Superdecomposable pure-injective modules over tubular algebras

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I report some results from the doctoral thesis [2] of my student Richard Harland. A preprint is in preparation.

Suppose that A is a tubular algebra and let r be a positive irrational. It has been known for a long time that the A-modules of slope r form a nonzero definable category, $\mathcal{D}_r$, of A-Mod; in particular there are indecomposable pure-injective modules of slope r. Until [2] however, nothing seems to have been known about the complexity of $\mathcal{D}_r$. Harland proved that the width, in the sense of Ziegler [10], of the lattice of pp formulas for $\mathcal{D}_r$ is undefined.

Theorem 0.1. *([2, Thm. 34]) Let A be a tubular algebra and let r be a positive irrational. Then the width of the lattice of pp formulas for the definable category $\mathcal{D}_r$, of modules of slope r, is undefined. If A is countable then there is a superdecomposable pure-injective module of slope r.*

The lattice of pp formulas is naturally isomorphic to that of pointed finitely presented modules, that is pairs (M, m) with M finite-dimensional and $m \in M$ under the (pre)order $(M, m) \geq (N, n)$ iff there is $f : M \to N$ with $f(m) = n$.

Over a finite-dimensional algebra a module is **pure-injective** if it is a direct summand of a direct product of finite-dimensional modules and a module is **superdecomposable** if it is nonzero and has no indecomposable direct summands.

For some time it had seemed that width being undefined (and the consequent existence, if the algebra is countable, of a superdecomposable pure-injective) might be an indication of wildness of the category of finite-dimensional modules but this turned out to be false: Puninski showed ([6], also see [7]) that the modules over any non-domestic string (so tame) algebra do have this degree of complexity. Harland's result shows this for another class of tame algebras.

The relation between the dimension and existence of superdecomposable pure-injectives is a result of Ziegler.

Theorem 0.2. *[10, 7.1] If $\mathcal{D}$ is a definable subcategory of* A-Mod *and if there is a superdecomposable pure-injective in $\mathcal{D}$ then the width of the lattice of pp formulas for $\mathcal{D}$ is undefined. The converse holds if A (or just the lattice of pp formulas for $\mathcal{D}$) is countable. [The converse in the case where A is uncountable is open.]*

Here width is a dimension defined in [10]; it takes values which are ordinals or ∞ - that is, undefined. We can give a quick definition of a dimension, breadth, which is essentially equivalent in the sense that, though the two dimensions may have somewhat different values on a particular lattice L, the one is undefined on L iff the other is (see [4, §10.2] or [5, §7.3.1]). The breadth of a modular lattice is defined by (transfinitely) inductively collapsing all intervals which are totally ordered. The value is the number of times this is done before we reach the trivial lattice, or is ∞ if at some stage we reach a nontrivial quotient lattice without any totally ordered intervals. It follows that if width is undefined then so is the Krull-Gabriel dimension - the dimension which is obtained by (transfinitely) inductively collapsing intervals which are of finite length, see [1], [3, p. 197ff.], [5, §13.2.1]) and, again in the countable case and by a result [10, 8.3] of Ziegler, it follows from Harland's theorem that there are $2^{\aleph_0}$ indecomposable pure-injective modules of slope r (we recall that Reiten and Ringel, [8, 13.1], showed that every indecomposable module over a tubular algebra has a slope).

The proof in [2] takes place mostly in the category of finitely presented modules, even though all nonzero modules in $\mathcal{D}_r$ are infinite-dimensional. This is in part because the dimension being measured can be expressed completely in terms of the category of (pointed) finite-dimensional modules but, also, a theme that runs right through the proof is the way that finite-dimensional modules approximate the modules with irrational slope r: what happens *at* r is strongly determined by what happens *near* r. For example, there is the following result.

Theorem 0.3. *[2, Thm. 30] Let ϕ/ψ be a pp-pair and let r be a positive irrational. Then the following are equivalent:*
(i) ϕ/ψ is closed near the left of r;
(ii) ϕ/ψ is closed near the right of r;
(iii) ϕ/ψ is closed at r.

By "closed at r" we mean closed on every module of slope r; by "(-) closed near the left of r" we mean that for every $\epsilon > 0$ there is a finite-dimensional module with slope in $(r-\epsilon, r)$ on which (-) is closed. Pp-pairs, which may be thought of as referring to intervals in the lattice of pp formulas, come from the model theory of modules, which provides techniques used throughout the proof, but direct reformulations can be made, as follows.

Theorem 0.4. *[2, Thm. 30] Let F be a functor from* A-mod *to* **Ab** *which is finitely presented and let r be a positive irrational. Then the following are equivalent:*
(i) F is closed near the left of r;
(ii) F is closed near the right of r;
(iii) $\overrightarrow{F}$ is closed at r where $\overrightarrow{F}$ denotes that extension of F to arbitrary modules which commutes with direct limits.

To reformulate purely in terms of finite-dimensional modules, we introduce some brief terminology. Suppose that $f : M \to N$ is a morphism of finite-dimensional modules, let $m \in M$ and set $n = f(m)$. For any module X say that the pair $(M, m)/(N, n)$ of pointed modules is closed on X if for every morphism $g : M \to X$ there is a morphism $g' : N \to X$ such that $g(m) = g'(n)$.

Theorem 0.5. *[2, Thm. 30] Let $f : (M, m) \to (N, n)$ be a morphism of pointed finite-dimensional modules and let r be a positive irrational. Then the following are equivalent:*

(i) $(M,m)/(N,n)$ *is closed near the left of* r*;*
(ii) $(M,m)/(N,n)$ *is closed near the right of* r*;*
(iii) $(M,m)/(N,n)$ *is closed at* r*.*

A good deal of work in the proof is involved in strengthening this, from cofinality to uniformity. The definition of a pp-pair being closed near the left of r is compatible with that pp-pair also being open on some modules with slope arbitrarily close to, and less than, r; it is shown that this does not happen. This, and the width being undefined, are shown first for some particular tubular algebras (the algebras $C(4,\lambda)$, $C(6)$, $C(7)$, $C(8)$ which are used by Ringel in [9, §5.6]), then certain tilting ("shrinking") functors are used to transfer results to the general tubular case.

References

[1] W. Geigle, The Krull-Gabriel dimension of the representation theory of a tame hereditary artin algebra and applications to the structure of exact sequences, Manus. Math., 54(1-2) (1985), 83-106.

[2] R. Harland, Pure-injective Modules over Tubular Algebras and String Algebras, Doctoral Thesis, University of Manchester, 2011, *available at* www.maths.manchester.ac.uk/∼mprest/publications.html

[3] C. U. Jensen and H. Lenzing, Model Theoretic Algebra; with particular emphasis on Fields, Ring and Modules, Gordon and Breach, 1989.

[4] M. Prest, Model Theory and Modules, London Math. Soc. Lect. Note Ser., Vol. 130, Cambridge University Press, 1988.

[5] M. Prest, Purity, Spectra and Localisation, Encyclopedia of Mathematics and its Applications, Vol. 121, Cambridge University Press, 2009.

[6] G. Puninski, Superdecomposable pure-injective modules exist over some string algebras, Proc. Amer. Math. Soc., 132(7) (2004), 1891-1898.

[7] G. Puninski, How to construct a 'concrete' superdecomposable pure-injective module over a string algebra, J. Algebra, 212(4) (2008), 704-717.

[8] I. Reiten and C. M. Ringel, Infinite dimensional representations of canonical algebras, Canad. J. Math., 58(1) (2006), 180-224.

[9] C. M. Ringel, Tame Algebras and Integral Quadratic Forms, Lecture Notes in Mathematics, Vol. 1099, Springer-Verlag, 1984.

[10] M. Ziegler, Model theory of modules, Ann. Pure Appl. Logic, 26(2) (1984), 149-213.