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Gorenstein-projective modules, monic representations, and quasi-CY singularities

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Plan

- Gorenstein-projective modules of algebra $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$

- Application I: A symmetric recollement of $D^b_{\mathrm{sg}}(\Lambda)$

- Monic representations

- Application II: $\mathcal{GP}(kQ \otimes_k A)$

- Application III: Quasi-CY singularity categories via the monomorphism categories

Basic notion

- A-mod: the category of fin. gen. modules of Artin algebra A
- **A complete A-projective resolution** is an exact sequence of projective A-modules

$$P^\bullet = \cdots \longrightarrow P^{-1} \longrightarrow P^0 \xrightarrow{d^0} P^1 \longrightarrow \cdots$$

such that $\mathrm{Hom}_A(P^\bullet, A)$ is also exact.

- An A-module M is **Gorenstein-projective**, if $\exists$ a complete A-projective resolution $P^\bullet$ such that $M \cong \mathrm{Ker}\, d^0$.
- $\mathcal{GP}(A)$: the category of Gorenstein-projective A-modules.

Basic Facts

Fact I: $\mathcal{GP}(A)$ is a resolving subcategory of A-mod;

$\mathcal{GP}(A)$ is a Frobenius category with relative projective-injective objects being projective A-modules, and hence the stable category $\underline{\mathcal{GP}(A)}$ is triangulated.

Fact II: If A is a Gorenstein algebra (i.e., $\mathrm{id.}_A A < \infty, \mathrm{id.} A_A < \infty$), then any exact sequence of projective A-modules is a complete A-projective resolution; and

$$\mathcal{GP}(A) = {}^{\perp}A = \{X \in A\text{-mod} \mid \mathrm{Ext}^i_A(X, A) = 0, \ \forall\, i \geq 1\}.$$

Basic Facts (continued)

Fact III: If A is Gorenstein, then $\mathcal{GP}(A)$ is functorially finite in A-mod; and hence $\mathcal{GP}(A)$ has relative Auslander-Reiten sequences; and then $\underline{\mathcal{GP}(A)}$ has Auslander-Reiten triangles and hence a Serre functor.

Fact IV: (R.-O. Buchweitz "$\Longrightarrow$"; D. Happel "$\Longrightarrow$";)

(A. Beligiannis "$\Longleftarrow$"; S.J.Zhu "$\Longleftarrow$"; P.A.Bergh-D.A.Jorgensen-S.Oppermann "$\Longleftarrow$")

A is a Gorenstein algebra $\Longleftrightarrow$ the natural functor gives a triangle-equivalence $\underline{\mathcal{GP}(A)} \cong D^b_{\rm sg}(A) := D^b(A)/K^b(A\text{-proj})$.

Gorensteinness of $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$

Fact I: (X.W.Chen) Let A and B be Gorenstein algebras. Then Λ is Gorenstein iff pd.$_AM < \infty$ and pd.$M_B < \infty$.

Fact II: (B.L.Xiong - $\sim$)

(i) If pd.$_AM < \infty$, then Λ is Gorenstein iff A and B are Gorenstein, and pd.$M_B < \infty$.

(ii) If pd.$M_B < \infty$, then Λ is Gorenstein iff A and B are Gorenstein, and pd.$_AM < \infty$.

(iii) If id.$A_A < \infty$ or if id.$_BB < \infty$, then Λ is Gorenstein iff A and B are Gorenstein, pd.$_AM < \infty$ and pd.$M_B < \infty$.

Main Result I

Thm. Let $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ and $\mathrm{pd.}_A M$, $\mathrm{pd.} M_B < \infty$. Then $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi \in \mathcal{GP}(\Lambda)$ iff $\phi : M \otimes_B Y \to X$ is injective, $\mathrm{Cok}\,\phi \in \mathcal{GP}(A)$, and $Y \in \mathcal{GP}(B)$. In this case, $X \in \mathcal{GP}(A)$ iff $M \otimes_B Y \in \mathcal{GP}(A)$.

Rem. The result fails if $\mathrm{pd.}_A M = \infty$ or $\mathrm{pd.} M_B = \infty$.

Cor. Let Λ be Gorenstein and $\mathrm{p.d.}_A M < \infty$. Then $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi \in \mathcal{GP}(\Lambda)$ iff $\phi : M \otimes_B Y \to X$ is injective, $\mathrm{Cok}\,\phi \in \mathcal{GP}(A)$, and $Y \in \mathcal{GP}(B)$. In this case, $X \in \mathcal{GP}(A)$ iff $M \otimes_B Y \in \mathcal{GP}(A)$.

If ${}_A M$ is projective, then $X, M \otimes_B Y \in \mathcal{GP}(A)$.

About the proof

- Complete projective resolution

- **A generalization of the Horseshoe Lemma**:

Let R be a ring and $0 \to Y \xrightarrow{f} X \xrightarrow{g} Z \to 0$ an exact sequence of R-modules. Given two R-complexes

$$0 \longrightarrow Y \xrightarrow{c^{-1}} C^0 \xrightarrow{c^0} C^1 \longrightarrow \cdots ,$$

$$0 \longrightarrow Z \xrightarrow{d^{-1}} D^0 \xrightarrow{d^0} D^1 \longrightarrow \cdots ,$$

if $\mathrm{Ext}^1_A(\mathrm{Im}d^i, C^{i+1}) = 0$ for each $i \geq -1$, then there is the following commutative diagram of R-modules

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & C^0 & \xrightarrow{c^0} & C^1 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f} & & {\scriptstyle \binom{0}{\mathrm{id}}}\downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & D^0 \oplus C^0 & \xrightarrow{\partial^0} & D^1 \oplus C^1 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle g} & & {\scriptstyle (\mathrm{Id},0)}\downarrow & & \downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & D^0 & \xrightarrow{d^0} & D^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

Moreover, if the top and the bottom rows are exact, so is the middle one.

Symmetric recollements

Definition A triangulated category recollement

$$\mathcal{C}' \begin{array}{c} \xleftarrow{\;i^*\;} \\ \xrightarrow{\;i_*\;} \\ \xleftarrow[\;i^!\;]{} \end{array} \mathcal{C} \begin{array}{c} \xleftarrow{\;j_!\;} \\ \xrightarrow{\;j^*\;} \\ \xleftarrow[\;j_*\;]{} \end{array} \mathcal{C}''$$

is lower-symmetric, if $\exists$ exact functors $j_?$ and $i_?$ such that

$$\mathcal{C}'' \begin{array}{c} \xleftarrow{\;j^*\;} \\ \xrightarrow{\;j_*\;} \\ \xleftarrow[\;j_?\;]{} \end{array} \mathcal{C} \begin{array}{c} \xleftarrow{\;i_*\;} \\ \xrightarrow{\;i^!\;} \\ \xleftarrow[\;i_?\;]{} \end{array} \mathcal{C}'$$

is a recollement; and it is upper-symmetric, if $\exists$ are exact functors $j^?$ and $i^?$ such that

$$\mathcal{C}'' \begin{array}{c} \xleftarrow{\;j^?\;} \\ \xrightarrow{\;j_!\;} \\ \xleftarrow[\;j^*\;]{} \end{array} \mathcal{C} \begin{array}{c} \xleftarrow{\;i^?\;} \\ \xrightarrow{\;i^*\;} \\ \xleftarrow[\;i_*\;]{} \end{array} \mathcal{C}'$$

is a recollement.

A recollement is symmetric if it is lower- and upper-symmetric.

Main result II

Theorem Let $\Lambda == \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ be Gorenstein and ${}_AM$ projective.

Then we have a symmetric recollement

$$D^b_{\mathrm{sg}}(A) \quad \begin{smallmatrix} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow[i^!]{} \end{smallmatrix} \quad D^b_{\mathrm{sg}}(\Lambda) \quad \begin{smallmatrix} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow[j_*]{} \end{smallmatrix} \quad D^b_{\mathrm{sg}}(B).$$

Remark Iyama-Kato-Miyachi's result on $T_2 = \left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right)$: If A is Gorenstein, then there is a recollement

$$\underline{\mathrm{CM}(A)} \quad \begin{smallmatrix} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow[i^!]{} \end{smallmatrix} \quad \underline{\mathrm{CM}(T_2)} \quad \begin{smallmatrix} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow[j_*]{} \end{smallmatrix} \quad \underline{\mathrm{CM}(A)}.$$

About the proof

- Gorenstein-projective modules of $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$

- I. Reiten and M. Van den Bergh: A Hom-finite Krull-Schmidt triangulated k-category has a Serre functor iff it has Auslander-Reiten triangles.

- P. Jϕrgensen: A recollement of a triangulated category with a Serre functor is symmetric.

What we also need

Theorem Let

$$\mathcal{C}' \underset{\overset{\xleftarrow{i^!}}{}}{\overset{\xleftarrow{i^*}}{\xrightarrow{i_*}}} \mathcal{C} \underset{\overset{\xleftarrow{j_*}}{}}{\overset{\xleftarrow{j_!}}{\xrightarrow{j^*}}} \mathcal{C}''$$

be a diagram of exact functors of triangulated categories. Then it is a recollement iff the following conditions are satisfied.

(R1) (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$, and (j^*, j_*) are adjoint pairs;

(R2) i_*, $j_!$ and j_* are fully faithful;

(R3) $\mathrm{Im} i_* = \mathrm{Ker} j^*$.

Quiver representations over an algebra

- $Q = (Q_0, Q_1, s, e)$: a finite quiver
- $\mathrm{Rep}(Q, k)$: the representation category of Q over field k
- $\mathrm{Rep}(Q, A)$: the representation cat. of Q over k-algebra A, whose objects are $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ where $X_i \in A$-mod, and for an arrow $\alpha : i \to j$, $X_\alpha : X_i \longrightarrow X_j$ is an A-map.
- $(kQ \otimes_k A)\text{-mod} \cong \mathrm{Rep}(Q, A)$.

Monic representations

Dafinition A representation $X = (X_i, X_\alpha, i \in Q_0, \alpha \in Q_1)$ of Q over A is **monic**, if for each $i \in Q_0$ we the following A-map is injective

$$(X_\alpha)_{\alpha\in Q_1,\ e(\alpha)=i} : \bigoplus_{\substack{\alpha\in Q_1\\ e(\alpha)=i}} X_{s(\alpha)} \longrightarrow X_i.$$

- Mon(Q, A): the full subcategory of Rep(Q, A) consisting of monic representations of Q over A.

- Let χ be a full subcategory of A-mod. Let Mon(Q, χ) denote the full subcategory of Rep(Q, A) consisting of monic representations X such that for $i \in Q_0$

$$X_i \in \chi, \quad X_i / \bigoplus_{\substack{\alpha\in Q_1\\ e(\alpha)=i}} X_{s(\alpha)} \in \chi.$$

A remark

- If $Q = \bullet \longrightarrow \bullet$, then $\mathrm{Mon}(Q, A)$ is called **the submodule category of** A by C.M.Ringel and M.Schmidmeier.

- If $Q = \bullet \longrightarrow \cdots \longrightarrow \bullet$, then $\mathrm{Mon}(Q, A)$ is called **the filtered chain category of** A by D.M.Arnold and D.Simson.

- This has been studied by C.M.Ringel, M.Schmidmeier, D.M.Arnold, D.Simson, D. Kussin, H. Lenzing, H. Meltzer, X. W. Chen, A. Moore, O.Iyama, K.Kato and J.I.Miyachi, $\cdots$, in the different settings.

Main result III

Theorem (X.H.Luo-$\sim$) Let Q be a finite acyclic quiver, A a fin. dim. algebra, and $\Lambda = kQ \otimes_k A$. Then

$$\mathcal{GP}(\Lambda) = \mathrm{Mon}(Q, \mathcal{GP}(A)).$$

In particular, A is self-injective iff $\mathcal{GP}(\Lambda) = \mathrm{Mon}(Q, A)$; iff $\mathrm{Mon}(Q, A)$ is a Frobenius category.

Remark If A is self-injective, and $Q = A_n$, we can generalize C.M.Ringel - M.Schmidmeier's work on the Auslander-Reiten theory of $\mathrm{Mon}(A_2, A)$; we can also compute the Serre functor of the singularity category via the stable monomorphism category.

Fractional quasi-CY singularity categories

Theorem (B.L.Xiong, $\sim$, Y.H.Zhang) Let $Q = A_n$, $A = A(m,t)$ be the self-injective algebra, $m \geq 1, t \geq 2$.

(i) If $t = 2$, then $\underline{\mathrm{Mon}(Q, A)}$ is quasi-CY of dimension $\frac{(m,n-1)}{n+1}$;

(ii) If$t \geq 3$, then $\underline{\mathrm{Mon}(Q, A)}$ is quasi-CY of dimension $\frac{2(m,t,n+1)}{(m,t)(n+1)}$.

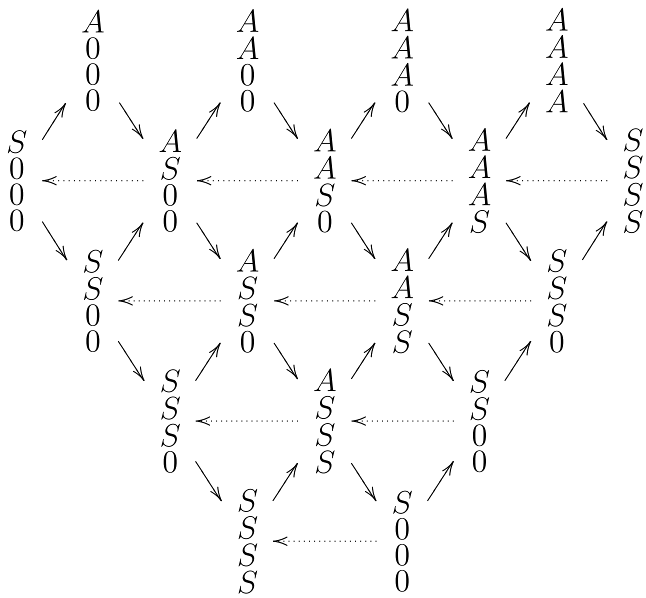
Auslander-Reiten quivers of $\mathrm{Mon}(A_n, \Lambda_t)$ **of finite type**

Let $Q = A_n$, $\Lambda_t = k[x]/\langle x^t\rangle$, $n \geq 2$, $t \geq 2$. By a work of D.Simson, $\mathrm{Mon}(A_n, \Lambda_t)$ is of finite type iff (n, t) are one of:

$$(n, 2); \quad (2, 3); \quad (2, 4); \quad (2, 5); \quad (3, 3); \quad (4, 3).$$

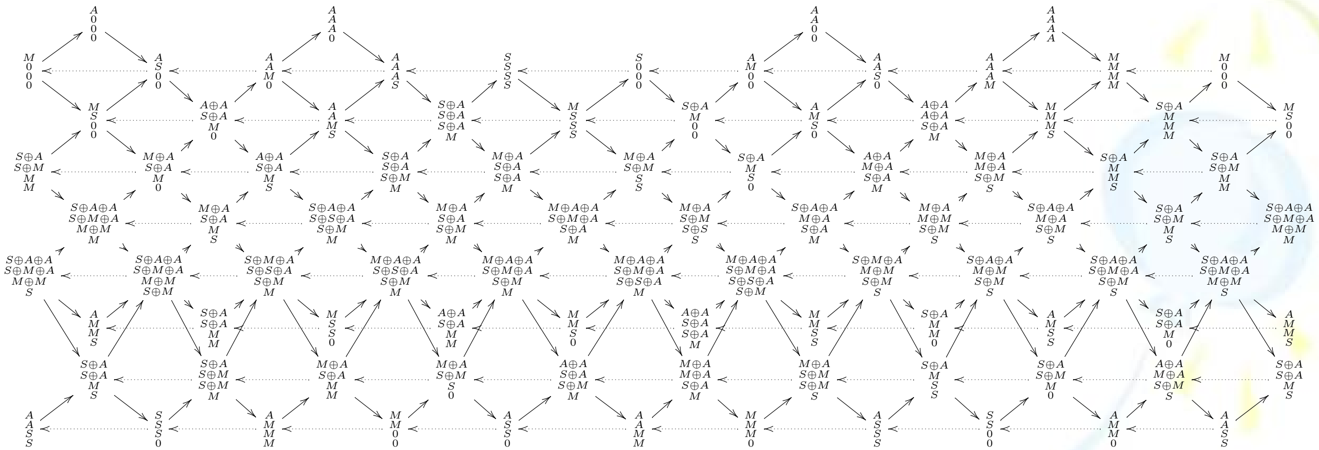
where the AR quivers of $\mathrm{Mon}(A_2, \Lambda_t)$ $(t = 2, 3, 4, 5)$ were given by C.M.Ringel and M.Schmidmeier; the ones for $(3, t)$ were done by A.Moore. The remaining were done by B.L.Xiong.

$\mathrm{Mon}(A_4, \Lambda_2)$ has $4 + 10$ indecomposables; its Auslander-Reiten quiver is a Möbius band):



In general, $\mathrm{Mon}(A_n, \Lambda_2)$ has $n + \frac{n(n+1)}{2}$ indecomposables; its Auslander-Reiten quiver is a Möbius band.

Mon(A_4, Λ_3) has 4 + 80 indecomposables; its Auslander-Reiten quiver is a tube:



Thank you very much

for your attention!