Homological Theory of Recollements of Abelian Categories

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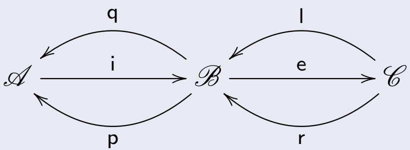
Definition

A recollement situation between abelian categories $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ is a diagram



$\mathsf{R_{ab}}(\mathscr{A}, \mathscr{B}, \mathscr{C})$

satisfying the following conditions:

1. $(\mathsf{l}, \mathsf{e}, \mathsf{r})$ is an adjoint triple.
2. $(\mathsf{q}, \mathsf{i}, \mathsf{p})$ is an adjoint triple.
3. The functors i, l, and r are fully faithful.
4. $\operatorname{Im} \mathsf{i} = \operatorname{Ker} \mathsf{e}$.

Properties of $\mathsf{R}_{\mathsf{ab}}(\mathscr{A},\mathscr{B},\mathscr{C})$

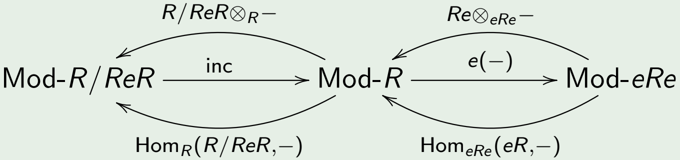
1. The functors $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ and $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ are exact.
2. The composition of functors $\mathsf{ql} = \mathsf{pr} = 0$.
3. $\mathsf{qi} \xrightarrow{\simeq} \mathsf{Id}_{\mathscr{A}}$, $\mathsf{Id}_{\mathscr{A}} \xrightarrow{\simeq} \mathsf{pi}$, $\mathsf{er} \xrightarrow{\simeq} \mathsf{Id}_{\mathscr{C}}$ and $\mathsf{Id}_{\mathscr{C}} \xrightarrow{\simeq} \mathsf{el}$.
4. The functor i induces an equivalence between $\mathscr{A}$ and the Serre subcategory $\operatorname{Ker}\mathsf{e} = \operatorname{Im}\mathsf{i}$ of $\mathscr{B}$.
5. $\mathscr{A}$ is a localizing and colocalizing subcategory of $\mathscr{B}$ and there is an equivalence $\mathscr{B}/\mathscr{A} \simeq \mathscr{C}$.
6. For every $B \in \mathscr{B}$ we have exact sequences:

$$0 \longrightarrow \operatorname{Ker}\mu_B \longrightarrow \mathsf{le}(B) \xrightarrow{\mu_B} B \xrightarrow{\lambda_B} \mathsf{iq}(B) \longrightarrow 0$$

$$0 \longrightarrow \mathsf{ip}(B) \xrightarrow{\kappa_B} B \xrightarrow{\nu_B} \mathsf{re}(B) \longrightarrow \operatorname{Coker}\nu_B \longrightarrow 0$$

Example: (One idempotent)

Let R be a ring and $e^2 = e \in R$ an idempotent. Then we have the recollement:



Example: (Generalized Matrix Rings)

Let R, S be rings, M a S-R-bimodule and N a R-S-bimodule. Let $\phi\colon M \otimes_R N \longrightarrow S$ be a S-S-bimodule homomorphism and let $\psi\colon N \otimes_S M \longrightarrow R$ be a R-R-bimodule homomorphism. Then the above data allow us to define the **generalized matrix ring**:

$$\Lambda_{(\phi,\psi)} = \begin{pmatrix} R & {}_RN_S \\ {}_SM_R & S \end{pmatrix}$$

where the multiplication is given by

$$\begin{pmatrix} r & n \\ m & s \end{pmatrix} \cdot \begin{pmatrix} r' & n' \\ m' & s' \end{pmatrix} = \begin{pmatrix} rr' + \psi(n \otimes m') & rn' + ns' \\ mr' + sm' & ss' + \phi(m \otimes n') \end{pmatrix}$$

$e_1 = \left(\begin{smallmatrix} 1_R & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_S \end{smallmatrix}\right)$ idempotents elements of $\Lambda_{(\phi,\psi)}$. Then:

Example: (Generalized Matrix Rings)

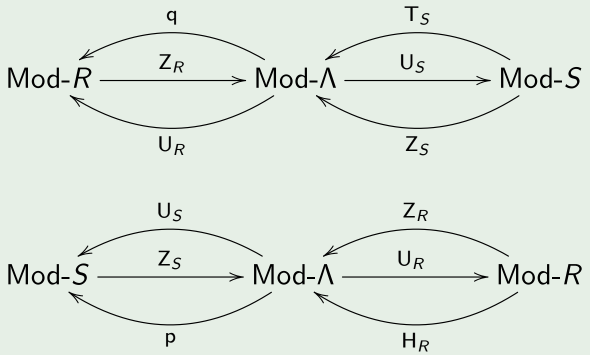
$$\xymatrix{\text{Mod-}\Lambda/\Lambda e_1\Lambda \ar[rr]^{\text{inc}} && \text{Mod-}\Lambda_{(\phi,\psi)} \ar@/_2pc/[ll]_{\Lambda/\Lambda e_1\Lambda\otimes_\Lambda -} \ar@/^2pc/[ll]^{\mathrm{Hom}_\Lambda(\Lambda/\Lambda e_1\Lambda,-)} \ar[rr]^{e_1(-)} && \text{Mod-}e_1\Lambda e_1 \ar@/_2pc/[ll]_{\Lambda e_1\otimes_{e_1\Lambda e_1}-} \ar@/^2pc/[ll]^{\mathrm{Hom}_{e_1\Lambda e_1}(e_1\Lambda,-)}}$$

$$\xymatrix{\text{Mod-}\Lambda/\Lambda e_2\Lambda \ar[rr]^{\text{inc}} && \text{Mod-}\Lambda_{(\phi,\psi)} \ar@/_2pc/[ll]_{\Lambda/\Lambda e_2\Lambda\otimes_\Lambda -} \ar@/^2pc/[ll]^{\mathrm{Hom}_\Lambda(\Lambda/\Lambda e_2\Lambda,-)} \ar[rr]^{e_2(-)} && \text{Mod-}e_2\Lambda e_2 \ar@/_2pc/[ll]_{\Lambda e_2\otimes_{e_2\Lambda e_2}-} \ar@/^2pc/[ll]^{\mathrm{Hom}_{e_2\Lambda e_2}(e_2\Lambda,-)}}$$

- $\text{Mod-}\Lambda/\Lambda e_1\Lambda \simeq \text{Mod-}S/\operatorname{Im}\phi$, $\text{Mod-}e_1\Lambda e_1 \simeq \text{Mod-}R$
- $\text{Mod-}\Lambda/\Lambda e_2\Lambda \simeq \text{Mod-}R/\operatorname{Im}\psi$, $\text{Mod-}S \simeq \text{Mod-}e_2\Lambda e_2$

Example: (Symmetric Recollement)

Let R, S rings and ${}_RN_S$ a bimodule. Then we have the triangular matrix ring $\Lambda = \begin{pmatrix} R & {}_RN_S \\ 0 & S \end{pmatrix}$ and the following recollements:



Let as before $\mathsf{R}_{\mathsf{ab}}(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories. Since the functors $\mathsf{i} \colon \mathscr{A} \longrightarrow \mathscr{B}$ and $\mathsf{e} \colon \mathscr{B} \longrightarrow \mathscr{C}$ are exact, they induce natural maps:

$$\mathsf{i}^n_{X,Y} \colon \operatorname{Ext}^n_{\mathscr{A}}(X, Y) \longrightarrow \operatorname{Ext}^n_{\mathscr{B}}(\mathsf{i}(X), \mathsf{i}(Y))$$

and

$$\mathsf{e}^n_{Z,W} \colon \operatorname{Ext}^n_{\mathscr{B}}(Z, W) \longrightarrow \operatorname{Ext}^n_{\mathscr{C}}(\mathsf{e}(Z), \mathsf{e}(W))$$

Let as before $\mathrm{R_{ab}}(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories. Since the functors $\mathrm{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ and $\mathrm{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ are exact, they induce natural maps:

$$\mathrm{i}^n_{X,Y}\colon\ \mathrm{Ext}^n_{\mathscr{A}}(X, Y) \longrightarrow \mathrm{Ext}^n_{\mathscr{B}}(\mathrm{i}(X), \mathrm{i}(Y))$$

and

$$\mathrm{e}^n_{Z,W}\colon\ \mathrm{Ext}^n_{\mathscr{B}}(Z, W) \longrightarrow \mathrm{Ext}^n_{\mathscr{C}}(\mathrm{e}(Z), \mathrm{e}(W))$$

Problem

- Find necessary and sufficient conditions such that the induced homomorphisms $\mathrm{i}^n_{X,Y}$ and $\mathrm{e}^n_{Z,W}$ are isomorphisms for $0 \le n \le k$.

Let as before $\mathsf{R}_{\mathrm{ab}}(\mathscr{A}, \mathscr{B}, \mathscr{C})$ be a recollement of abelian categories. Since the functors $\mathsf{i}\colon \mathscr{A} \longrightarrow \mathscr{B}$ and $\mathsf{e}\colon \mathscr{B} \longrightarrow \mathscr{C}$ are exact, they induce natural maps:

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Problem

- Find necessary and sufficient conditions such that the induced homomorphisms $\mathsf{i}^n_{X,Y}$ and $\mathsf{e}^n_{Z,W}$ are isomorphisms for $0 \leq n \leq k$.
- Relate (if possible) the global/finitistic dimension of the categories involved in $\mathsf{R}_{\mathrm{ab}}(\mathscr{A}, \mathscr{B}, \mathscr{C})$.

For $0 \leq n \leq \infty$ we define the following full subcategories of $\mathscr{B}$:

- $\mathcal{X}_n = \{B \in \mathscr{B} \mid \exists \ \mathsf{l}(P_n) \longrightarrow \cdots \longrightarrow \mathsf{l}(P_0) \longrightarrow B \longrightarrow 0$ exact where $P_i \in \operatorname{Proj}\mathscr{C}, \ 0 \leq i \leq n\}$
- $\mathcal{Y}_n = \{B \in \mathscr{B} \mid \exists \ 0 \longrightarrow B \longrightarrow \mathsf{r}(I_0) \longrightarrow \cdots \longrightarrow \mathsf{r}(I_n)$ exact where $I_i \in \operatorname{Inj}\mathscr{C}, \ 0 \leq i \leq n\}$

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Note that $\mathsf{l}(P_i) \in \operatorname{Proj}\mathscr{B}$ and $\mathsf{r}(I_i) \in \operatorname{Inj}\mathscr{B}$.

Theorem

Let $\mathrm{R_{ab}}(\mathscr{A}, \mathscr{B}, \mathscr{C})$. Then the following statements are equivalent:

1. The map $\mathrm{i}^n_{X,Y}\colon \operatorname{Ext}^n_{\mathscr{A}}(X, Y) \longrightarrow \operatorname{Ext}^n_{\mathscr{B}}(\mathrm{i}(X), \mathrm{i}(Y))$ is an isomorphism, $\forall X, Y \in \mathscr{A}$ and $0 \leq n \leq k$.
2. $\operatorname{Im} \mu_P \in \mathfrak{X}_{k-1}$, $\forall P \in \operatorname{Proj} \mathscr{B}$, where $\mu\colon \mathrm{le} \longrightarrow \mathrm{Id}_{\mathscr{B}}$.
3. $\operatorname{Im} \nu_I \in \mathfrak{Y}_{k-1}$, $\forall I \in \operatorname{Inj} \mathscr{B}$, where $\nu\colon \mathrm{Id}_{\mathscr{B}} \longrightarrow \mathrm{re}$.

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Theorem

Let $\mathsf{R}_{\mathsf{ab}}(\mathscr{A}, \mathscr{B}, \mathscr{C})$. Then the following statements are equivalent:

1. The map $\mathsf{e}^n_{Z,W}\colon \operatorname{Ext}^n_{\mathscr{B}}(Z, W) \longrightarrow \operatorname{Ext}^n_{\mathscr{C}}(\mathsf{e}(Z), \mathsf{e}(W))$ is an isomorphism, $\forall\, W \in \mathscr{B}$, (resp. $\forall\, Z \in \mathscr{B}$), and $0 \leq n \leq k$.
2. $Z \in \mathfrak{X}_{k+1}$ (resp. $W \in \mathfrak{Y}_{k+1}$).

Theorem

Let $\mathsf{R}_{\mathsf{ab}}(\mathscr{A}, \mathscr{B}, \mathscr{C})$.

1. We have:

$$\operatorname{gl.dim} \mathscr{B} \leq \operatorname{gl.dim} \mathscr{A} + \operatorname{gl.dim} \mathscr{C}$$
$$+ \sup\{\operatorname{pd}_{\mathscr{B}} \mathsf{i}(P) \mid P \in \operatorname{Proj} \mathscr{A}\} + 1$$

2. If $\operatorname{gl.dim} \mathscr{B} \ \leq \ 1$ then:

$$\operatorname{gl.dim} \mathscr{A} \ \leq \ 1 \quad \text{and} \quad \operatorname{gl.dim} \mathscr{C} \ \leq \ 1$$

3. If $\sup\{\operatorname{pd}_{\mathscr{B}} \mathsf{i}(P) \mid P \in \operatorname{Proj} \mathscr{A}\} \leq 1$, then the following are equivalent:
    1. $\operatorname{gl.dim} \mathscr{B} < \infty$.
    2. $\operatorname{gl.dim} \mathscr{A} < \infty$ and $\operatorname{gl.dim} \mathscr{C} < \infty$.

Let $\mathsf{F}\colon \mathscr{D} \longrightarrow \mathcal{G}$ be a right exact functor between abelian categories where we assume that $\mathscr{D}$ has enough projectives.

We say that F has locally bounded homological dimension, if there exists $n \geq 0$ such that whenever $\mathsf{L}_m\mathsf{F}(A) = 0$ for $m \gg 0$ then $\mathsf{L}_m\mathsf{F}(A) = 0$ for every $m \geq n+1$.

The minimum such n (if it exists) is called the **locally bounded homological dimension** of F and is denoted by $\mathrm{l.b.hom.dim}\,\mathsf{F}$.

Theorem

Let $\mathsf{R}_{\mathsf{ab}}(\mathscr{A}, \mathscr{B}, \mathscr{C})$.

1. If the functor $\mathsf{l} \colon \mathscr{C} \longrightarrow \mathscr{B}$ has locally bounded homological dimension, then:

$$\mathsf{FPD}\,(\mathscr{C}) \ \leq \ \mathsf{FPD}\,(\mathscr{B}) + \mathsf{l.b.hom.dim}\,\mathsf{l}$$

Theorem

Let $\mathsf{R}_{\mathrm{ab}}(\mathscr{A}, \mathscr{B}, \mathscr{C})$.

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2. If the functors $\mathsf{r} \colon \mathscr{C} \longrightarrow \mathscr{B}$ and $\mathsf{p} \colon \mathscr{B} \longrightarrow \mathscr{A}$ are exact, then:

$$\mathsf{FPD}(\mathscr{A}) \ \leq \ \mathsf{FPD}(\mathscr{B}) \ \leq \ \mathsf{FPD}(\mathscr{A}) + \mathsf{FPD}(\mathscr{C}) + 1$$

Theorem

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$$\mathsf{FPD}(\mathscr{A}) \ \leq \ \mathsf{FPD}(\mathscr{B}) \ \leq \ \mathsf{FPD}(\mathscr{A}) + \mathsf{FPD}(\mathscr{C}) + 1$$

Corollary

Let R be a ring and $e^2 = e \in R$. If the functor $Re \otimes_{eRe} -$ has locally bounded homological dimension then:

$$\mathsf{Fin.\,dim}\, eRe \ \leq \ \mathsf{Fin.\,dim}\, R + \mathsf{l.b.hom.dim}\, Re \otimes_{eRe} -$$

Corollary

Let Λ be an Artin algebra with $\operatorname{rep.dim}\Lambda \leq 3$ and e an idempotent element of Λ. Then:

1. $\operatorname{rep.dim} e\Lambda e \leq 3$.
2. If the Λ-module $\Lambda/\Lambda e\Lambda$ is projective, then:

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Corollary

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Corollary

Let Λ be an Artin algebra. Then:

$$\operatorname{rep.dim} \Lambda \leq 3 \quad \Longleftrightarrow \quad \operatorname{rep.dim} \operatorname{End}_{\Lambda}(P) \leq 3$$

for any finitely generated projective Λ-module P.

Corollary

Let Λ be an Artin algebra with $\operatorname{rep.dim} \Lambda \leq 3$ and $e^2 = e \in \Lambda$. Then:

$$\operatorname{fin.dim} e\Lambda e < \infty$$

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Proof.

$$\operatorname{rep.dim} \Lambda \leq 3 \implies \operatorname{rep.dim} e\Lambda e \leq 3 \xrightarrow[\text{Todorov}]{\text{Igusa}} \operatorname{fin.dim} e\Lambda e < \infty$$

□

Corollary

Let Λ be an Artin algebra and $\Gamma = \mathrm{End}_\Lambda(\Lambda \oplus \mathrm{D}\,\Lambda)$. Then:

$$\mathrm{rep.\,dim}\,\Lambda \;\leq\; \mathrm{gl.\,dim}\,\Lambda + \mathrm{gl.\,dim}_{\Gamma/\Gamma e_\Lambda \Gamma}\Gamma + 1$$

and

$$\mathrm{gl.\,dim}_{\Gamma/\Gamma e_\Lambda \Gamma}\Gamma \;\leq\; \mathrm{gl.\,dim}\,\Gamma/\Gamma e_\Lambda\Gamma + \mathrm{pd}_\Gamma\,\Gamma/\Gamma e_\Lambda\Gamma$$

where $\mathrm{gl.\,dim}_{\Gamma/\Gamma e_\Lambda \Gamma}\Gamma = \sup\{\mathrm{pd}_\Gamma X \mid X \in \text{mod-}\Gamma/\Gamma e_\Lambda\Gamma\}$.

Corollary

Let Λ be an Artin algebra and $\Gamma = \operatorname{End}_\Lambda(\Lambda \oplus \mathrm{D}\,\Lambda)$. Then:

$$\operatorname{rep.dim}\Lambda \;\leq\; \operatorname{gl.dim}\Lambda + \operatorname{gl.dim}_{\Gamma/\Gamma e_\Lambda \Gamma}\Gamma + 1$$

and

$$\operatorname{gl.dim}_{\Gamma/\Gamma e_\Lambda \Gamma}\Gamma \;\leq\; \operatorname{gl.dim}\Gamma/\Gamma e_\Lambda \Gamma + \operatorname{pd}_\Gamma \Gamma/\Gamma e_\Lambda \Gamma$$

where $\operatorname{gl.dim}_{\Gamma/\Gamma e_\Lambda \Gamma}\Gamma = \sup\{\operatorname{pd}_\Gamma X \mid X \in \text{mod-}\Gamma/\Gamma e_\Lambda \Gamma\}$.

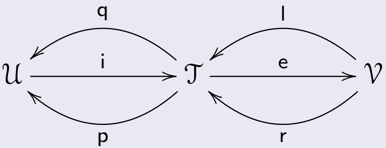
Proof.

1. $\Gamma = \operatorname{End}_\Lambda(\Lambda \oplus \mathrm{D}\,\Lambda) \simeq \begin{pmatrix} \Lambda & \mathrm{D}\,\Lambda \\ \operatorname{Hom}_\Lambda(\mathrm{D}\,\Lambda, \Lambda) & \Lambda \end{pmatrix}$.
2. $e_\Lambda = \begin{pmatrix} 1_\Lambda & 0 \\ 0 & 0 \end{pmatrix}$ idempotent element of Γ.
3. $\text{mod-}\Gamma/\Gamma e_\Lambda \Gamma \longrightarrow \text{mod-}\Gamma \longrightarrow \text{mod-}\Lambda$: recollement.

$\square$

Definition

A recollement situation between triangulated categories $\mathcal{U}, \mathcal{T}$ and $\mathcal{V}$ is a diagram



$\mathsf{R}_{\mathsf{tr}}(\mathcal{U}, \mathcal{T}, \mathcal{V})$

of triangulated functors satisfying the following conditions:

1. $(\mathsf{l}, \mathsf{e}, \mathsf{r})$ is an adjoint triple.
2. $(\mathsf{q}, \mathsf{i}, \mathsf{p})$ is an adjoint triple.
3. The functors i, l, and r are fully faithful.
4. $\operatorname{Im} \mathsf{i} = \operatorname{Ker} \mathsf{e}$.

Let $\mathcal{T}$ be a triangulated category and $X \in \mathcal{T}$. We write:

$$\langle X \rangle = \langle X \rangle_1 = \mathrm{add}\{X[i] \mid i \in \mathbb{Z}\}$$

$$\langle X \rangle_{n+1} = \mathrm{add}\{Y \in \mathcal{T} \mid \exists \;\; M \longrightarrow Y \longrightarrow N \longrightarrow M[1] \;\; \text{triangle with } M \in \langle X \rangle \text{ and } N \in \langle X \rangle_n\}$$

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$$\langle X \rangle_{n+1} = \mathsf{add}\{Y \in \mathcal{T} \mid \exists \ \ M \longrightarrow Y \longrightarrow N \longrightarrow M[1] \ \ \text{triangle}$$
$$\text{with } M \in \langle X \rangle \text{ and } N \in \langle X \rangle_n\}$$

Then the **Rouquier dimension** of $\mathcal{T}$ is defined as follows:

$$\dim \mathcal{T} = \min\{n \geq 0 \mid \exists \ X \in \mathcal{T} \text{ such that } \langle X \rangle_{n+1} = \mathcal{T}\}$$

Problem

Given a $\mathsf{R}_{\mathsf{tr}}(\mathcal{U}, \mathcal{T}, \mathcal{V})$, can we give bounds for the dimension of $\mathcal{T}$ in terms of the dimensions of $\mathcal{U}$ and $\mathcal{V}$?

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Given a $\mathrm{R}_{\mathrm{tr}}(\mathcal{U}, \mathcal{T}, \mathcal{V})$, can we give bounds for the dimension of $\mathcal{T}$ in terms of the dimensions of $\mathcal{U}$ and $\mathcal{V}$?

Theorem

Let $(\mathcal{U}, \mathcal{T}, \mathcal{V})$ be a recollement of triangulated categories. Then:

$$\max\{\dim \mathcal{U}, \dim \mathcal{V}\} \ \leq \ \dim \mathcal{T} \ \leq \ \dim \mathcal{U} + \dim \mathcal{V} + 1$$

Corollary

Let $\Lambda = \left(\begin{smallmatrix} R & {}_RN_S \\ 0 & S \end{smallmatrix}\right)$ be a triangular matrix ring. Then:

$$\begin{aligned} \max\{\dim \mathbf{D}^b(R), \dim \mathbf{D}^b(S)\} &\leq \dim \mathbf{D}^b(\Lambda) \\ &\leq \dim \mathbf{D}^b(R) + \dim \mathbf{D}^b(S) + 1 \end{aligned}$$