm{H}^*(G,M)$ is a finite module over $\mathrm{H}^*(G,k)$.

Precursor: Spectrum of the cohomology of a finite group

- G - **finite group.**

Theorem (Quillen stratification theorem '71)

Let G be a finite group.

$$\operatorname{Spec} \mathrm{H}^\bullet(G, k) = \bigcup_{E \subset G} \operatorname{Spec} \mathrm{H}^\bullet(E, k),$$

where E runs through all elementary abelian p-subgroups of G.

Finite generation of cohomology was known since late 50s-early 60s (Golod '59, Venkov '61, Evens '61).

• $\mathfrak{g}$ - **restricted Lie algebra,** a k-Lie algebra endowed with a $[p]^{\mathrm{th}}$ power map

$$[p] : \mathfrak{g} \to \mathfrak{g}$$

satisfying some natural p^{th}-power conditions.

Theorem (Suslin-Friedlander-Bendel '97)

$$\operatorname{Spec} \mathrm{H}^\bullet(\mathfrak{g}, k) = \mathcal{N}_p,$$

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For $\mathfrak{g} = \operatorname{Lie} \mathcal{G}$, $\mathcal{G}$ - reductive algebraic group, $p > h$, there is a much stronger result: $\boxed{\mathrm{H}^\bullet(\mathfrak{g}, k) \simeq k[\mathcal{N}]}$ for $\mathcal{N} \subset \mathfrak{g}$ the nilpotent cone (Friedlander-Parshall, Andersen-Jantzen, '87)

- G - **infinitesimal (=connected) finite group scheme**. Then

$$\operatorname{Spec} \mathrm{H}^{\bullet}(G, k) \simeq V(G),$$

where $V(G)$ is the scheme of *one-parameter subgroups* of G (Suslin-Friedlander-Bendel '97). The Lie algebra result is a special case of this more general identification.

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- G - **arbitrary finite group scheme.**

Definition

A p-**point** α of a finite group scheme G is a flat map of algebras

$$k[x]/x^p \xrightarrow{\alpha} kG$$

which factors through some unipotent abelian subgroup scheme $A \subset G$.

p-points $\sim$ "one-parameter subgroups" of G.

Definition

We say that two p-points $\alpha, \beta : k[x]/x^p \to kG$ are equivalent, $\alpha \sim \beta$, if the following condition holds: for any finite-dimensional kG-module M, $\alpha^*(M)$ is free if and only if $\beta^*(M)$ is free (as $k[x]/x^p$-modules).

$$\Pi(G) := \frac{\langle p - \text{points} \rangle}{\sim}$$

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Theorem (Friedlander-P., '07)

$$\operatorname{Proj} \mathrm{H}^\bullet(G,k) \simeq \Pi(G)$$

Applications

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$$\mathrm{Spec}(\mathrm{stmod}\, kG) \simeq \mathrm{Proj}\, \mathrm{H}^\bullet(G, k)$$

Homeomorphism $\sim$ classification of thick tensor ideal subcategories of stmod kG. For G a finite group, classification was proved by Benson-Carlson-Rickard, '97.

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Theorem (Farnsteiner, '07)

Let G be a finite group scheme. If $\dim \mathrm{Spec}\, \mathrm{H}^{\bullet}(G, k) \geq 3$, *then the representation theory of G is wild.*

JType(α, M)

A p-**point** α is a flat map of algebras $\boxed{\alpha : k[x]/x^p \to kG}$. Let $\alpha^*(M)$ be the restriction of a kG-module M to $k[x]/x^p$ via α.

p-points $\rightsquigarrow$ local methods / local invariants of modules. Study M by considering $\alpha^*(M)$ for all p-points α.

Definition

Let M be a finite dimensional kG-module. $\mathrm{JType}(\alpha, M) =$ Jordan type of $\alpha(x)$ as a p-nilpotent operator on $M =$

$$[p]^{a_p} \dots [1]^{a_1},$$

where a_i is the number of Jordan blocks of size i.

Detection of projectivity

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"Dade's lemma" for finite group schemes:

Theorem

Let G be a finite group scheme, and M be a finite dimensional kG-module. M is projective as a kG-module if and only if $\mathrm{JType}(\alpha, M) = [p]^{a_p}$ *for any p-point* $\alpha : k[x]/x^p \to kG$.

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Local Jordan type also detects "endo-trivial modules".

New invariant: generic Jordan type

Recall: $\frac{\langle p\text{−points}\rangle}{\sim} = \Pi(G) \simeq \operatorname{Proj} \mathrm{H}^\bullet(G, k)$.
A p-point $\alpha : K[x]/x^p \to KG$ is "generic" if the equivalence class $[\alpha]$ of α is a generic point of $\operatorname{Proj} \mathrm{H}^\bullet(G, k)$ (K/k is a field extension).

Theorem (Friedlander-P.-Suslin, '07)

Let $\alpha : K[x]/x^p \to KG$ be a generic p-point of G. Then

$$[\alpha]^* : \mathrm{stmod}\, kG \to \mathrm{stmod}\, K[x]/x^p$$

is a tensor-triangulated functor which is independent of a representative of the equivalence class $[\alpha]$.

Modules of constant Jordan type

Definition

Let G be a finite group scheme, and M be a finite dimensional kG-module. M has *constant Jordan type* if $\mathrm{JType}(\alpha, M)$ is independent of the p-point $\alpha : k[x]/x^p \to kG$.

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Remark

This definition might look different from the one Dave Benson gave since Dave's definition depended on a choice of generators of E. Theorem on the previous slide $\Rightarrow$ SAME.

Modules of CJT $\Rightarrow$ vector bundles on $\operatorname{Proj} \mathrm{H}^\bullet(G, k)$

To construct vector bundles on $\operatorname{Proj} \mathrm{H}^\bullet(G, k)$ we need to restrict to infinitesimal (=connected) finite group schemes. To avoid technical details, we restrict further to **restricted Lie algebras** (equivalently, infinitesimal group schemes of height 1).

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Resolution: E is a Lie algebra in disguise. Let

$$\mathfrak{g}_a := \operatorname{Lie} \mathbb{G}_a, \mathfrak{g} = (\mathfrak{g}_a)^{\oplus n}.$$

Then

$$\mathfrak{u}(\mathfrak{g}) \simeq k[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p) \simeq kE.$$

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$$\boxed{kE\text{-modules}} \sim \boxed{\mathfrak{u}(\mathfrak{g})\text{-modules}}.$$

Let $\mathfrak{g}$ be a restricted Lie algebra. Recall $\mathcal{N}_p = \{x \in \mathfrak{g} \mid x^{[p]} = 0\}$;

$$x \in \mathcal{N}_p(\mathfrak{g}) \quad \rightsquigarrow \quad k[x]/x^p \to \mathfrak{u}(\mathfrak{g})$$

$\{$equiv. classes of p-points $\} \sim \{$lines of p-nilpotent elements in $\mathfrak{g}\}$.
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Definition

$$\Theta = x_1 \otimes Y_1 + \ldots + x_n \otimes Y_n \in u(\mathfrak{g}) \otimes k[\mathcal{N}_p]$$

- the universal p-nilpotent operator

Any $x = a_1 x_1 + \ldots + a_n x_n \in \mathcal{N}_p(\mathfrak{g})$ is a specialization of Θ for some values $(a_1, \ldots, a_n)$ of $(Y_1, \ldots, Y_n)$.

Universal p-nilpotent operator

For an $\mathfrak{g}$-module M, Θ determines a "global" p-nilpotent homogeneous operator

$$\Theta_M : M \otimes k[\mathcal{N}_p(\mathfrak{g})] \to M \otimes k[\mathcal{N}_p(\mathfrak{g})]$$

$$m \otimes f \mapsto \sum x_i m \otimes Y_i f$$

Let $\mathbb{P}(\mathfrak{g}) = \operatorname{Proj} k[\mathcal{N}_p(\mathfrak{g})]$,

$$\widetilde{\Theta}_M : M \otimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})} \to M \otimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)$$

Vector bundles

Theorem (Friedlander-P. '11)

Let M be a $\mathfrak{g}$-module of constant Jordan type. Then

$$\mathcal{K}er\{\widetilde{\Theta}^i_M\}, \mathcal{I}m\{\widetilde{\Theta}^i_M\},$$

for $1 \leq i \leq p-1$, and their various allowable quotients, are algebraic vector bundles on $\mathbb{P}(\mathfrak{g})$.

Dave Benson's lectures $\rightsquigarrow$ Examples, constructions, realization questions, dreams...

Elementary subalgebras of restricted Lie algebras

Definition

A restricted subalgebra $\epsilon \subset \mathfrak{g}$ is called *elementary* of dimension r if $\epsilon \simeq (\mathfrak{g}_a)^{\oplus r}$ (that is, ϵ is abelian with trivial restriction).

$\mathfrak{u}(\epsilon) \simeq k[x_1, \ldots, x_r]/(x_1^p, \ldots, x_r^p)$.

Definition

$\mathbb{E}(r, \mathfrak{g})$ is a (projective) variety of elementary subalgebras of $\mathfrak{g}$.

Question: what can we say about the geometry of $\mathbb{E}(r, \mathfrak{g})$?

- $r = 1$. $\mathbb{E}(1, \mathfrak{g}) = \operatorname{Proj} k[\mathcal{N}_p(\mathfrak{g})] \simeq \operatorname{Proj} \mathrm{H}^\bullet(\mathfrak{g}, k)$.

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  For $\mathfrak{g} = \operatorname{Lie} \mathcal{G}$ with $\mathcal{G}$ a connected reductive group, $\mathcal{N}_p(\mathfrak{g})$ is irreducible.

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- $\mathfrak{g} = \operatorname{Lie}\mathcal{G}$. $\mathbb{E}(r, \mathfrak{g})$ is closely related to $\operatorname{Proj} \mathrm{H}^\bullet(\mathcal{G}_{(r)}, k)$ for many classes of connected algebraic groups $\mathcal{G}$ (Suslin-Friedlander-Bendel $'97$, McNinch $'02$, Sobaje $'12$).

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??Geometry of $\mathbb{E}(r, \operatorname{Lie}\mathcal{G})$ for $r = 3$??

Maximal elementary subalgebras

Notation: $\mathrm{rk}_{el}(\mathfrak{g}) = \max\{r \mid \exists \epsilon \subset \mathfrak{g}, \dim \epsilon = r\}$

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- $\mathfrak{gl}_n$ (or $\mathfrak{sl}_n$).
  - $n = 2m$. Then $\mathrm{rk}_{el}(\mathfrak{gl}_{2m}) = m^2$, $\mathbb{E}(m^2, \mathfrak{gl}_{2m}) \simeq \mathrm{Grass}_{m,2m}$.

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  - $n = 2m + 1$. Then $\mathrm{rk}_{el}(\mathfrak{gl}_{2m+1}) = m(m+1)$, $\mathbb{E}(m(m+1), \mathfrak{gl}_{2m+1}) \simeq \mathrm{Grass}_{m,2m+1} \sqcup \mathrm{Grass}_{m,2m+1}$

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- $\mathfrak{sp}_{2n}$. Then $\mathrm{rk}_{el}(\mathfrak{sp}_{2n}) = \frac{n(n+1)}{2}$, $\mathbb{E}\left(\frac{(n(n+1)}{2}, \mathfrak{sp}_{2n}\right) \simeq \mathrm{LG}_{n,2n}$, the Lagrangian Grassmannian.

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In classical cases the elementary rank $\mathrm{rk}_{el}(\mathfrak{g})$ and the corresponding variety $\mathbb{E}(r, \mathfrak{g})$ are related to cominuscule parabolics; hence, they are governed by the combinatorics of the Dynkin diagram/root system.

Constant radical/socle rank modules

M - $\mathfrak{g}$-module. Consider $M \downarrow_\epsilon$ where ϵ runs through elementary subalgebras of $\mathfrak{g}$ of dimension r.
Numerical invariants: $\dim \operatorname{Rad}^j(M \downarrow_\epsilon)$, $\dim \operatorname{Soc}^j(M \downarrow_\epsilon)$.

Definition

- M is a module of constant (r, j) radical rank if the dimension of $\operatorname{Rad}^j(M \downarrow_\epsilon)$ is independent of $\epsilon \in \mathbb{E}(r, \mathfrak{g})$.
- M is a module of constant (r, j) socle rank if the dimension of $\operatorname{Soc}^j(M \downarrow_\epsilon)$ is independent of $\epsilon \in \mathbb{E}(r, \mathfrak{g})$.

Constant radical/socle rank modules

M - $\mathfrak{g}$-module. Consider $M \downarrow_\epsilon$ where ϵ runs through elementary subalgebras of $\mathfrak{g}$ of dimension r.
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Take $r = 1$, and let $j = 1, \dots, p-1 \rightsquigarrow$ modules of CJT.
For $r > 1$ the conditions of constant socle and constant radical rank are independent.

Representations of $\mathfrak{g} \Rightarrow$ coherent sheaves on $\mathbb{E}(r, \mathfrak{g})$.

We replace the operator Θ for $r = 1$ with a vector $(\Theta_1, \ldots, \Theta_r)$. Two (equivalent) constructions:

(1) Via patching local constructions on an affine covering of $\mathbb{E}(r, \mathfrak{g})$.

(2) Via equivariant descent, using

$$\Theta_i : M \otimes k[\mathcal{N}_p^r(\mathfrak{g})] \to M \otimes k[\mathcal{N}_p^r(\mathfrak{g})][1]$$

for $i = 1, \ldots, r$ where $\mathcal{N}_p^r(\mathfrak{g})$ is the variety of p-nilpotent commuting elements of $\mathfrak{g}$.

Theorem (Carlson-Friedlander-P.)

There exist functors

$$\mathcal{I}m^j, \mathcal{K}er^j : \mathfrak{u}(\mathfrak{g}) - \mathrm{mod} \to \mathsf{Coh}(\mathbb{E}(r, \mathfrak{g}))$$

such that the fiber of $\mathcal{I}m^j(M)$ *(resp.* $\mathcal{K}er^j(M)$*) for a restricted* $\mathfrak{u}(\mathfrak{g})$*-module* M *at a generic point* $\epsilon \in \mathbb{E}(r, \mathfrak{g})$ *is naturally identified with* $\mathsf{Rad}^j(M\downarrow_\epsilon)$ *(resp.* $\mathsf{Soc}^j(M\downarrow_\epsilon)$*).*

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M - a $\mathfrak{g}$-module of constant (r, j) radical rank $\Rightarrow$
$\mathcal{I}m^j(M)$ is an *algebraic vector bundle* on $\mathbb{E}(r, \mathfrak{g})$
with fiber at $\epsilon \in \mathbb{E}(r, \mathfrak{g})$ naturally isomorphic to $\mathrm{Rad}^j(M\downarrow_\epsilon)$

M - a $\mathfrak{g}$-module of constant (r, j) socle rank $\Rightarrow$
$\mathcal{K}er^j(M)$ is an *algebraic vector bundle* on $\mathbb{E}(r, \mathfrak{g})$
with fiber at $\epsilon \in \mathbb{E}_r(\mathfrak{g})$ naturally isomorphic to $\mathrm{Soc}^j(M\downarrow_\epsilon)$

Examples

$\mathfrak{g} = \mathfrak{sl}_{2n}$ (resp. $\mathfrak{sp}_{2n}$), V - standard representation of $\mathfrak{g}$.

$X = \mathbb{E}(n^2, \mathfrak{g}) = \mathrm{Grass}_n(V)$ (resp. $\mathbb{E}(\binom{n+1}{2}, \mathfrak{g}) = \mathrm{LG}_n(V)$).

γ_n - tautological rank n bundle on X.

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γ_n - tautological rank n bundle on X.

1. $\mathcal{I}m(V) = \gamma_n$

   For any $m \leq n$,
2. $\mathcal{I}m^m(V^{\otimes m}) \simeq \gamma_n^{\otimes m}$,
3. $\mathcal{I}m^m(S^m(V)) \simeq S^m(\gamma_n)$,
4. $\mathcal{I}m^m(\Lambda^m(V)) \simeq \Lambda^m(\gamma_n)$.

Tangent and cotangent bundles

$\mathfrak{g} = \mathfrak{sl}_{2n}$ (or $\mathfrak{g} = \mathfrak{sp}_{2n}$). Consider $\mathfrak{g}$ acting as an adjoint representation on itself.

$X = \mathbb{E}(n^2, \mathfrak{sl}_{2n}) \simeq \mathrm{Grass}_{n,2n}$ (or $X = \mathbb{E}(\binom{n+1}{2}, \mathfrak{sp}_{2n}) \simeq \mathrm{LG}_{n,2n}$).

1. $\mathrm{Coker}(\mathfrak{g}) \simeq T_X$
2. $\mathcal{I}m^2(\mathfrak{g}) \simeq \Omega_X$

For a simple algebraic group G we can make analogous definitions and calculations for bundles on homogeneous spaces associated with cominuscule parabolics by considering G-orbits of $\mathbb{E}(r, \mathfrak{g})$.

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For a simple algebraic group G we can make analogous definitions and calculations for bundles on homogeneous spaces associated with cominuscule parabolics by considering G-orbits of $\mathbb{E}(r, \mathfrak{g})$.
Challenge: Come up with interesting representations of constant radical/socle rank (e.g., constant Jordan type) to exhibit interesting "new" vector bundles on homogeneous spaces.

$\boxed{\textbf{THANK YOU}}$