Averaging t-structures

Original motivation: G finite group acts on algebra A (variety X)
$\Rightarrow$ G-actions on $D^b(A)$ or $D^b(X)$

Can more generally consider abstract G-actions on the categories.
Q: Are such actions abelian in nature?
I.e. induced by action on derived equivalent algebra or variety?

This is too hard, and answer is probably negative. Better question:
Q: Is there a G-invariant t-structure?

Reasonable to assume we start with a t-structure (X, Y). Then
Q: Can (X, Y) be averaged into an invariant t-structure?

Remark: G-invariant t-structure on triangulated category T induces t-structure on equivariant category T^G (always triangulated).

E.g. G acts on $X \Rightarrow \mathrm{coh}(X)^G = \mathrm{coh}[X/G]$ (quotient stack); special case $\mathbb{X}$ weighted projective line, then often $\mathbb{X} = [C/G]$ for some finite group action on a smooth, projective curve C.

Rephrasing: Given t-structures $(\mathsf{X}_1, \mathsf{Y}_1), \ldots, (\mathsf{X}_n, \mathsf{Y}_n)$
(get back group situation by $(g\mathsf{X}, g\mathsf{Y})$ with $g \in G$)

Q: Are the following two t-structures?

$(\langle \mathsf{X}_1, \ldots, \mathsf{X}_n \rangle, \mathsf{Y}_1 \cap \cdots \cap \mathsf{Y}_n)$

$(\mathsf{X}_1 \cap \cdots \cap \mathsf{X}_n, \langle \mathsf{Y}_1, \ldots, \mathsf{Y}_n \rangle)$

Equivalently:

- Is the extension closure of aisles an aisle?
- Is the intersection of aisles an aisle?

Easy answer: big categories

T *big* triangulated category (i.e. has arbitrary coproducts), e.g. $D(\mathrm{Mod}A)$ or $D(\mathrm{Qcoh}(X))$

t-structure (X, Y) on T *compactly generated* (c.g.) if there is a subset $C \subset \mathsf{T}^c$ with

$\mathsf{X} = \mathrm{Susp}(C)$ (suspensions, extensions, all coproducts, summands)

$\mathsf{Y} = \{t \in \mathsf{T} \mid \mathrm{Hom}^{\leq 0}(C, t) = 0\}$

Proposition: T big, $(\mathsf{X}_i, \mathsf{Y}_i)$ compactly generated

$\Rightarrow \langle \mathsf{X}_i \rangle_i$ and $\bigcap_i \mathsf{X}_i$ are aisles

$\langle \mathsf{Y}_i \rangle_i$ and $\bigcap_i \mathsf{Y}_i$ are co-aisles

So averaging is always possible in the big world!

Only non-trivial part: truncation triangles. Use **naive algorithm**:

Given $t \in \mathsf{T}$.
Want sequences $\mathsf{NX}(t)_k \to t \to \mathsf{NY}(t)_k$ with $\mathsf{NX}(t)_k \in \langle \mathsf{X}_i \rangle_i \ \forall k$.

Start ($k = 1$): truncation of t w.r.t. $(\mathsf{X}_1, \mathsf{Y}_1)$.

Recursion: Being with triangle for $k - 1$

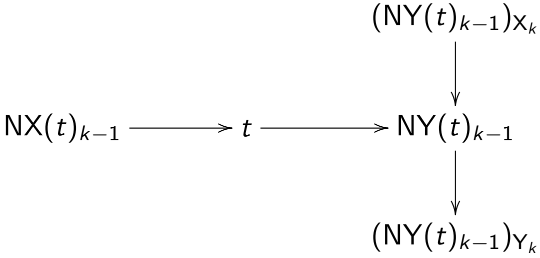
$$\mathsf{NX}(t)_{k-1} \longrightarrow t \longrightarrow \mathsf{NY}(t)_{k-1}$$

Only non-trivial part: truncation triangles. Use **naive algorithm**:

Given $t \in \mathsf{T}$.
Want sequences $\mathsf{NX}(t)_k \to t \to \mathsf{NY}(t)_k$ with $\mathsf{NX}(t)_k \in \langle \mathsf{X}_i \rangle_i \ \forall k$.

Start $(k = 1)$: truncation of t w.r.t. $(\mathsf{X}_1, \mathsf{Y}_1)$.

Recursion: Then truncate $\mathsf{NY}(t)_{k-1}$ w.r.t. $(\mathsf{X}_k, \mathsf{Y}_k$



Only non-trivial part: truncation triangles. Use **naive algorithm**:

Given $t \in \mathsf{T}$.
Want sequences $\mathsf{NX}(t)_k \to t \to \mathsf{NY}(t)_k$ with $\mathsf{NX}(t)_k \in \langle \mathsf{X}_i \rangle_i \ \forall k$.

Start ($k = 1$): truncation of t w.r.t. $(\mathsf{X}_1, \mathsf{Y}_1)$.

Recursion: Define $\mathsf{NY}(t)_k := (\mathsf{NY}(t)_{k-1})_{\mathsf{Y}_k}$

$$\begin{array}{ccccc}
 & & & & (\mathsf{NY}(t)_{k-1})_{\mathsf{X}_k} \\
 & & & & \downarrow \\
\mathsf{NX}(t)_{k-1} & \longrightarrow & t & \longrightarrow & \mathsf{NY}(t)_{k-1} \\
 & & \downarrow & & \downarrow \\
 & & \mathsf{NY}(t)_k & \overset{\text{def}}{=\!=\!=} & (\mathsf{NY}(t)_{k-1})_{\mathsf{Y}_k}
\end{array}$$

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Given $t \in \mathsf{T}$.
Want sequences $\mathsf{NX}(t)_k \to t \to \mathsf{NY}(t)_k$ with $\mathsf{NX}(t)_k \in \langle \mathsf{X}_i \rangle_i \ \forall k$.

Start ($k = 1$): truncation of t w.r.t. $(\mathsf{X}_1, \mathsf{Y}_1)$.

Recursion: Finally, complete octahedron diagram

$$\begin{array}{ccccc}
\mathsf{NX}(t)_{k-1} & \longrightarrow & \mathsf{NX}(t)_k & \longrightarrow & (\mathsf{NY}(t)_{k-1})_{\mathsf{X}_k} \\
\| & & \downarrow & & \downarrow \\
\mathsf{NX}(t)_{k-1} & \longrightarrow & t & \longrightarrow & \mathsf{NY}(t)_{k-1} \\
 & & \downarrow & & \downarrow \\
 & & \mathsf{NY}(t)_k & \overset{\text{def}}{=\!=} & (\mathsf{NY}(t)_{k-1})_{\mathsf{Y}_k}
\end{array}$$

Only non-trivial part: truncation triangles. Use **naive algorithm**:

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\mathsf{NX}(t)_{k-1} & \longrightarrow & t & \longrightarrow & \mathsf{NY}(t)_{k-1} \\
 & & \downarrow & & \downarrow \\
 & & \mathsf{NY}(t)_k & \overset{\text{def}}{=\!=} & (\mathsf{NY}(t)_{k-1})_{\mathsf{Y}_k}
\end{array}$$

$\mathsf{NX}(t)_k \in \langle \mathsf{X}_i \rangle_i$ from the top row.

$\rightsquigarrow$ chain

$$t \longrightarrow \mathsf{NY}(t)_1 \to \mathsf{NY}(t)_2 \to \mathsf{NY}(t)_3 \to \cdots$$

holim $\mathsf{NY}(t)_k$

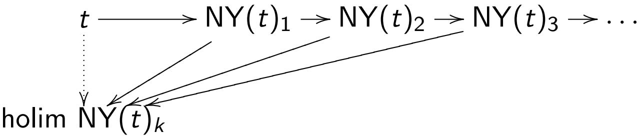
$\rightsquigarrow$ chain

$$
\begin{array}{ccccccccc}
t & \longrightarrow & \mathsf{NY}(t)_1 & \to & \mathsf{NY}(t)_2 & \to & \mathsf{NY}(t)_3 & \to & \cdots \\
& \swarrow & & \swarrow & & \swarrow & & & \\
\operatorname{holim} \mathsf{NY}(t)_k & & & & & & & &
\end{array}
$$

$\rightsquigarrow$ chain

$$\begin{tikzcd} t \arrow[r] \arrow[d, dotted] & \mathsf{NY}(t)_1 \arrow[r] \arrow[dl] & \mathsf{NY}(t)_2 \arrow[r] \arrow[dll] & \mathsf{NY}(t)_3 \arrow[r] \arrow[dlll] & \cdots \\ \operatorname{holim} \mathsf{NY}(t)_k & & & & \end{tikzcd}$$

$\rightsquigarrow$ chain



$\rightsquigarrow$ chain

$$\begin{array}{ccccccc} t & \longrightarrow & \mathrm{NY}(t)_1 & \to & \mathrm{NY}(t)_2 & \to & \mathrm{NY}(t)_3 \to \cdots \\ \vdots & \swarrow & & \swarrow & & \swarrow & \\ \text{holim } \mathrm{NY}(t)_k & & & & & & \end{array}$$

$y := \text{holim } \mathrm{NY}(t)_k \in \bigcap_i \mathrm{Y}_i$ (homotopy limit properties)

$\rightsquigarrow$ chain

$$\begin{array}{ccccccc} t & \longrightarrow & \mathsf{NY}(t)_1 & \to & \mathsf{NY}(t)_2 & \to & \mathsf{NY}(t)_3 \to \cdots \\ \vdots & \swarrow & & \swarrow & & \swarrow & \\ \mathrm{holim}\ \mathsf{NY}(t)_k & & & & & & \end{array}$$

$y := \mathrm{holim}\ \mathsf{NY}(t)_k \in \bigcap_i \mathsf{Y}_i$ (homotopy limit properties)

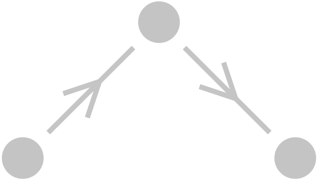
$x \to t \to y$ with $x \in \langle \mathsf{X}_i \rangle_i$ (general nonsense)

Subtleties: small categories

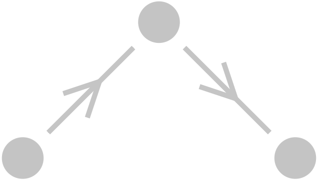
Example: Three stable t-structures on $D^b(\mathbb{C}A_2)$ s.t.

- naïve algorithm does not terminate
- $\langle \mathsf{X}_1, \mathsf{X}_2, \mathsf{X}_3 \rangle$ is an aisle.

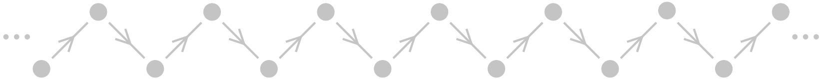
Auslander-Reiten quiver of the abelian category mod($\mathbb{C}A_2$):



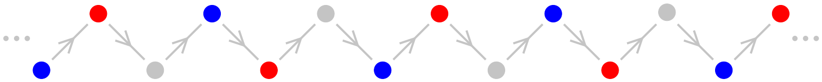
Auslander-Reiten quiver of the abelian category $\text{mod}(\mathbb{C}A_2)$:



Auslander-Reiten quiver of the derived category $D^b(\mathbb{C}A_2)$:



A stable t-structure (X, Y) on $\mathsf{T} := D^b(\mathbb{C}A_2)$:



X aisle

Y co-aisle

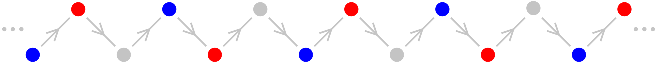
$\Sigma\mathsf{X} \subseteq \mathsf{X}$, $\Sigma^{-1}\mathsf{Y} \subseteq \mathsf{Y}$

$\mathrm{Hom}(\mathsf{X}, \mathsf{Y}) = 0$

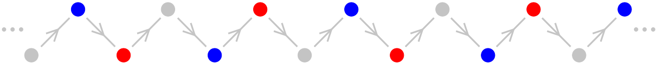
$\forall t \in \mathsf{T}\ \exists$ triangle $t_{\mathsf{X}} \to t \to t_{\mathsf{Y}} \to \Sigma t_{\mathsf{X}}$ with $t_{\mathsf{X}} \in \mathsf{X}$, $t_{\mathsf{Y}} \in \mathsf{Y}$

Three stable t-structures on $D^b(\mathbb{C}A_2)$:

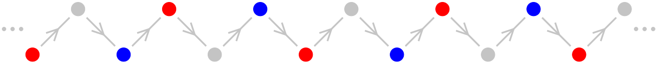
(X_1, Y_1):

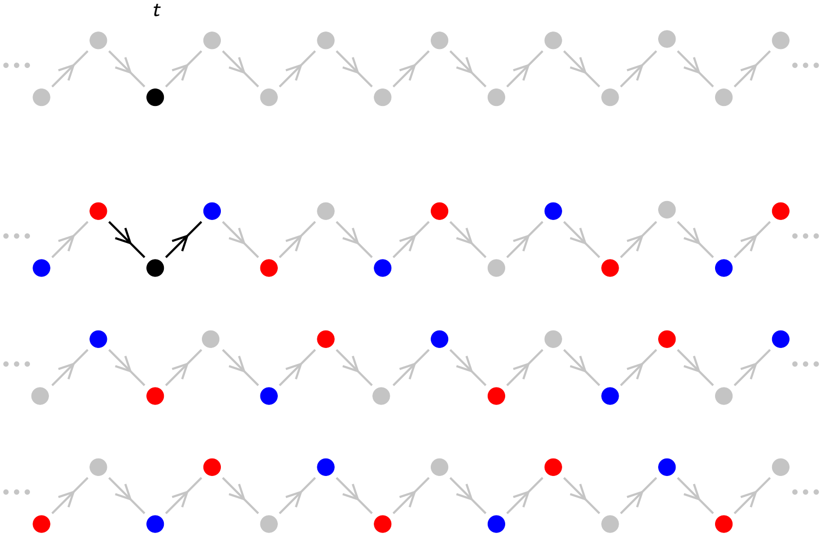


(X_2, Y_2):



(X_3, Y_3):

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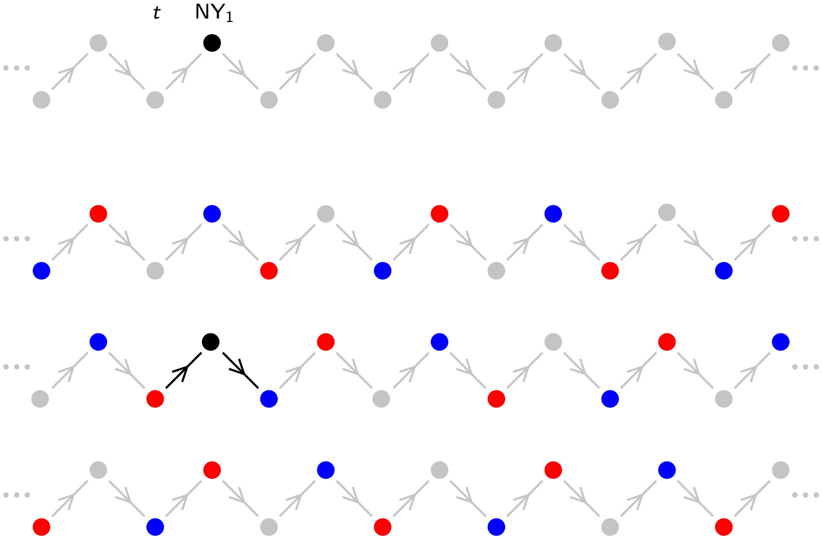

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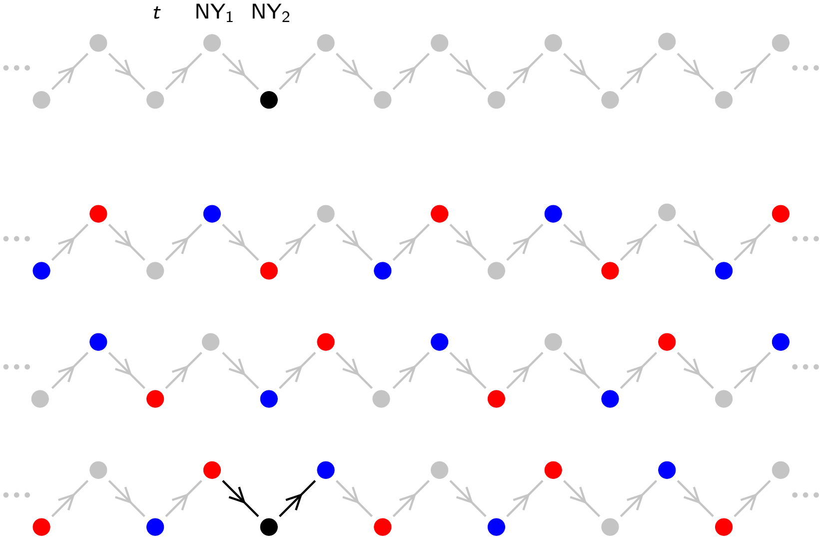
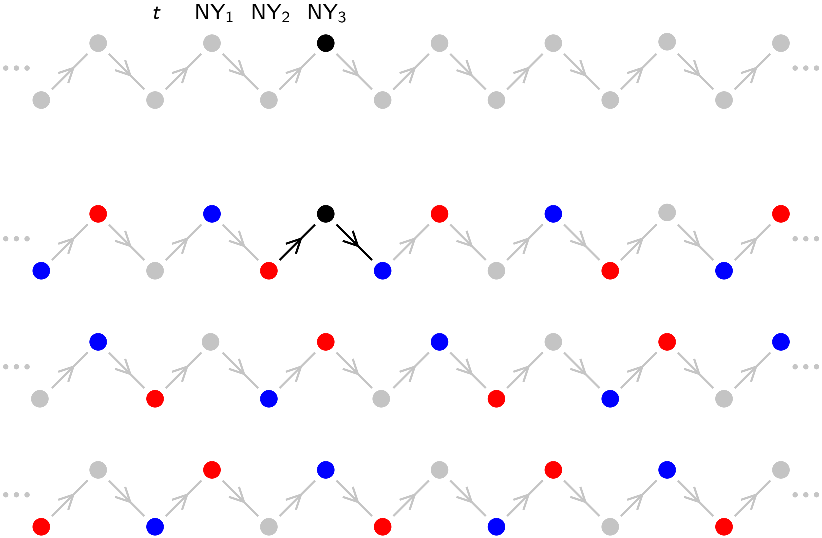
NY1



t
NY_1
NY_2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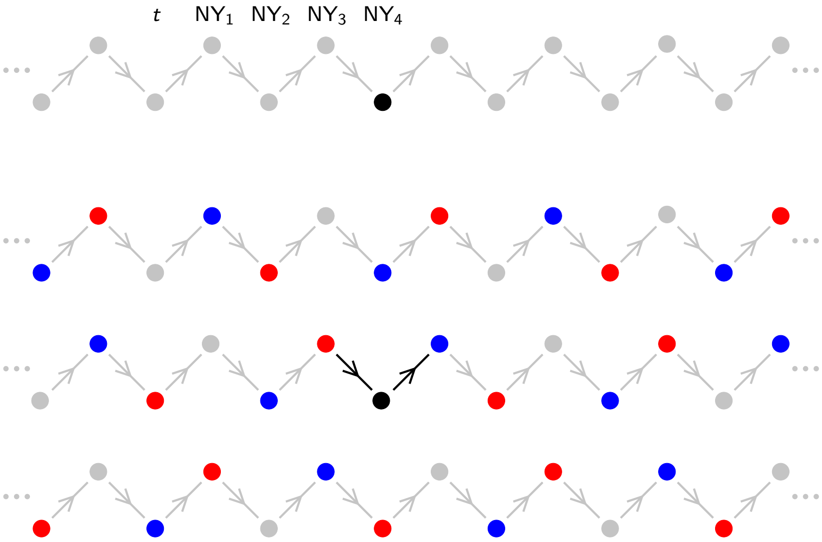
NY1
NY2
NY3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NY_2
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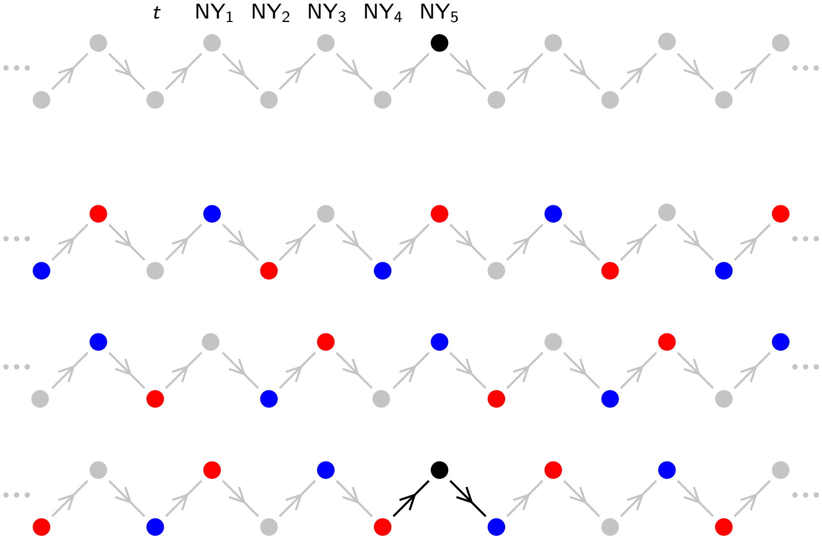
NY_4



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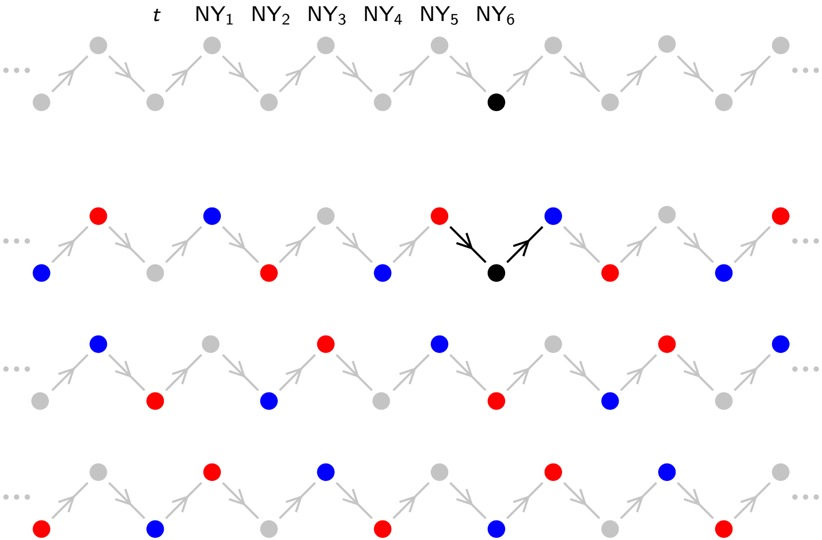
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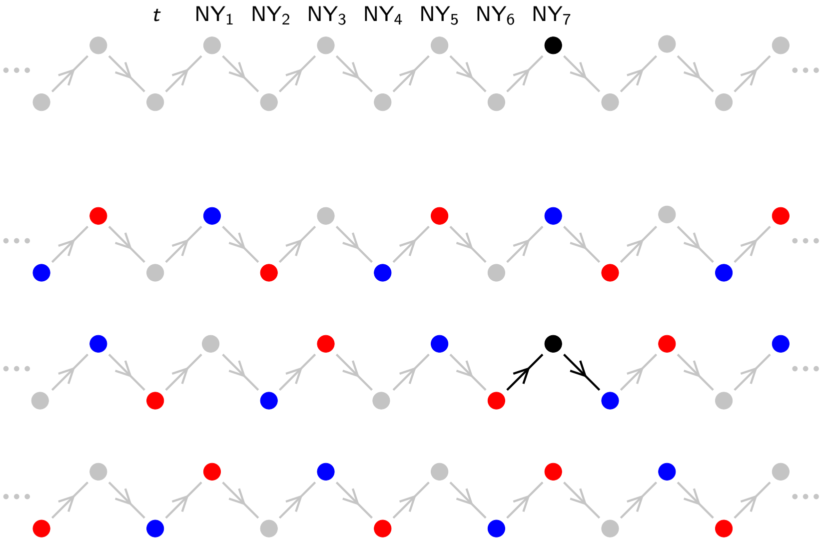

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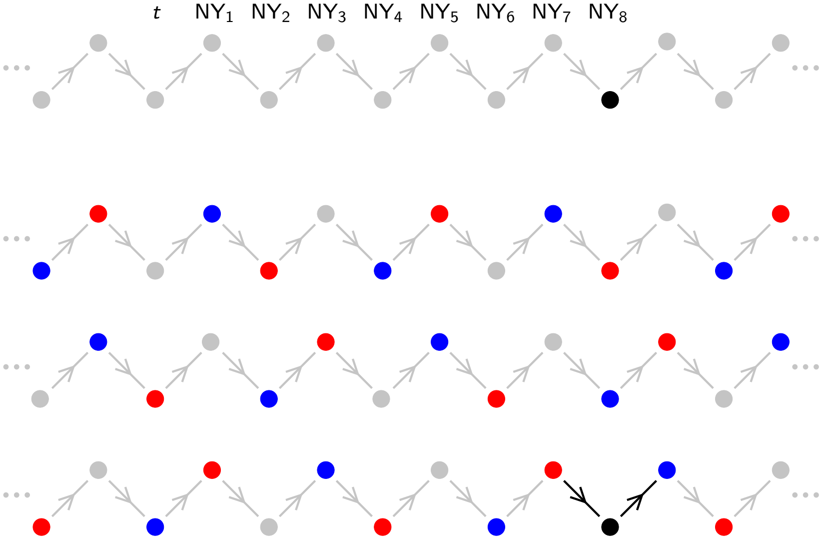

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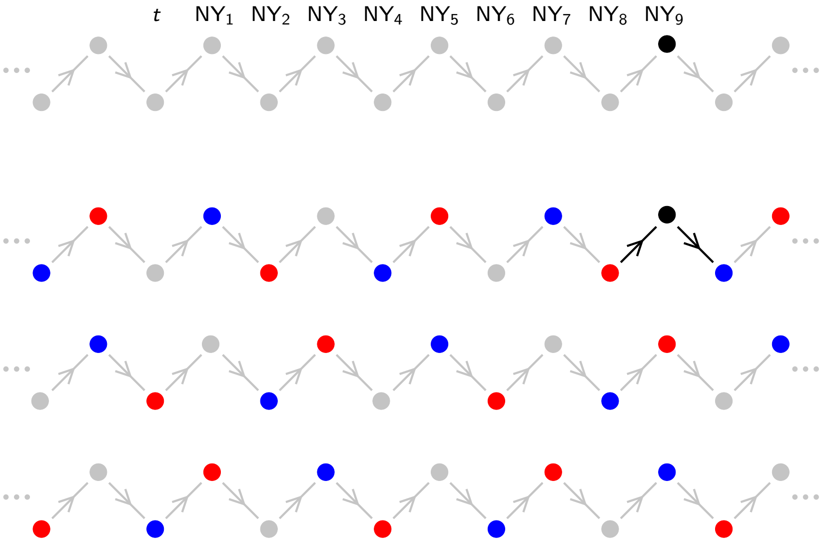
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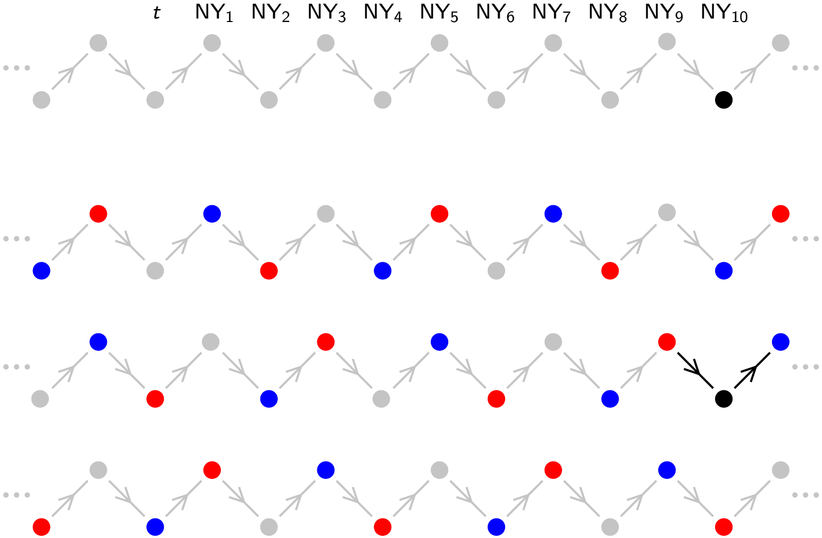

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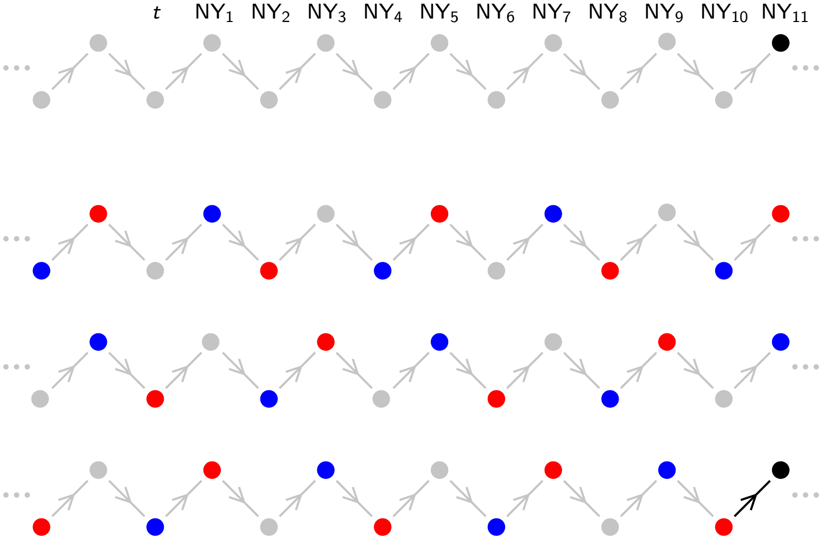

t
NY_1 NY_2 NY_3 NY_4 NY_5 NY_6 NY_7 NY_8 NY_9 NY_{10} NY_{11}

What went wrong: $\mathrm{NY}(t)_k \to \mathrm{NY}(t)_{k+2}$ zero map $\forall k$.
(holim $\mathrm{NY}(t)_k = 0$ in big category.)

Improvement: **Refined algorithm**. (Assume T Krull-Schmidt.)
Strip off superfluous summands of $\mathrm{NY}(t)_k$: want sequences
$\mathrm{RX}(t)_k \longrightarrow t \longrightarrow \mathrm{RY}(t)_k$ with

- $\mathrm{RX}(t)_k \in \langle \mathrm{X}_i \rangle_i$
- $t \to \mathrm{RY}(t)_k$ non-zero on all summands

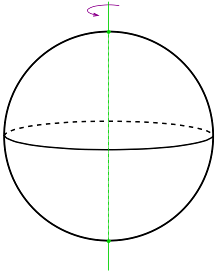
Theorem: refined algorithm terminates for all $t \in \mathrm{T}$
$\langle \mathrm{X}_i \rangle_i$ is an aisle.

Proposition: Refined algorithm terminates for piecewise hereditary T of finite representation type.

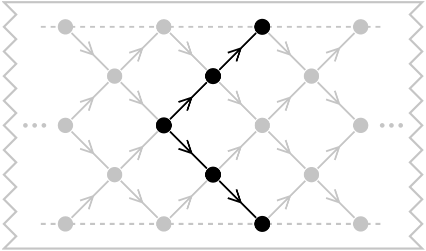
More subtleties

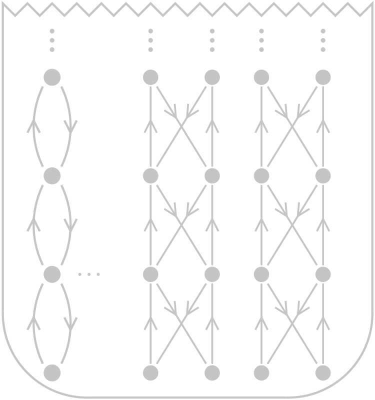
$$\mathsf{T} := D^b(\mathbb{C}\tilde{A}_{2,2}) \cong D^b(\mathbb{X}(2,2))$$
$$= D^b([\mathbb{P}^1/G]) = D^b(\mathbb{P}^1)^G$$

$G = \mathbb{Z}/2$ rotation by π on $S^2 = \mathbb{P}^1_{\mathbb{C}}$

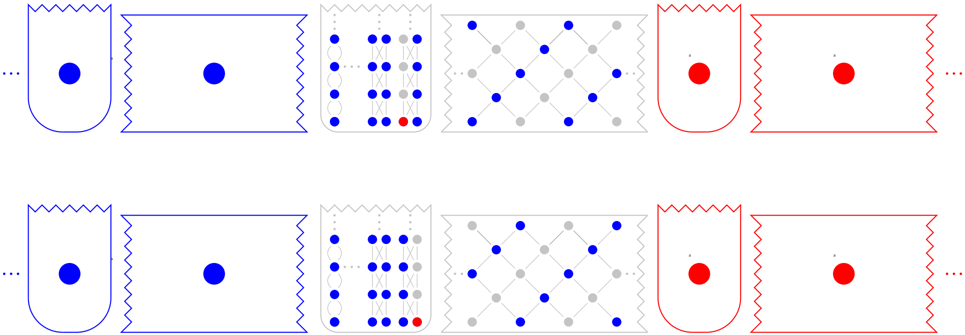


The Auslander-Reiten quiver of the abelian category $\mathrm{coh}(\mathbb{X}(2,2))$:

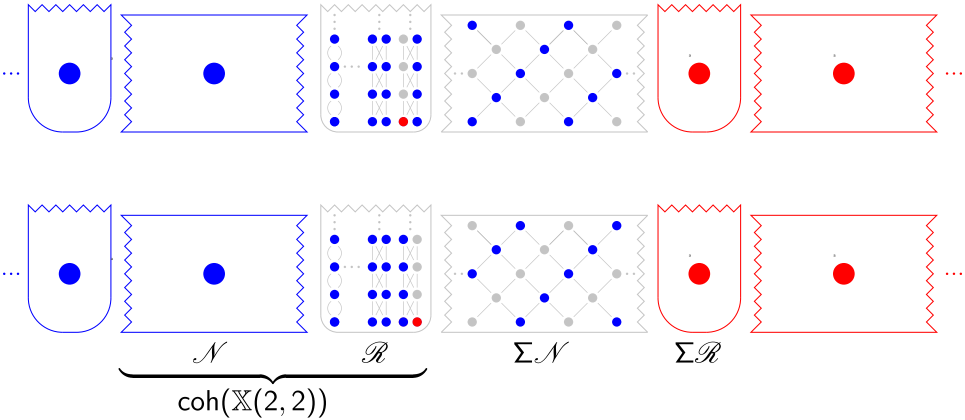




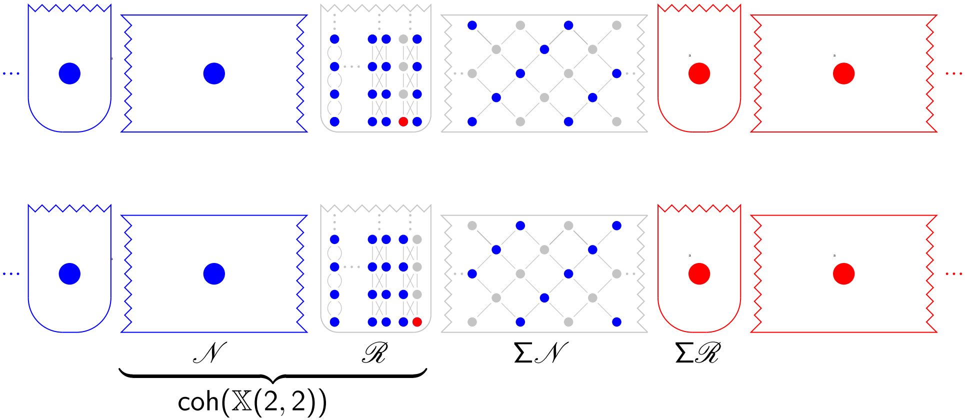
Two t-structures $(\color{red}{X_1}, \color{blue}{Y_1})$ and $(\color{red}{X_2}, \color{blue}{Y_2})$ on $D^b(\mathbb{X}(2,2))$:



Two t-structures $(\mathsf{X}_1, \mathsf{Y}_1)$ and $(\mathsf{X}_2, \mathsf{Y}_2)$ on $D^b(\mathbb{X}(2,2))$:



Two t-structures (X_1, Y_1) and (X_2, Y_2) on $D^b(\mathbb{X}(2,2))$:



$$Y_1 \cap Y_2 \cap \Sigma\mathscr{N} = 0$$
$$\langle X_1, X_2 \rangle \cap \Sigma\mathscr{N} = 0$$

Bad news:

- no truncation triangles for $t \in \Sigma\mathscr{N}$
- extension closure of the two aisles is not an aisle
- averaging is not possible!

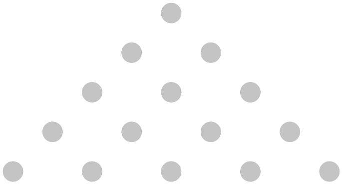
Theorem: T piecewise tame hereditary;
$(X_1, Y_1), \ldots, (X_1, Y_1)$ t-structures on T. Then

	(i)	$\langle X_1, \ldots, X_n \rangle$ is an aisle.
$\iff$	(ii)	Refined algorithm always terminates.
$\iff$	(iii)	For any non-regular component $\mathscr{N} \subset$ T with monotone sequences in all $Y_i \cap \mathscr{N}$, there is a monotone sequence in $Y_1 \cap \cdots \cap Y_n \cap \mathscr{N}$.

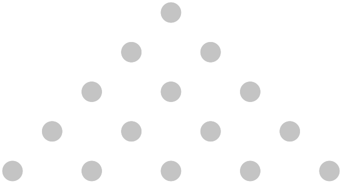
Because (iii) is a criterion only in non-regular components, one might expect the same result in the wild case.

Final example: averaging two t-structures in $D^b(\mathbb{C}A_5)$, and the induced t-structure on the equivariant category.

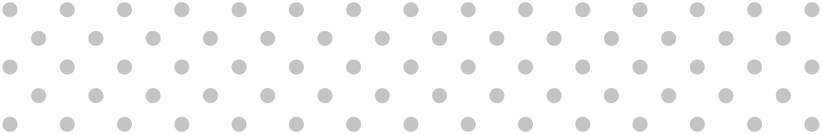
Auslander-Reiten quiver of the abelian category mod($\mathbb{C}A_5$):



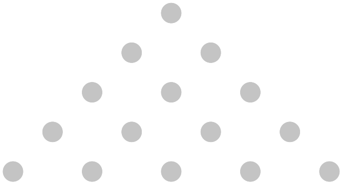
Auslander-Reiten quiver of the abelian category $\mathrm{mod}(\mathbb{C}A_5)$:



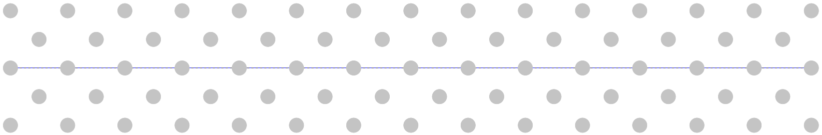
Auslander-Reiten quiver of the derived category $D^b(\mathbb{C}A_5)$:



Auslander-Reiten quiver of the abelian category mod($\mathbb{C}A_5$):

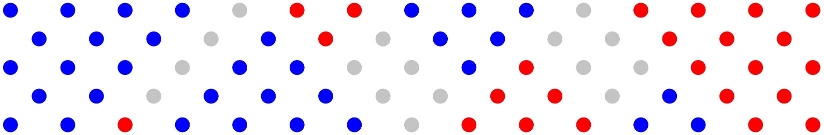


Auslander-Reiten quiver of the derived category $D^b(\mathbb{C}A_5)$:

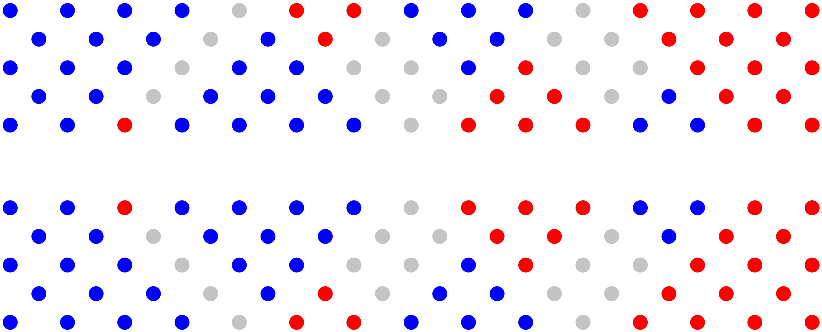


Involution $\Sigma^{-1}\iota$ of $D^b(\mathbb{C}A_5)$ where ι is the involution of A_5.

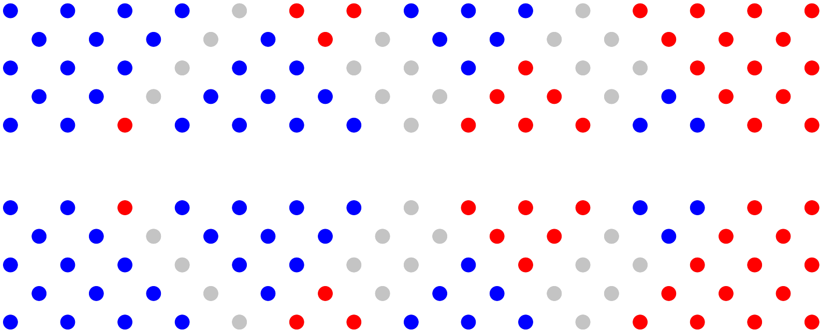
A t-structure (X, Y) on $D^b(\mathbb{C}A_5)$:



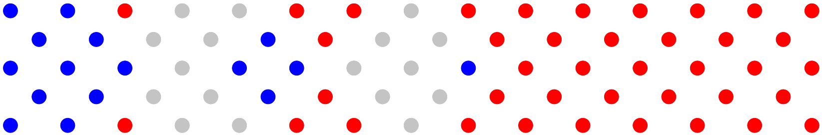
The t-structure and its image under the involution:



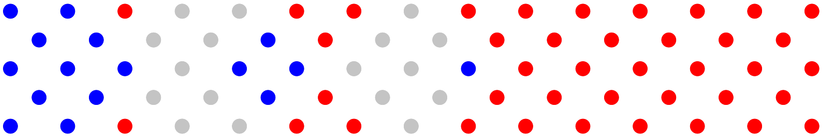
The t-structure and its image under the involution:



And the t-structure averaged by aisle extension:



The G-invariant t-structure on $\mathsf{T} := D^b(\mathbb{C}A_5)$:



and the induced t-structure on $\mathsf{T}^G = D^b(\mathbb{C}D_4)$: