Quiver and relations of $\mathcal{O}_0(\mathfrak{sl}_{n+1})$ induced from $\mathcal{O}_0(\mathfrak{sl}_n)$

Daiva Pučinskaitė

University of Kiel
Germany

ICRA XV

August 16, 2012
Bielefeld

Lie-algebra $\mathfrak{sl}_n(\mathbb{C})$

Lie-algebra $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) = \{C = (c_{ij})_{ij} \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(C) = 0\}$

Lie-algebra $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) = \{C = (c_{ij})_{ij} \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(C) = 0\}$
- $\mathfrak{sl}_n = \underbrace{\left(\bigoplus_{i<j} \mathbb{C} \cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_+} \oplus \underbrace{\left(\bigoplus_{i=1}^{n-1} \mathbb{C} \cdot (E_{ii} - E_{i+1,i+1})\right)}_{\mathfrak{h}_n} \oplus \underbrace{\left(\bigoplus_{i>j} \mathbb{C} \cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_-}$

Lie-algebra $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) = \{C = (c_{ij})_{ij} \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(C) = 0\}$
- $\mathfrak{sl}_n = \underbrace{\left(\bigoplus_{i<j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_+} \oplus \underbrace{\left(\bigoplus_{i=1}^{n-1} \mathbb{C}\cdot(E_{ii} - E_{i+1,i+1})\right)}_{\mathfrak{h}_n} \oplus \underbrace{\left(\bigoplus_{i>j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_-}$

$$\begin{array}{cccc} \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\ & \alpha_{23} & \cdots & \alpha_{2n} \\ & & \ddots & \vdots \\ & & & \alpha_{n-1n} \end{array}$$

Lie-algebra $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) = \{C = (c_{ij})_{ij} \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(C) = 0\}$
- $\mathfrak{sl}_n = \underbrace{\left(\bigoplus_{i<j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_+} \oplus \underbrace{\left(\bigoplus_{i=1}^{n-1} \mathbb{C}\cdot(E_{ii} - E_{i+1,i+1})\right)}_{\mathfrak{h}_n} \oplus \underbrace{\left(\bigoplus_{i>j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_-}$
- $W(n) = \left\langle \left\{ s_{ij} := s_{\alpha_{ij}} \mid i < j \right\} \right\rangle$

$$\begin{array}{cccc}
\alpha_{12} & \alpha_{13} & \cdots & \alpha_{1n} \\
 & \alpha_{23} & \cdots & \alpha_{2n} \\
 & & \ddots & \vdots \\
 & & & \alpha_{n-1n}
\end{array}$$

Lie-algebra $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) = \{C = (c_{ij})_{ij} \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(C) = 0\}$
- $\mathfrak{sl}_n = \underbrace{\left(\bigoplus_{i<j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_+} \oplus \underbrace{\left(\bigoplus_{i=1}^{n-1} \mathbb{C}\cdot(E_{ii} - E_{i+1,i+1})\right)}_{\mathfrak{h}_n} \oplus \underbrace{\left(\bigoplus_{i>j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_-}$
- $W(n) = \left\langle \left\{ s_{ij} := s_{\alpha_{ij}} \mid i < j \right\} \right\rangle$

$$\begin{array}{cccc} s_{12} & s_{13} & \cdots & s_{1n} \\ & s_{23} & \cdots & s_{2n} \\ & & \ddots & \vdots \\ & & & s_{n-1n} \end{array}$$

Lie-algebra $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) = \{C = (c_{ij})_{ij} \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(C) = 0\}$
- $\mathfrak{sl}_n = \underbrace{\left(\bigoplus_{i<j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_+} \oplus \underbrace{\left(\bigoplus_{i=1}^{n-1} \mathbb{C}\cdot(E_{ii} - E_{i+1,i+1})\right)}_{\mathfrak{h}_n} \oplus \underbrace{\left(\bigoplus_{i>j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_-}$
- $W(n) = \left\langle \left\{ s_{ij} := s_{\alpha_{ij}} \mid i < j \right\} \right\rangle \longleftrightarrow \mathrm{Sym}(n) = \langle \{(i,j) \mid i < j\} \rangle$

$$\begin{array}{cccc}
s_{12} & s_{13} & \cdots & s_{1n} \\
 & s_{23} & \cdots & s_{2n} \\
 & & \ddots & \vdots \\
 & & & s_{n-1n}
\end{array}$$

Lie-algebra $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) = \{C = (c_{ij})_{ij} \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(C) = 0\}$
- $\mathfrak{sl}_n = \underbrace{\left(\bigoplus_{i<j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_+} \oplus \underbrace{\left(\bigoplus_{i=1}^{n-1} \mathbb{C}\cdot(E_{ii} - E_{i+1,i+1})\right)}_{\mathfrak{h}_n} \oplus \underbrace{\left(\bigoplus_{i>j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_-}$
- $W(n) = \left\langle \left\{ s_{ij} := s_{\alpha_{ij}} \mid i < j \right\} \right\rangle \longleftrightarrow \mathrm{Sym}(n) = \langle\{(i,j) \mid i < j\}\rangle$

$$\begin{array}{cccc} s_{12} & s_{13} & \cdots & s_{1n} \\ & s_{23} & \cdots & s_{2n} \\ & & \ddots & \vdots \\ & & & s_{n-1n} \end{array} \qquad s_{ij} \mapsto (i,j) \qquad \begin{array}{cccc} (1,2) & (1,3) & \cdots & (1,n) \\ & (2,3) & \cdots & (2,n) \\ & & \ddots & \vdots \\ & & & (n\text{-}1,n) \end{array}$$

Lie-algebra $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) = \{C = (c_{ij})_{ij} \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(C) = 0\}$
- $\mathfrak{sl}_n = \underbrace{\left(\bigoplus_{i<j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_+} \oplus \underbrace{\left(\bigoplus_{i=1}^{n-1} \mathbb{C}\cdot(E_{ii} - E_{i+1,i+1})\right)}_{\mathfrak{h}_n} \oplus \underbrace{\left(\bigoplus_{i>j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_-}$
- $W(n) = \left\langle \left\{ s_{ij} := s_{\alpha_{ij}} \mid i < j \right\} \right\rangle \longleftrightarrow \mathrm{Sym}(n) = \langle\{(i,j) \mid i < j\}\rangle$

$$\begin{array}{cccc} s_{12} & s_{13} & \cdots & s_{1n} \\ & s_{23} & \cdots & s_{2n} \\ & & \ddots & \vdots \\ & & & s_{n-1n} \end{array} \qquad \overset{s_{ij} \mapsto (i,j)}{\longleftrightarrow} \qquad \begin{array}{cccc} (1,2) & (1,3) & \cdots & (1,n) \\ & (2,3) & \cdots & (2,n) \\ & & \ddots & \vdots \\ & & & (n\text{-}1,n) \end{array}$$

- Bruhat order on $\mathrm{Sym}(n)$:

Lie-algebra $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) = \{C = (c_{ij})_{ij} \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(C) = 0\}$
- $\mathfrak{sl}_n = \underbrace{\left(\bigoplus_{i<j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_+} \oplus \underbrace{\left(\bigoplus_{i=1}^{n-1} \mathbb{C}\cdot(E_{ii} - E_{i+1,i+1})\right)}_{\mathfrak{h}_n} \oplus \underbrace{\left(\bigoplus_{i>j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_-}$
- $W(n) = \left\langle \left\{ s_{ij} := s_{\alpha_{ij}} \mid i < j \right\} \right\rangle \longleftrightarrow \mathrm{Sym}(n) = \langle\{(i,j) \mid i < j\}\rangle$

$$\begin{array}{cccc} s_{12} & s_{13} & \cdots & s_{1n} \\ & s_{23} & \cdots & s_{2n} \\ & & \ddots & \vdots \\ & & & s_{n-1n} \end{array} \quad \overset{s_{ij} \mapsto (i,j)}{\longleftrightarrow} \quad \begin{array}{cccc} (1,2) & (1,3) & \cdots & (1,n) \\ & (2,3) & \cdots & (2,n) \\ & & \ddots & \vdots \\ & & & (n\text{-}1,n) \end{array}$$

- Bruhat order on $\mathrm{Sym}(n)$: $\sigma < \tau$ if $\sigma \lhd \nu_1 \lhd \cdots \lhd \nu_{k-1} \lhd \nu_k = \tau$

Lie-algebra $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) = \{C = (c_{ij})_{ij} \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(C) = 0\}$
- $$\mathfrak{sl}_n = \underbrace{\left(\bigoplus_{i<j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_+} \oplus \underbrace{\left(\bigoplus_{i=1}^{n-1} \mathbb{C}\cdot(E_{ii} - E_{i+1,i+1})\right)}_{\mathfrak{h}_n} \oplus \underbrace{\left(\bigoplus_{i>j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_-}$$
- $W(n) = \left\langle \left\{ s_{ij} := s_{\alpha_{ij}} \mid i < j \right\} \right\rangle \longleftrightarrow \mathrm{Sym}(n) = \langle\{(i,j) \mid i < j\}\rangle$

$$\begin{matrix} s_{12} & s_{13} & \cdots & s_{1n} \\ & s_{23} & \cdots & s_{2n} \\ & & \ddots & \vdots \\ & & & s_{n-1n} \end{matrix} \qquad \overset{s_{ij} \mapsto (i,j)}{\longleftrightarrow} \qquad \begin{matrix} (1,2) & (1,3) & \cdots & (1,n) \\ & (2,3) & \cdots & (2,n) \\ & & \ddots & \vdots \\ & & & (n\text{-}1,n) \end{matrix}$$

- Bruhat order on $\mathrm{Sym}(n)$: $\sigma < \tau$ if $\sigma \lhd \nu_1 \lhd \cdots \lhd \nu_{k-1} \lhd \nu_k = \tau$

  $\sigma \lhd \nu$

Lie-algebra $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) = \{C = (c_{ij})_{ij} \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(C) = 0\}$
- $\mathfrak{sl}_n = \underbrace{\left(\bigoplus_{i<j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_+} \oplus \underbrace{\left(\bigoplus_{i=1}^{n-1} \mathbb{C}\cdot(E_{ii} - E_{i+1,i+1})\right)}_{\mathfrak{h}_n} \oplus \underbrace{\left(\bigoplus_{i>j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_-}$
- $W(n) = \left\langle \left\{ s_{ij} := s_{\alpha_{ij}} \mid i < j \right\} \right\rangle \longleftrightarrow \mathrm{Sym}(n) = \langle \{(i,j) \mid i < j\} \rangle$

$$\begin{array}{cccc} s_{12} & s_{13} & \cdots & s_{1n} \\ & s_{23} & \cdots & s_{2n} \\ & & \ddots & \vdots \\ & & & s_{n-1n} \end{array} \quad \overset{s_{ij} \mapsto (i,j)}{\longleftrightarrow} \quad \begin{array}{cccc} (1,2) & (1,3) & \cdots & (1,n) \\ & (2,3) & \cdots & (2,n) \\ & & \ddots & \vdots \\ & & & (n\text{-}1,n) \end{array}$$

- Bruhat order on $\mathrm{Sym}(n)$: $\sigma < \tau$ if $\sigma \lhd \nu_1 \lhd \cdots \lhd \nu_{k-1} \lhd \nu_k = \tau$

$$\sigma \lhd \nu \quad \Leftrightarrow \quad \nu = (i,j) \cdot \sigma$$

Lie-algebra $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) = \{C = (c_{ij})_{ij} \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(C) = 0\}$
- $$\mathfrak{sl}_n = \underbrace{\left(\bigoplus_{i<j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_+} \oplus \underbrace{\left(\bigoplus_{i=1}^{n-1} \mathbb{C}\cdot(E_{ii} - E_{i+1,i+1})\right)}_{\mathfrak{h}_n} \oplus \underbrace{\left(\bigoplus_{i>j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_-}$$
- $W(n) = \left\langle \left\{ s_{ij} := s_{\alpha_{ij}} \mid i < j \right\} \right\rangle \longleftrightarrow \mathrm{Sym}(n) = \langle \{(i,j) \mid i < j\} \rangle$

$$\begin{array}{cccc} s_{12} & s_{13} & \cdots & s_{1n} \\ & s_{23} & \cdots & s_{2n} \\ & & \ddots & \vdots \\ & & & s_{n-1n} \end{array} \qquad \overset{s_{ij} \mapsto (i,j)}{\longleftrightarrow} \qquad \begin{array}{cccc} (1,2) & (1,3) & \cdots & (1,n) \\ & (2,3) & \cdots & (2,n) \\ & & \ddots & \vdots \\ & & & (n\text{-}1,n) \end{array}$$

- Bruhat order on $\mathrm{Sym}(n)$: $\sigma < \tau$ if $\sigma \lhd \nu_1 \lhd \cdots \lhd \nu_{k-1} \lhd \nu_k = \tau$

$$\sigma \lhd \nu \quad \Leftrightarrow \quad \nu = (i,j)\cdot\sigma \quad \text{and} \quad l(\nu) = l(\sigma) + 1$$

Lie-algebra $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) = \{C = (c_{ij})_{ij} \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}(C) = 0\}$
- $\mathfrak{sl}_n = \underbrace{\left(\bigoplus_{i<j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_+} \oplus \underbrace{\left(\bigoplus_{i=1}^{n-1} \mathbb{C}\cdot(E_{ii} - E_{i+1,i+1})\right)}_{\mathfrak{h}_n} \oplus \underbrace{\left(\bigoplus_{i>j} \mathbb{C}\cdot E_{ij}\right)}_{(\mathfrak{sl}_n)_-}$
- $W(n) = \left\langle \left\{ s_{ij} := s_{\alpha_{ij}} \mid i < j \right\} \right\rangle \longleftrightarrow \mathrm{Sym}(n) = \langle \{(i,j) \mid i < j\} \rangle$

$$\begin{array}{cccc} s_{12} & s_{13} & \cdots & s_{1n} \\ & s_{23} & \cdots & s_{2n} \\ & & \ddots & \vdots \\ & & & s_{n-1n} \end{array} \qquad \overset{s_{ij} \mapsto (i,j)}{\longleftrightarrow} \qquad \begin{array}{cccc} (1,2) & (1,3) & \cdots & (1,n) \\ & (2,3) & \cdots & (2,n) \\ & & \ddots & \vdots \\ & & & (n\text{-}1,n) \end{array}$$

- Bruhat order on $\mathrm{Sym}(n)$: $\sigma < \tau$ if $\sigma \lhd \nu_1 \lhd \cdots \lhd \nu_{k-1} \lhd \nu_k = \tau$

$$\sigma \lhd \nu \quad \Leftrightarrow \quad \nu = (i,j)\cdot\sigma \quad \text{and} \quad l(\nu) = l(\sigma) + 1$$

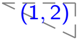
$\sigma = (\sigma(1), \ldots, \sigma(n))$

Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(2), \leqslant)$

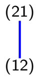
Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(2), \leqslant)$



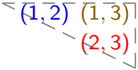
Bruhat order on Sym(n)

Example. Hasse-diagram on (Sym(2), $\leqslant$) $(1,2)$



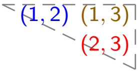
Bruhat order on Sym(n)

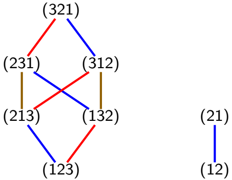
Example. Hasse-diagram on $(\mathrm{Sym}(3), \leqslant)$



Bruhat order on Sym(n)

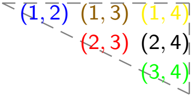
Example. Hasse-diagram on $(\mathrm{Sym}(3), \leqslant)$

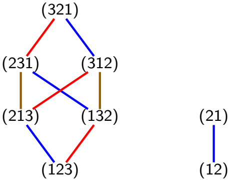




Bruhat order on Sym(n)

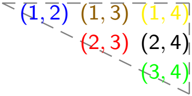
Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

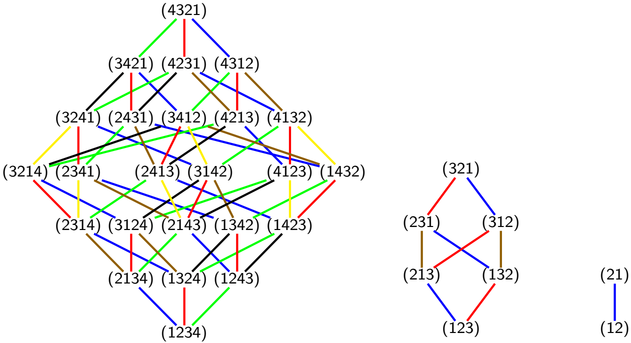




Bruhat order on Sym(n)

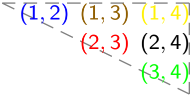
Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

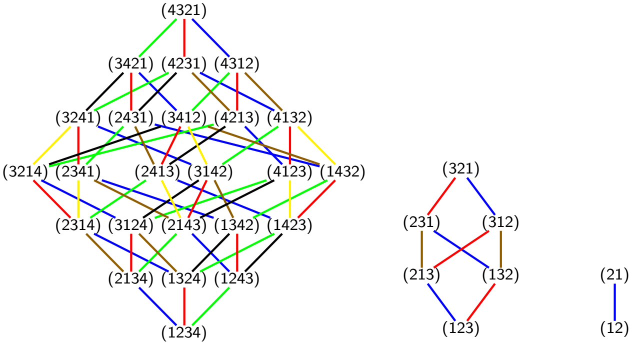




Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

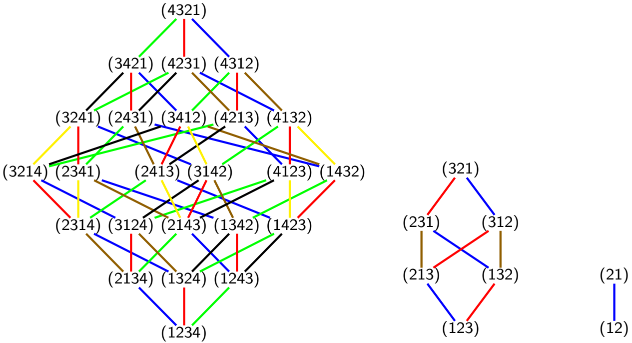




Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

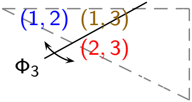
$(1,2)$ $(1,3)$ $(1,4)$
$(2,3)$ $(2,4)$
$(3,4)$

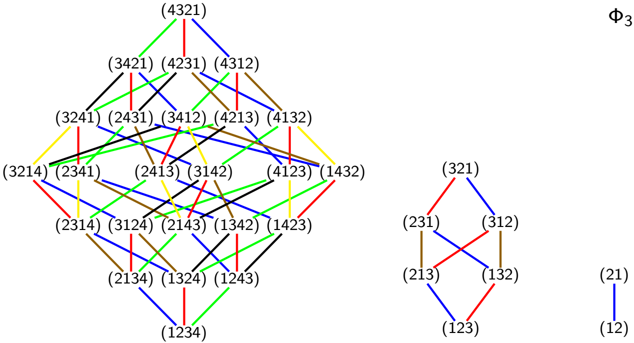


$$\begin{array}{ccc} \mathrm{Sym}(n) & \xrightarrow{\Phi_n} & \mathrm{Sym}(n) \\ \sigma & \mapsto & \omega_n \cdot \sigma \cdot \omega_n \end{array}$$

Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(3), \leqslant)$

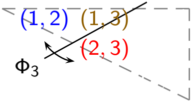


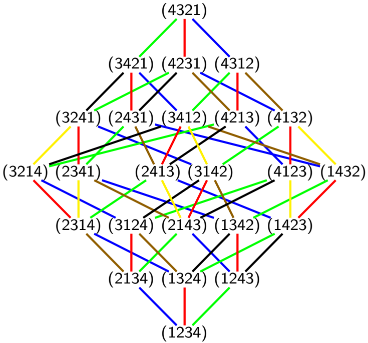


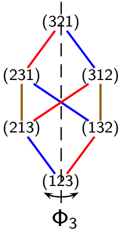
$$\begin{array}{ccc} \mathrm{Sym}(n) & \xrightarrow{\Phi_n} & \mathrm{Sym}(n) \\ \sigma & \mapsto & \omega_n \cdot \sigma \cdot \omega_n \end{array}$$

Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(3), \leqslant)$



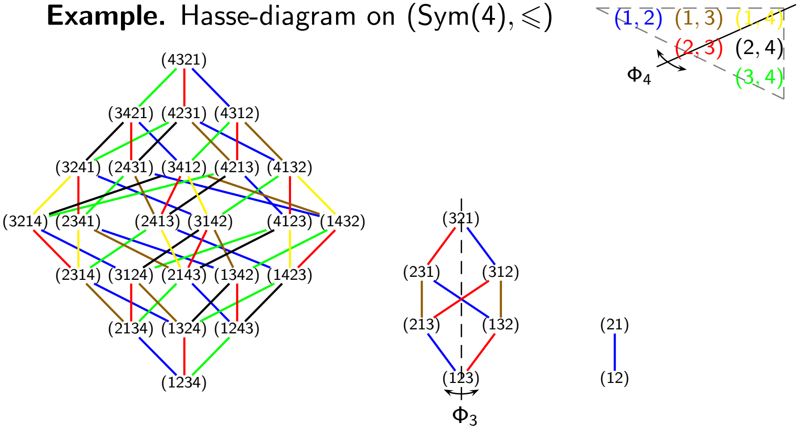




$$\begin{array}{ccc} \mathrm{Sym}(n) & \xrightarrow{\Phi_n} & \mathrm{Sym}(n) \\ \sigma & \mapsto & \omega_n \cdot \sigma \cdot \omega_n \end{array}$$

Bruhat order on Sym(n)

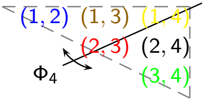
Example. Hasse-diagram on (Sym(4), $\leqslant$)

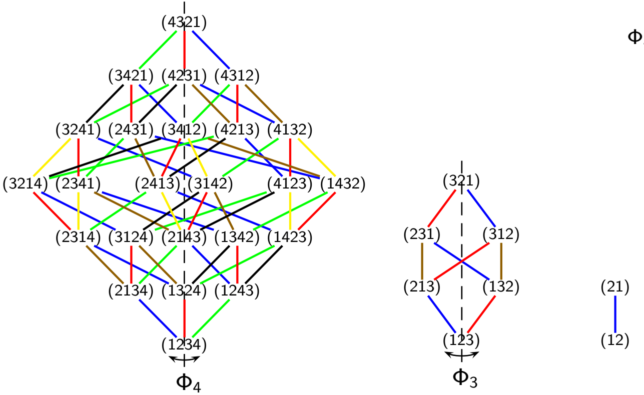


$$\begin{array}{ccc} \mathrm{Sym}(n) & \xrightarrow{\Phi_n} & \mathrm{Sym}(n) \\ \sigma & \mapsto & \omega_n \cdot \sigma \cdot \omega_n \end{array}$$

Bruhat order on Sym(n)

Example. Hasse-diagram on (Sym(4), $\leqslant$)

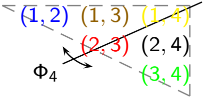


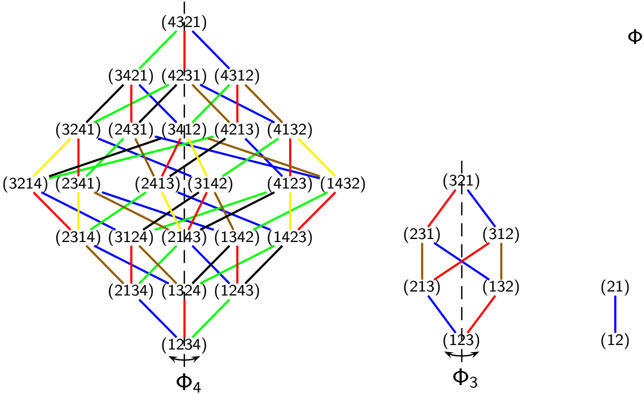


$$\begin{array}{ccc} \mathrm{Sym}(n) & \xrightarrow{\Phi_n} & \mathrm{Sym}(n) \\ \sigma & \mapsto & \omega_n \cdot \sigma \cdot \omega_n \end{array}$$

Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

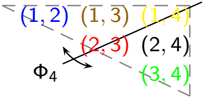


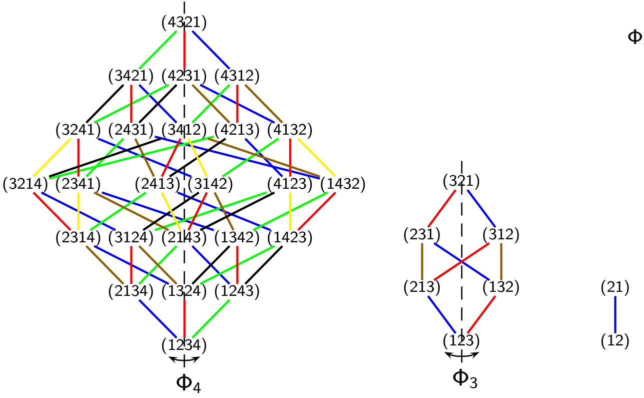


$$\begin{array}{ccc} \mathrm{Sym}(n) & \xrightarrow{\Phi_n} & \mathrm{Sym}(n) \\ \sigma & \mapsto & \omega_n \cdot \sigma \cdot \omega_n \end{array}, \qquad \sigma \lhd \tau \Leftrightarrow \Phi_n(\sigma) \lhd \Phi_n(\tau)$$

Bruhat order on Sym(n)

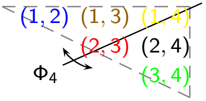
Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

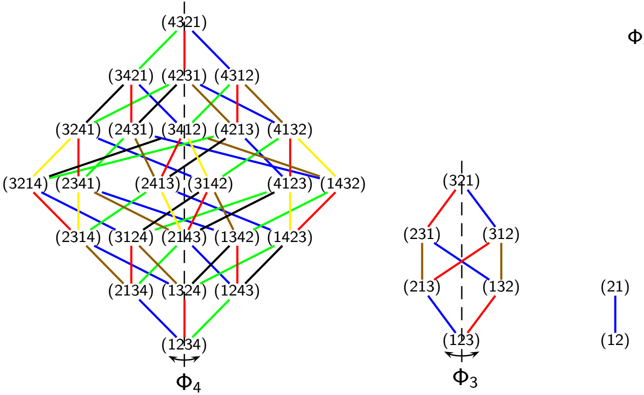




Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

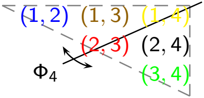


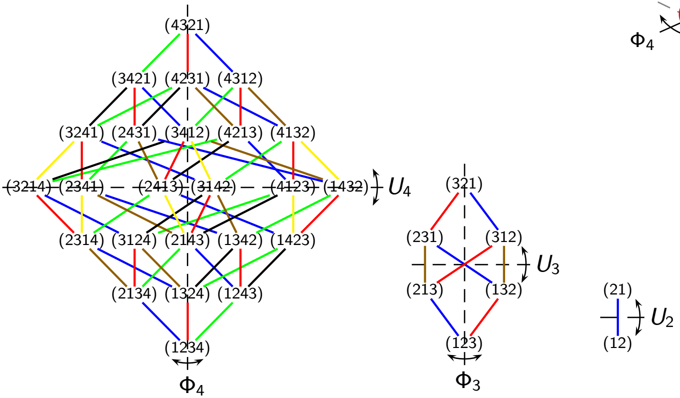


$$\begin{array}{ccc} \mathrm{Sym}(n) & \xrightarrow{U_n} & \mathrm{Sym}(n) \\ \sigma & \mapsto & \omega_n \cdot \sigma \end{array}$$

Bruhat order on Sym(n)

Example. Hasse-diagram on (Sym(4), $\leqslant$)

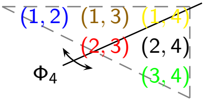


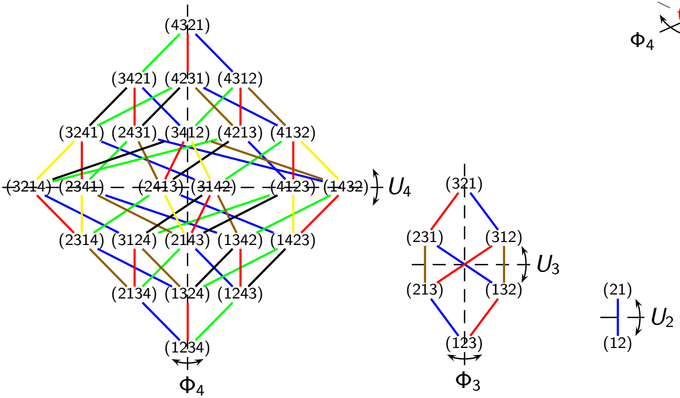


$$\begin{array}{ccc} \mathrm{Sym}(n) & \xrightarrow{U_n} & \mathrm{Sym}(n) \\ \sigma & \mapsto & \omega_n \cdot \sigma \end{array}$$

Bruhat order on $\mathrm{Sym}(n)$

Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

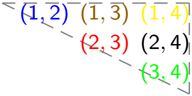


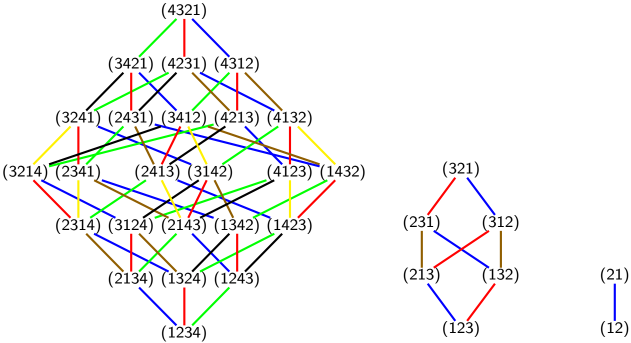


$$\begin{array}{ccc} \mathrm{Sym}(n) & \xrightarrow{U_n} & \mathrm{Sym}(n) \\ \sigma & \mapsto & \omega_n \cdot \sigma \end{array}, \qquad \sigma \lhd \tau \Leftrightarrow U_n(\sigma) \rhd U_n(\tau)$$

Bruhat order on Sym(n)

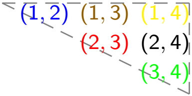
Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

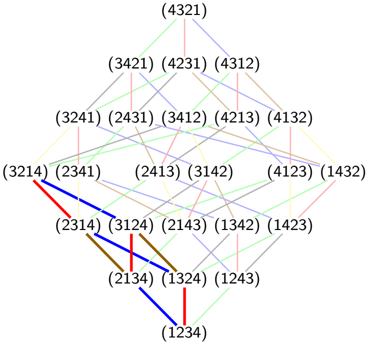


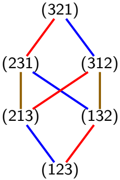


Bruhat order on Sym(n)

Example. Hasse-diagram on (Sym(4), $\leqslant$)

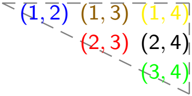


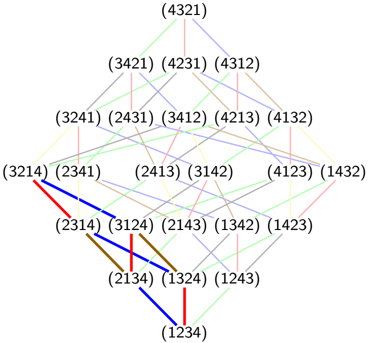


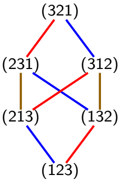


Bruhat order on Sym(n)

Example. Hasse-diagram on (Sym(4), $\leqslant$)

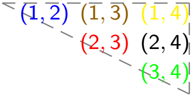


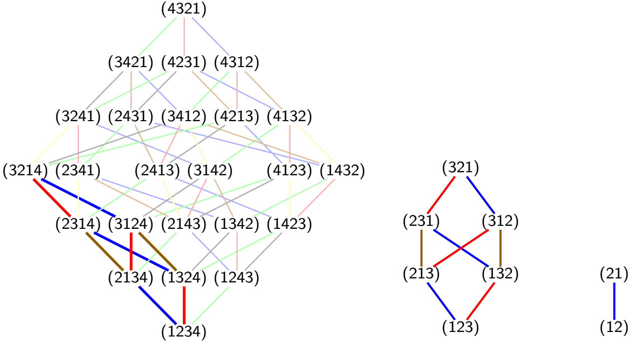




Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

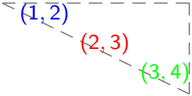


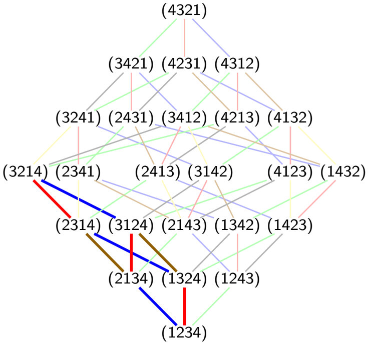


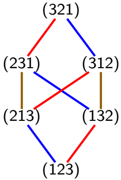
$$\mathrm{Sym}(n+1) = \dot{\bigcup_{1\leq k\leq n}} \underbrace{(k,\ldots,n+1)\cdot \mathrm{Sym}(n)}_{(*,\cdots,*,k)}$$

Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$



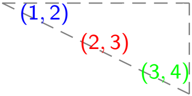


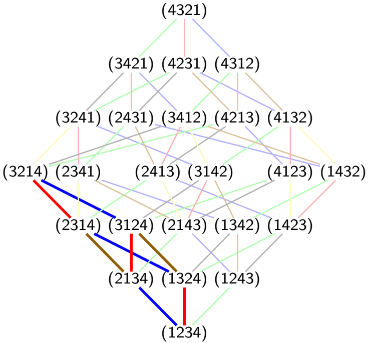


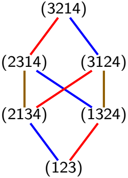
$$\mathrm{Sym}(n+1) = \dot{\bigcup_{1 \leq k \leq n}} \underbrace{(k, \ldots, n+1) \cdot \mathrm{Sym}(n)}_{(*,\cdots,*,k)}$$

Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$



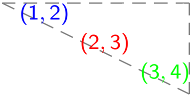


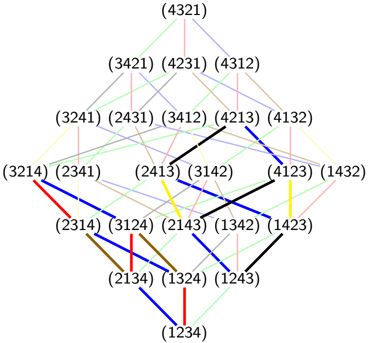


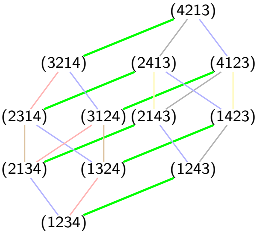
$$\mathrm{Sym}(n+1) = \dot{\bigcup_{1\leq k\leq n}} \underbrace{(k,\ldots,n+1)\cdot \mathrm{Sym}(n)}_{(*,\cdots,*,k)}$$

Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$



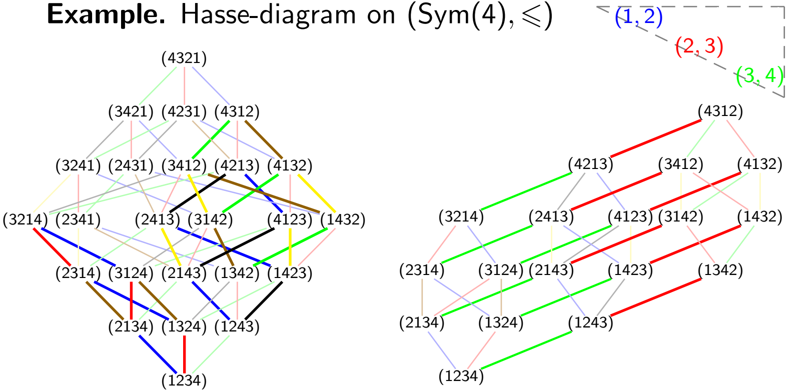




$$\mathrm{Sym}(n+1) = \dot{\bigcup_{1 \le k \le n}} \underbrace{(k, \ldots, n+1) \cdot \mathrm{Sym}(n)}_{(*, \cdots, *, k)}$$

Bruhat order on Sym(n)

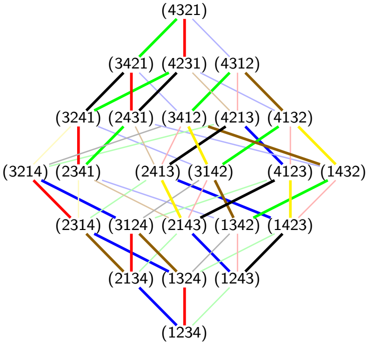
Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

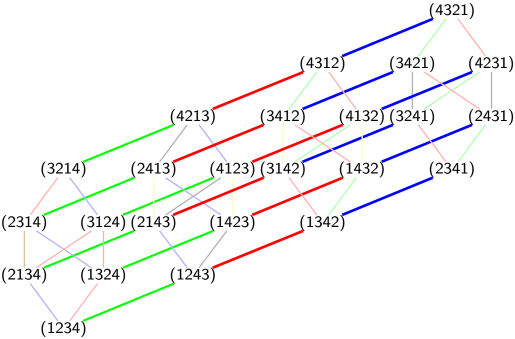


$$\mathrm{Sym}(n{+}1) = \dot{\bigcup_{1\leq k\leq n}} \underbrace{(k,\ldots,n{+}1)\cdot\mathrm{Sym}(n)}_{(*,\cdots,*,k)}$$

Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

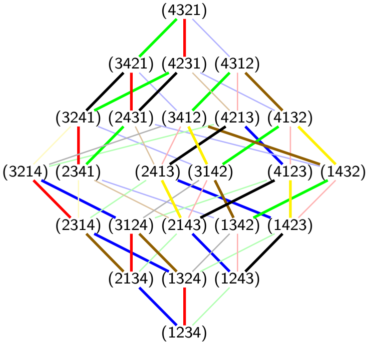


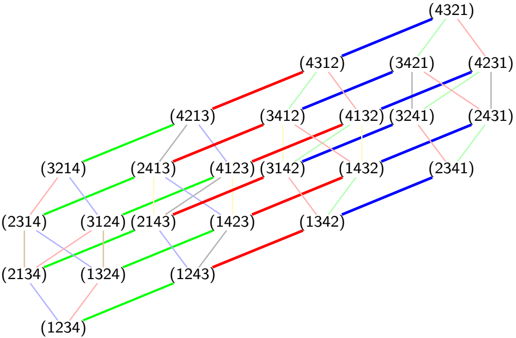


$$\mathrm{Sym}(n{+}1) = \dot{\bigcup_{1\leq k\leq n}} \underbrace{(k,\ldots,n{+}1)\cdot \mathrm{Sym}(n)}_{(*,\cdots,*,k)}$$

Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

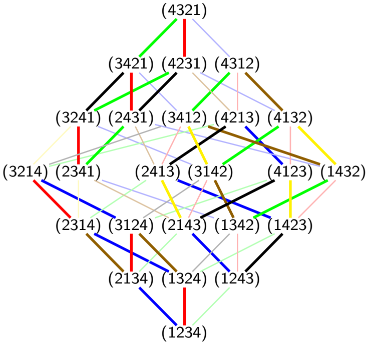


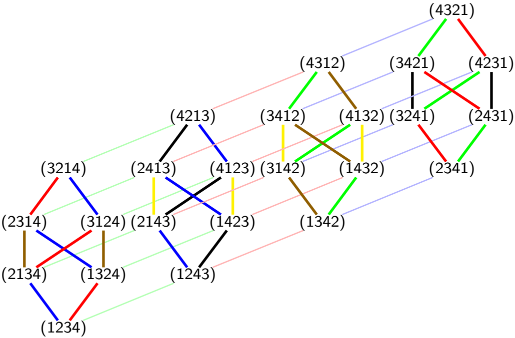


$$\mathrm{Sym}(n{+}1) = \dot{\bigcup_{1\le k\le n}} \underbrace{(k,\ldots,n{+}1)\cdot \mathrm{Sym}(n)}_{(*,\cdots,*,k)}, \qquad \frac{\sigma \lhd \tau}{(k,\ldots,n{+}1)\cdot\sigma \lhd (k,\ldots,n{+}1)\cdot\tau}$$

Bruhat order on Sym(n)

Example. Hasse-diagram on (Sym(4), $\leqslant$)

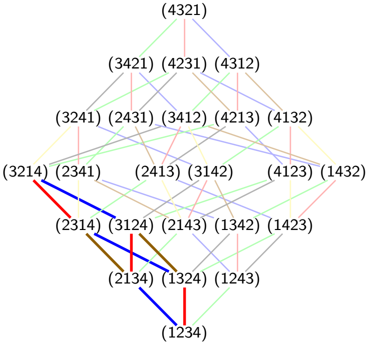


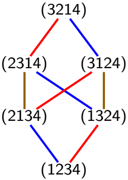


$$\mathrm{Sym}(n+1) = \dot{\bigcup_{1\le k\le n}} \underbrace{(k,\ldots,n+1)\cdot \mathrm{Sym}(n)}_{(*,\cdots,*,k)}, \quad \begin{array}{c}\sigma \lhd \tau \\ \hline (k,\ldots,n+1)\cdot\sigma \lhd (k,\ldots,n+1)\cdot\tau\end{array}$$

Bruhat order on Sym(n)

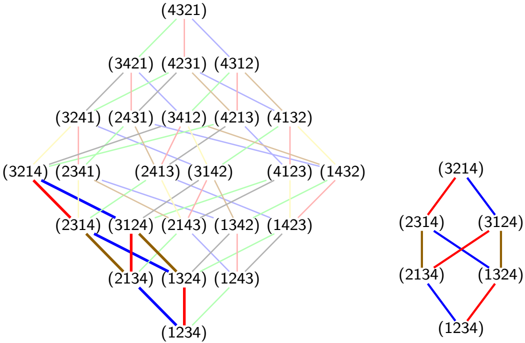
Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$





Bruhat order on Sym(n)

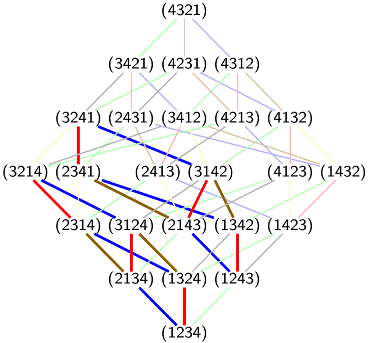
Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

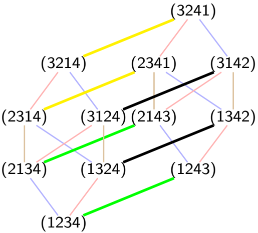


$$\mathrm{Sym}(n+1) = \dot{\bigcup_{1\leq k\leq n}} \underbrace{\mathrm{Sym}(n)\cdot(k,\ldots,n+1)}_{(\ldots *,n+1,*\ldots)}$$

Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

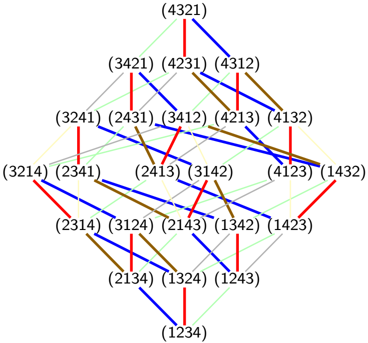


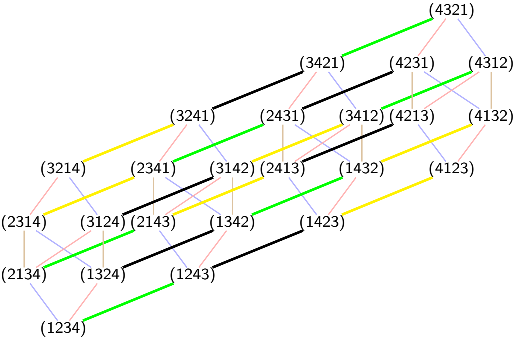


$$\mathrm{Sym}(n+1) = \dot{\bigcup_{1 \leq k \leq n}} \underbrace{\mathrm{Sym}(n) \cdot (k, \ldots, n+1)}_{(\ldots *, n+1, * \ldots)}$$

Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$

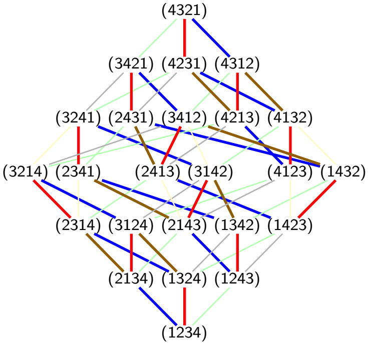


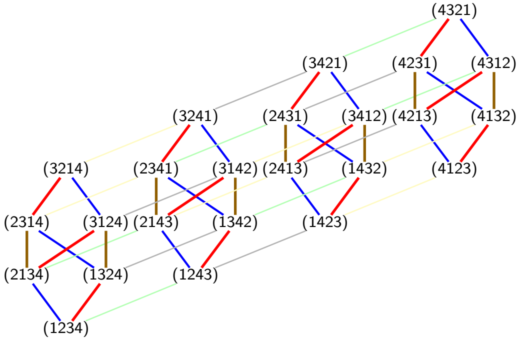


$$\mathrm{Sym}(n+1) = \dot{\bigcup_{1 \leq k \leq n}} \underbrace{\mathrm{Sym}(n) \cdot (k, \ldots, n+1)}_{(\ldots *, n+1, * \ldots)}$$

Bruhat order on Sym(n)

Example. Hasse-diagram on $(\mathrm{Sym}(4), \leqslant)$





$$\mathrm{Sym}(n+1) = \dot{\bigcup_{1\leq k\leq n}} \underbrace{\mathrm{Sym}(n)\cdot(k,\ldots,n+1)}_{(\ldots *, n+1, * \ldots)}, \qquad \begin{array}{c} \sigma \lhd \tau \\ \sigma\cdot(k,\ldots,n+1) \lhd \tau\cdot(k,\ldots,n+1) \end{array}$$

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

Category $\mathcal{O}(\mathfrak{sl}_n)$

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

$$\text{Category } \mathcal{O}(\mathfrak{sl}_n) = \left\{ M \in \text{mod-}\mathfrak{U}(\mathfrak{sl}_n) \right.$$

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

$$\text{Category } \mathcal{O}(\mathfrak{sl}_n) = \left\{ M \in \text{mod-}\mathfrak{U}(\mathfrak{sl}_n) \;\middle|\; M \text{ is finitely generated} \right.$$

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

$$\text{Category } \mathcal{O}(\mathfrak{sl}_n) = \left\{ M \in \text{mod-}\mathfrak{U}(\mathfrak{sl}_n) \;\middle|\; \begin{array}{l} M \text{ is finitely generated} \\ M \text{ is } \mathfrak{h}_n \text{ semisimple} \\ \\ \end{array} \right.$$

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

$$\text{Category } \mathcal{O}(\mathfrak{sl}_n) = \left\{ M \in \text{mod-}\mathfrak{U}(\mathfrak{sl}_n) \;\middle|\; \begin{array}{l} M \text{ is finitely generated} \\ M \text{ is } \mathfrak{h}_n \text{ semisimple} \\ \dim_{\mathbb{C}} \mathfrak{U}((\mathfrak{sl}_n)_+).m < \infty \end{array} \right\}$$

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

$$\text{Category } \mathcal{O}(\mathfrak{sl}_n) = \left\{ M \in \text{mod-}\mathfrak{U}(\mathfrak{sl}_n) \;\middle|\; \begin{array}{l} M \text{ is finitely generated} \\ M \text{ is } \mathfrak{h}_n \text{ semisimple} \\ \dim_{\mathbb{C}} \mathfrak{U}((\mathfrak{sl}_n)_+).m < \infty \end{array} \right\}$$

- $\mathcal{O}(\mathfrak{sl}_n)$ decomposes into blocks $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ for some $\lambda \in \mathfrak{h}_n^*$

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

$$\text{Category } \mathcal{O}(\mathfrak{sl}_n) = \left\{ M \in \text{mod-}\mathfrak{U}(\mathfrak{sl}_n) \;\middle|\; \begin{array}{l} M \text{ is finitely generated} \\ M \text{ is } \mathfrak{h}_n \text{ semisimple} \\ \dim_{\mathbb{C}} \mathfrak{U}((\mathfrak{sl}_n)_+).m < \infty \end{array} \right\}$$

- $\mathcal{O}(\mathfrak{sl}_n)$ decomposes into blocks $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ for some $\lambda \in \mathfrak{h}_n^*$

$$\mathfrak{h}_n^* = \{\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \mid \lambda_i = \lambda(E_{ii} - E_{i+1,i+1}) \in \mathbb{C}\}$$

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

$$\text{Category } \mathcal{O}(\mathfrak{sl}_n) = \left\{ M \in \text{mod-}\mathfrak{U}(\mathfrak{sl}_n) \;\middle|\; \begin{array}{l} M \text{ is finitely generated} \\ M \text{ is } \mathfrak{h}_n \text{ semisimple} \\ \dim_{\mathbb{C}} \mathfrak{U}((\mathfrak{sl}_n)_+).m < \infty \end{array} \right\}$$

- $\mathcal{O}(\mathfrak{sl}_n)$ decomposes into blocks $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ for some $\lambda \in \mathfrak{h}_n^*$

  $$\mathfrak{h}_n^* = \{\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \mid \lambda_i = \lambda(E_{ii} - E_{i+1,i+1}) \in \mathbb{C}\}$$

- $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ is regular and integer

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

$$\text{Category } \mathcal{O}(\mathfrak{sl}_n) = \left\{ M \in \text{mod-}\mathfrak{U}(\mathfrak{sl}_n) \;\middle|\; \begin{array}{l} M \text{ is finitely generated} \\ M \text{ is } \mathfrak{h}_n \text{ semisimple} \\ \dim_{\mathbb{C}} \mathfrak{U}((\mathfrak{sl}_n)_+).m < \infty \end{array} \right\}$$

- $\mathcal{O}(\mathfrak{sl}_n)$ decomposes into blocks $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ for some $\lambda \in \mathfrak{h}_n^*$

  $$\mathfrak{h}_n^* = \{\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \mid \lambda_i = \lambda(E_{ii} - E_{i+1,i+1}) \in \mathbb{C}\}$$

- $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ is regular and integer if $\lambda_i \in \mathbb{N}_0, \; 1 \leq i \leq n-1$

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

$$\text{Category } \mathcal{O}(\mathfrak{sl}_n) = \left\{ M \in \text{mod-}\mathfrak{U}(\mathfrak{sl}_n) \;\middle|\; \begin{array}{l} M \text{ is finitely generated} \\ M \text{ is } \mathfrak{h}_n \text{ semisimple} \\ \dim_{\mathbb{C}} \mathfrak{U}((\mathfrak{sl}_n)_+).m < \infty \end{array} \right\}$$

- $\mathcal{O}(\mathfrak{sl}_n)$ decomposes into blocks $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ for some $\lambda \in \mathfrak{h}_n^*$

  $$\mathfrak{h}_n^* = \{\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \mid \lambda_i = \lambda(E_{ii} - E_{i+1,i+1}) \in \mathbb{C}\}$$
- $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ is regular and integer if $\lambda_i \in \mathbb{N}_0,\ 1 \leq i \leq n-1$
  - $\text{Sym}(n) \leftrightarrow \{\text{indecomposable projectives of } \mathcal{O}_\lambda(\mathfrak{sl}_n)\}$

    $\sigma \qquad\qquad [P(\sigma \cdot \lambda)]$

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

$$\text{Category } \mathcal{O}(\mathfrak{sl}_n) = \left\{ M \in \text{mod-}\mathfrak{U}(\mathfrak{sl}_n) \;\middle|\; \begin{array}{l} M \text{ is finitely generated} \\ M \text{ is } \mathfrak{h}_n \text{ semisimple} \\ \dim_{\mathbb{C}} \mathfrak{U}((\mathfrak{sl}_n)_+).m < \infty \end{array} \right\}$$

- $\mathcal{O}(\mathfrak{sl}_n)$ decomposes into blocks $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ for some $\lambda \in \mathfrak{h}_n^*$

  $$\mathfrak{h}_n^* = \{\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \mid \lambda_i = \lambda(E_{ii} - E_{i+1,i+1}) \in \mathbb{C}\}$$

- $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ is regular and integer if $\lambda_i \in \mathbb{N}_0, \; 1 \leq i \leq n-1$
  - $\text{Sym}(n) \leftrightarrow \{\text{indecomposable projectives of } \mathcal{O}_\lambda(\mathfrak{sl}_n)\}$

    $\sigma \qquad\qquad [P(\sigma \cdot \lambda)]$
  - $\text{End}_{\mathfrak{sl}_n}\left(\bigoplus_{\sigma \in \text{Sym}(n)} P(\sigma \cdot \lambda)\right)$

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

$$\text{Category } \mathcal{O}(\mathfrak{sl}_n) = \left\{ M \in \text{mod-}\mathfrak{U}(\mathfrak{sl}_n) \;\middle|\; \begin{array}{l} M \text{ is finitely generated} \\ M \text{ is } \mathfrak{h}_n \text{ semisimple} \\ \dim_{\mathbb{C}} \mathfrak{U}((\mathfrak{sl}_n)_+).m < \infty \end{array} \right\}$$

- $\mathcal{O}(\mathfrak{sl}_n)$ decomposes into blocks $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ for some $\lambda \in \mathfrak{h}_n^*$

  $$\mathfrak{h}_n^* = \{\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \mid \lambda_i = \lambda(E_{ii} - E_{i+1,i+1}) \in \mathbb{C}\}$$
- $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ is regular and integer if $\lambda_i \in \mathbb{N}_0, \;\; 1 \leq i \leq n-1$
  - $\mathrm{Sym}(n) \leftrightarrow \{\text{indecomposable projectives of } \mathcal{O}_\lambda(\mathfrak{sl}_n)\}$

    $\sigma \qquad\qquad [P(\sigma \cdot \lambda)]$
  - $A(n) := \mathrm{End}_{\mathfrak{sl}_n}\left(\bigoplus_{\sigma \in \mathrm{Sym}(n)} P(\sigma \cdot \lambda)\right)$

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

$$\text{Category } \mathcal{O}(\mathfrak{sl}_n) = \left\{ M \in \text{mod-}\mathfrak{U}(\mathfrak{sl}_n) \;\middle|\; \begin{array}{l} M \text{ is finitely generated} \\ M \text{ is } \mathfrak{h}_n \text{ semisimple} \\ \dim_{\mathbb{C}} \mathfrak{U}((\mathfrak{sl}_n)_+).m < \infty \end{array} \right\}$$

- $\mathcal{O}(\mathfrak{sl}_n)$ decomposes into blocks $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ for some $\lambda \in \mathfrak{h}_n^*$

  $$\mathfrak{h}_n^* = \{\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \mid \lambda_i = \lambda(E_{ii} - E_{i+1,i+1}) \in \mathbb{C}\}$$

- $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ is regular and integer if $\lambda_i \in \mathbb{N}_0, \; 1 \leq i \leq n-1$
  - $\text{Sym}(n) \leftrightarrow \{\text{indecomposable projectives of } \mathcal{O}_\lambda(\mathfrak{sl}_n)\}$
    $\sigma \qquad\qquad [P(\sigma \cdot \lambda)]$
  - $A(n) := \text{End}_{\mathfrak{sl}_n}\left(\bigoplus_{\sigma \in \text{Sym}(n)} P(\sigma \cdot \lambda)\right)$

$$\boxed{\mathcal{O}_\lambda(\mathfrak{sl}_n) \sim \text{mod} - A(n)}$$

Regular and integer block of $\mathcal{O}(\mathfrak{sl}_n)$

Definition (Bernstein, Gelfand, Gelfand)

$$\text{Category } \mathcal{O}(\mathfrak{sl}_n) = \left\{ M \in \text{mod-}\mathfrak{U}(\mathfrak{sl}_n) \;\middle|\; \begin{array}{l} M \text{ is finitely generated} \\ M \text{ is } \mathfrak{h}_n \text{ semisimple} \\ \dim_{\mathbb{C}} \mathfrak{U}((\mathfrak{sl}_n)_+).m < \infty \end{array} \right\}$$

- $\mathcal{O}(\mathfrak{sl}_n)$ decomposes into blocks $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ for some $\lambda \in \mathfrak{h}_n^*$
  $$\mathfrak{h}_n^* = \{\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \mid \lambda_i = \lambda(E_{ii} - E_{i+1,i+1}) \in \mathbb{C}\}$$
- $\mathcal{O}_\lambda(\mathfrak{sl}_n)$ is regular and integer if $\lambda_i \in \mathbb{N}_0, \; 1 \le i \le n-1$
  - $\text{Sym}(n) \leftrightarrow \{\text{indecomposable projectives of } \mathcal{O}_\lambda(\mathfrak{sl}_n)\}$
    $\sigma \qquad\qquad [P(\sigma \cdot \lambda)]$
  - $A(n) := \text{End}_{\mathfrak{sl}_n}\left(\bigoplus_{\sigma \in \text{Sym}(n)} P(\sigma \cdot \lambda)\right) = \mathbb{C}Q(n)/\mathcal{I}(n)$

$$\boxed{\mathcal{O}_\lambda(\mathfrak{sl}_n) \quad \sim \quad \text{mod} - A(n)}$$

The quiver $Q(n)$ of $A(n)$

The quiver $Q(n)$ of $A(n)$

$$\boxed{Q(n) \quad = \quad (Q_0(n),\ Q_1(n))}$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$

The quiver $Q(n)$ of $A(n)$

$$\boxed{Q(n) \;=\; (Q_0(n),\; Q_1(n))}$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$
- $Q_1(n) \leftrightsquigarrow$ coefficients $\mu(\sigma, \tau)$ of Kazhdan-Lusztig polynomials

The quiver $Q(n)$ of $A(n)$

$$\boxed{Q(n) \quad = \quad (Q_0(n),\ Q_1(n))}$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$
- $Q_1(n) \leftrightsquigarrow$ coefficients $\mu(\sigma, \tau)$ of Kazhdan-Lusztig polynomials

  $\sigma, \tau \in Q_0(n)$

The quiver $Q(n)$ of $A(n)$

$$\boxed{Q(n) \;=\; (Q_0(n),\ Q_1(n))}$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$
- $Q_1(n) \leftrightsquigarrow$ coefficients $\mu(\sigma, \tau)$ of Kazhdan-Lusztig polynomials

$$\sigma, \tau \in Q_0(n) \;\Rightarrow\; \mu(\sigma, \tau) \;=\; \left|\left\{\beta \in Q_1(n) \mid \sigma \xrightarrow{\beta} \tau\right\}\right|$$

The quiver $Q(n)$ of $A(n)$

$$\boxed{Q(n) \;=\; (Q_0(n),\ Q_1(n))}$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$
- $Q_1(n) \leftrightsquigarrow$ coefficients $\mu(\sigma,\tau)$ of Kazhdan-Lusztig polynomials

$$\sigma,\tau \in Q_0(n) \;\Rightarrow\; \mu(\sigma,\tau) \;=\; \left|\left\{\beta \in Q_1(n) \mid \sigma \xrightarrow{\beta} \tau\right\}\right| \;=\; \mu(\tau,\sigma)$$

The quiver $Q(n)$ of $A(n)$

$$\boxed{Q(n) \;=\; (Q_0(n),\; Q_1(n))}$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$
- $Q_1(n) \leftrightsquigarrow$ coefficients $\mu(\sigma,\tau)$ of Kazhdan-Lusztig polynomials

$$\sigma,\tau \in Q_0(n) \;\Rightarrow\; \mu(\sigma,\tau) \;=\; \left|\left\{\beta \in Q_1(n) \mid \sigma \xrightarrow{\beta} \tau\right\}\right| \;=\; \mu(\tau,\sigma)$$

$$=$$

The quiver $Q(n)$ of $A(n)$

$$\boxed{Q(n) \quad = \quad (Q_0(n),\ Q_1(n))}$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$
- $Q_1(n) \leftrightsquigarrow$ coefficients $\mu(\sigma,\tau)$ of Kazhdan-Lusztig polynomials

$$\begin{aligned} \sigma,\tau \in Q_0(n) \quad \Rightarrow \quad \mu(\sigma,\tau) \quad &= \quad \left|\left\{\beta \in Q_1(n) \mid \sigma \xrightarrow{\beta} \tau\right\}\right| \\ &= \quad \mu(\tau,\sigma) \\ &= \quad \begin{cases} 1 & \text{if } \sigma \lhd \tau \quad \text{(Bruhat order),} \\ & \end{cases} \end{aligned}$$

The quiver $Q(n)$ of $A(n)$

$$\boxed{Q(n) \;=\; (Q_0(n),\; Q_1(n))}$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$
- $Q_1(n) \leftrightsquigarrow$ coefficients $\mu(\sigma,\tau)$ of Kazhdan-Lusztig polynomials

$$\sigma,\tau \in Q_0(n) \;\Rightarrow\; \mu(\sigma,\tau) \;=\; \left|\left\{\beta \in Q_1(n) \mid \sigma \xrightarrow{\beta} \tau\right\}\right| \;=\; \mu(\tau,\sigma) \;=\; \begin{cases} 1 & \text{if } \sigma \lhd \tau \;\; \text{(Bruhat order)}, \\ 0 & \text{if } \sigma \text{ and } \tau \text{ are incomparable}, \end{cases}$$

The quiver $Q(n)$ of $A(n)$

$$\boxed{Q(n) \;=\; (Q_0(n),\ Q_1(n))}$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$
- $Q_1(n) \leftrightsquigarrow$ coefficients $\mu(\sigma,\tau)$ of Kazhdan-Lusztig polynomials

$$\sigma,\tau \in Q_0(n) \;\Rightarrow\; \mu(\sigma,\tau) \;=\; \left|\left\{\beta \in Q_1(n) \mid \sigma \xrightarrow{\beta} \tau\right\}\right| \;=\; \mu(\tau,\sigma) \;=\; \begin{cases} 1 & \text{if } \sigma \lhd \tau \quad \text{(Bruhat order)},\\ 0 & \text{if } \sigma \text{ and } \tau \text{ are incomparable},\\ ? & \text{in general.} \end{cases}$$

The quiver $Q(n)$ of $A(n)$

$$\boxed{Q(n) \;=\; (Q_0(n),\ Q_1(n))}$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$
- $Q_1(n) \leftrightsquigarrow$ coefficients $\mu(\sigma,\tau)$ of Kazhdan-Lusztig polynomials

$$\sigma,\tau \in Q_0(n) \;\Rightarrow\; \mu(\sigma,\tau) = \left|\left\{\beta \in Q_1(n) \mid \sigma \xrightarrow{\beta} \tau\right\}\right|$$

$$= \mu(\tau,\sigma)$$

$$= \begin{cases} 1 & \text{if } \sigma \lhd \tau \quad \text{(Bruhat order)}, \\ 0 & \text{if } \sigma \text{ and } \tau \text{ are incomparable}, \\ ? & \text{in general.} \end{cases}$$

$$= \mu(\Phi_n(\sigma), \Phi_n(\tau))$$

The quiver $Q(n)$ of $A(n)$

$$\boxed{Q(n) \;=\; (Q_0(n),\; Q_1(n))}$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$
- $Q_1(n) \leftrightsquigarrow$ coefficients $\mu(\sigma, \tau)$ of Kazhdan-Lusztig polynomials

$$\sigma, \tau \in Q_0(n) \;\Rightarrow\; \mu(\sigma,\tau) \begin{array}[t]{cl} = & \left|\left\{\beta \in Q_1(n) \mid \sigma \xrightarrow{\beta} \tau\right\}\right| \\ = & \mu(\tau,\sigma) \\ = & \begin{cases} 1 & \text{if } \sigma \lhd \tau \;\; \text{(Bruhat order)}, \\ 0 & \text{if } \sigma \text{ and } \tau \text{ are incomparable}, \\ ? & \text{in general.} \end{cases} \\ = & \mu(\Phi_n(\sigma), \Phi_n(\tau)) \\ = & \mu(U_n(\sigma), U_n(\tau)) \end{array}$$

The quiver $Q(n)$ of $A(n)$

$$\boxed{Q(n) \;=\; (Q_0(n),\; Q_1(n))}$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$
- $Q_1(n) \leftrightsquigarrow$ coefficients $\mu(\sigma,\tau)$ of Kazhdan-Lusztig polynomials

$$\sigma,\tau \in Q_0(n) \;\Rightarrow\; \mu(\sigma,\tau) \;=\; \left|\left\{\beta \in Q_1(n) \mid \sigma \xrightarrow{\beta} \tau\right\}\right| \;=\; \mu(\tau,\sigma) \;=\; \begin{cases} 1 & \text{if } \sigma \lhd \tau \quad \text{(Bruhat order)}, \\ 0 & \text{if } \sigma \text{ and } \tau \text{ are incomparable}, \\ ? & \text{in general.} \end{cases} \;=\; \mu(\Phi_n(\sigma),\Phi_n(\tau)) \;=\; \mu(U_n(\sigma),U_n(\tau))$$

- $Q_1(n) \quad \rightsquigarrow \quad Q_1(n+1)$

The quiver $Q(n)$ of $A(n)$

$$\boxed{Q(n) \quad = \quad (Q_0(n),\ Q_1(n))}$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$
- $Q_1(n) \leftrightsquigarrow$ coefficients $\mu(\sigma,\tau)$ of Kazhdan-Lusztig polynomials

$$\begin{aligned} \sigma,\tau \in Q_0(n) \quad \Rightarrow \quad \mu(\sigma,\tau) \quad &= \quad \left|\left\{\beta \in Q_1(n) \mid \sigma \xrightarrow{\beta} \tau\right\}\right| \\ &= \quad \mu(\tau,\sigma) \\ &= \quad \begin{cases} 1 & \text{if } \sigma \lhd \tau \quad \text{(Bruhat order)}, \\ 0 & \text{if } \sigma \text{ and } \tau \text{ are incomparable}, \\ ? & \text{in general.} \end{cases} \\ &= \quad \mu(\Phi_n(\sigma),\Phi_n(\tau)) \\ &= \quad \mu(U_n(\sigma),U_n(\tau)) \end{aligned}$$

- $Q_1(n) \quad \rightsquigarrow \quad Q_1(n+1)$

  $\mu(\sigma,\tau)$

The quiver $Q(n)$ of $A(n)$

$$Q(n) \;=\; (Q_0(n),\; Q_1(n))$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$
- $Q_1(n) \leftrightsquigarrow$ coefficients $\mu(\sigma,\tau)$ of Kazhdan-Lusztig polynomials

$$\sigma,\tau \in Q_0(n) \;\Rightarrow\; \begin{aligned} \mu(\sigma,\tau) &= \left|\left\{\beta \in Q_1(n) \mid \sigma \xrightarrow{\beta} \tau\right\}\right| \\ &= \mu(\tau,\sigma) \\ &= \begin{cases} 1 & \text{if } \sigma \lhd \tau \;\; \text{(Bruhat order)}, \\ 0 & \text{if } \sigma \text{ and } \tau \text{ are incomparable}, \\ ? & \text{in general.} \end{cases} \\ &= \mu(\Phi_n(\sigma), \Phi_n(\tau)) \\ &= \mu(U_n(\sigma), U_n(\tau)) \end{aligned}$$

- $Q_1(n) \;\rightsquigarrow\; Q_1(n+1)$

$$\mu(\sigma,\tau) \;=\; \mu((k,\ldots,n+1)\cdot\sigma, (k,\ldots,n+1)\cdot\tau)$$

The quiver $Q(n)$ of $A(n)$

$$\boxed{Q(n) \quad = \quad (Q_0(n),\ Q_1(n))}$$

- $Q_0(n) \longleftrightarrow \mathrm{Sym}(n)$
- $Q_1(n) \leftrightsquigarrow$ coefficients $\mu(\sigma,\tau)$ of Kazhdan-Lusztig polynomials

$$\begin{aligned}
\sigma,\tau \in Q_0(n) \quad \Rightarrow \quad \mu(\sigma,\tau) \quad &= \quad \left|\left\{\beta \in Q_1(n) \mid \sigma \xrightarrow{\beta} \tau\right\}\right| \\
&= \quad \mu(\tau,\sigma) \\
&= \quad \begin{cases} 1 & \text{if } \sigma \lhd \tau \quad \text{(Bruhat order)}, \\ 0 & \text{if } \sigma \text{ and } \tau \text{ are incomparable}, \\ ? & \text{in general.} \end{cases} \\
&= \quad \mu(\Phi_n(\sigma), \Phi_n(\tau)) \\
&= \quad \mu(U_n(\sigma), U_n(\tau))
\end{aligned}$$

- $Q_1(n) \quad \rightsquigarrow \quad Q_1(n+1)$

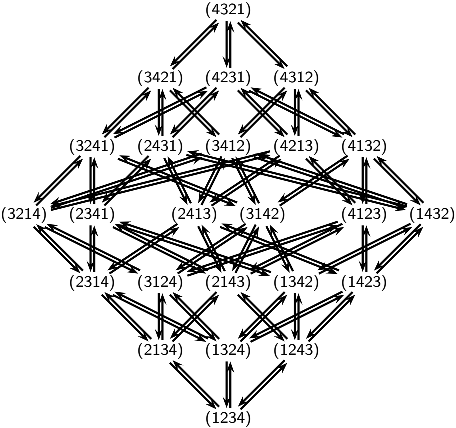
$$\begin{aligned}
\mu(\sigma,\tau) \quad &= \quad \mu((k,\ldots,n+1)\cdot\sigma, (k,\ldots,n+1)\cdot\tau) \\
&= \quad \mu(\sigma\cdot(k,\ldots,n+1), \tau\cdot(k,\ldots,n+1))
\end{aligned}$$

Example.

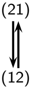
Example. The quiver $Q(n)$ of $A(n)$ for $n = 2, 3, 4$

Example. The quiver $Q(n)$ of $A(n)$ for $n = 2, 3, 4$

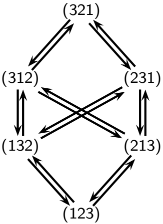
$Q(4)$



$Q(2)$

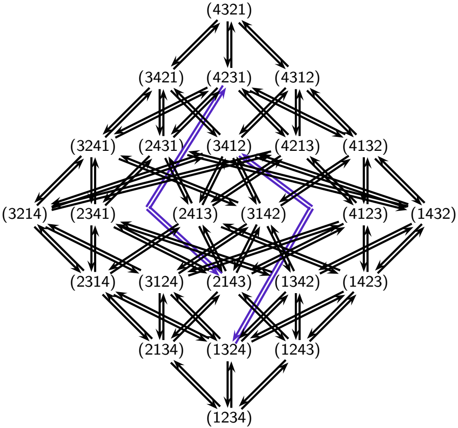


$Q(3)$

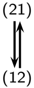


Example. The quiver $Q(n)$ of $A(n)$ for $n = 2, 3, 4$

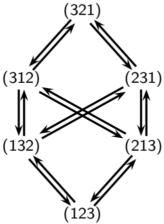
$Q(4)$



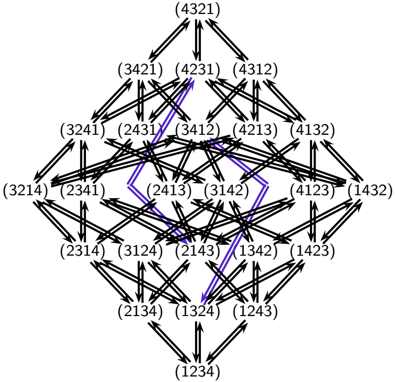
$Q(2)$



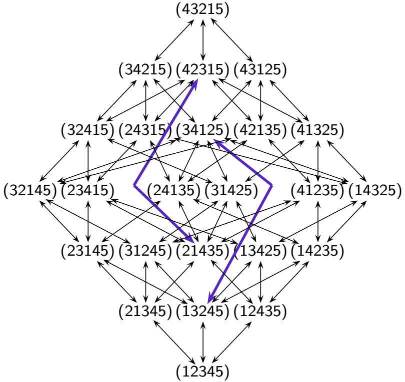
$Q(3)$



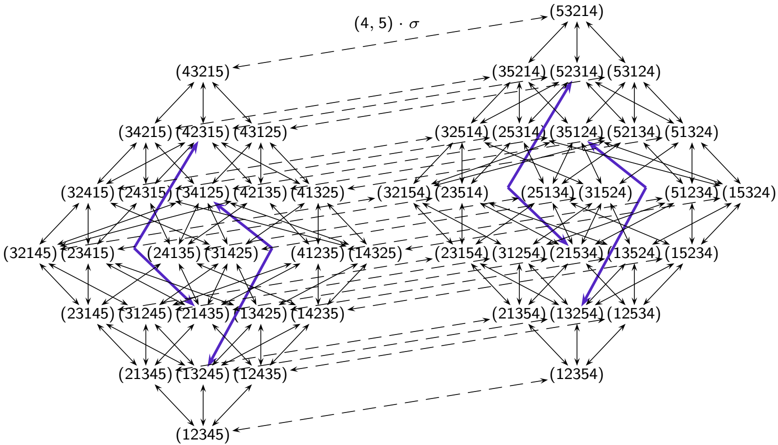
Example. The quiver $Q(5)$ of $A(5)$ from $Q(4)$



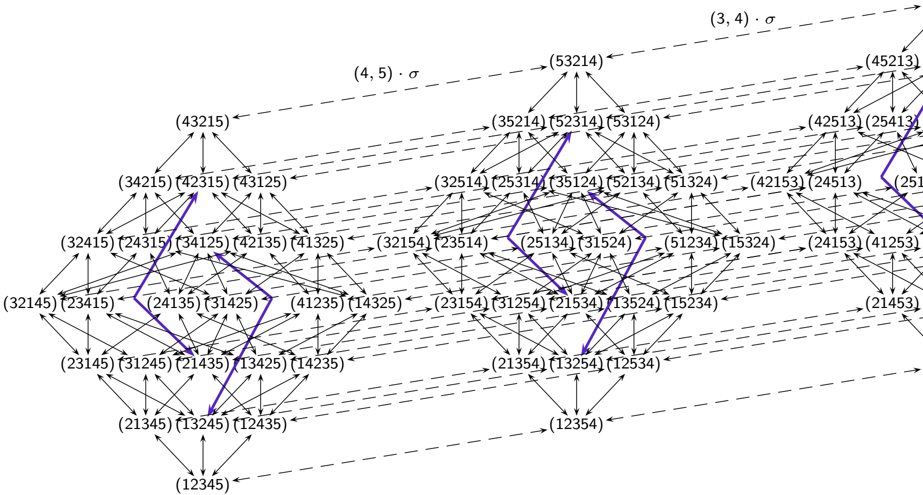
Example. The quiver $Q(5)$ of $A(5)$ from $Q(4)$



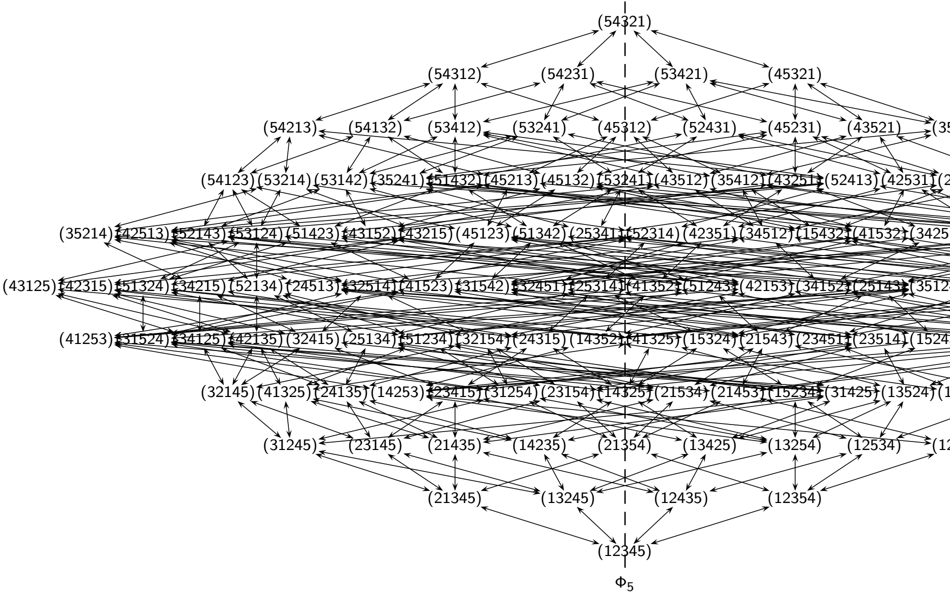
Example. The quiver $Q(5)$ of $A(5)$ from $Q(4)$



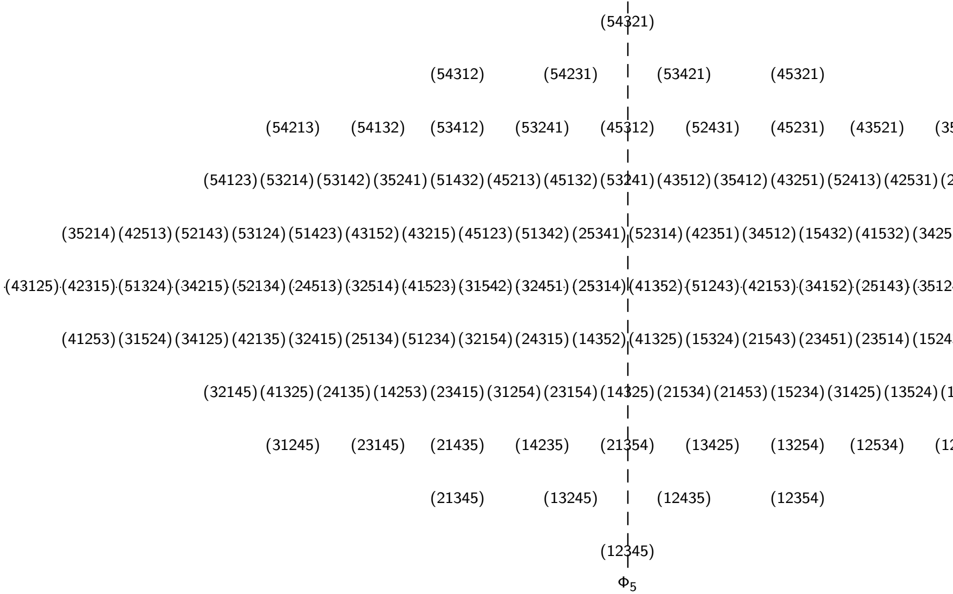
Example. The quiver $Q(5)$ of $A(5)$ from $Q(4)$



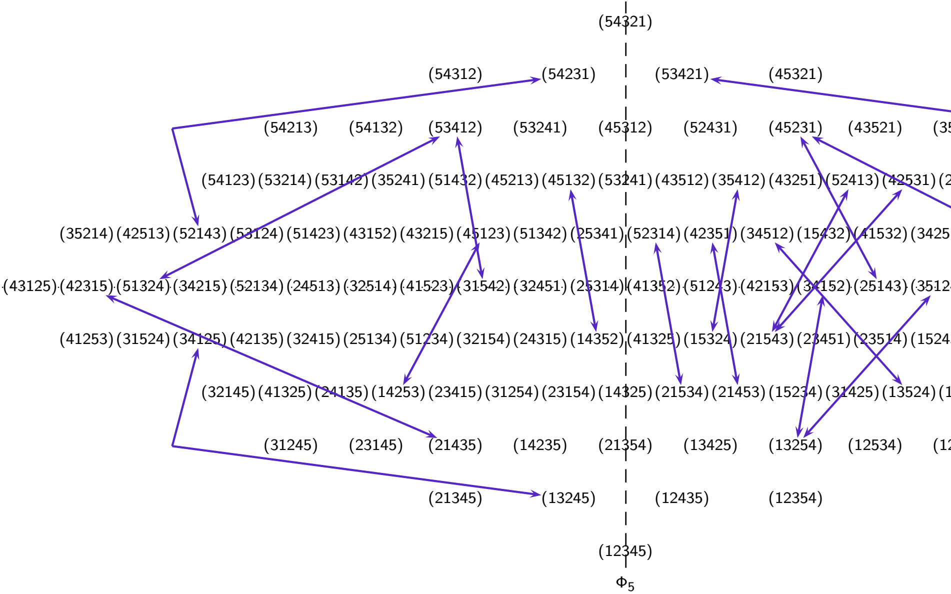
Example. The quiver $Q(5)$ of $A(5)$ from $Q(4)$



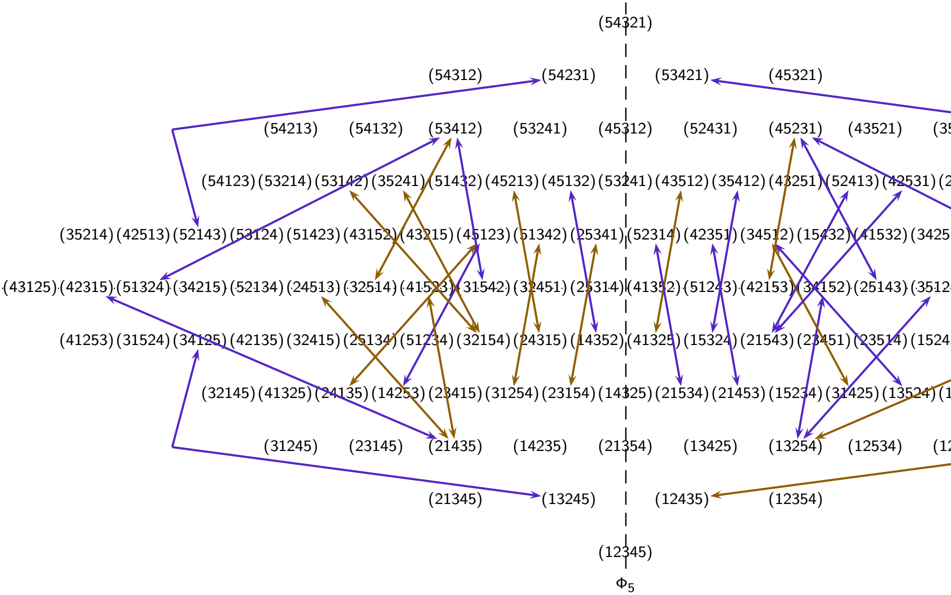
Example. The quiver $Q(5)$ of $A(5)$ from $Q(4)$



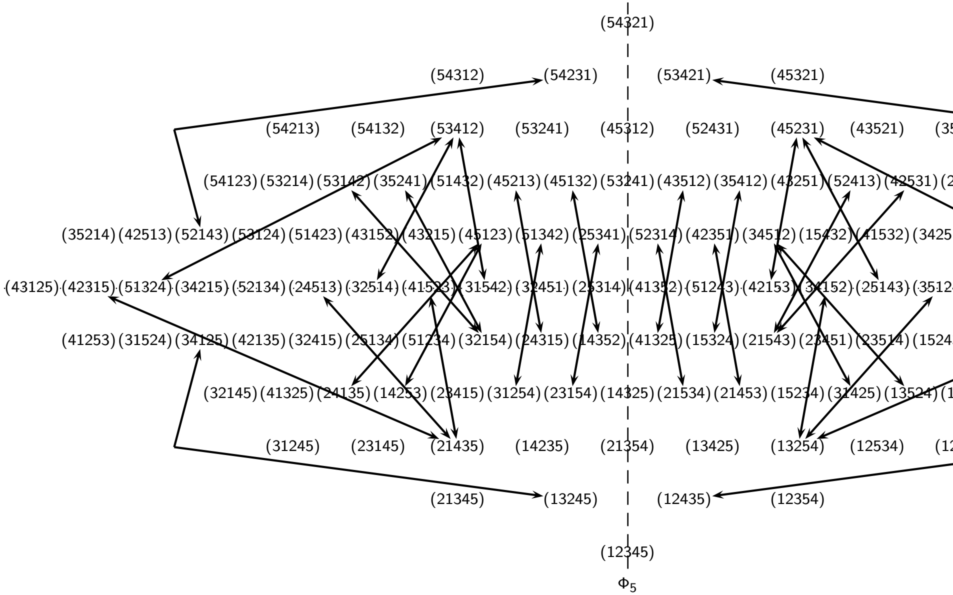
Example. The quiver $Q(5)$ of $A(5)$ from $Q(4)$



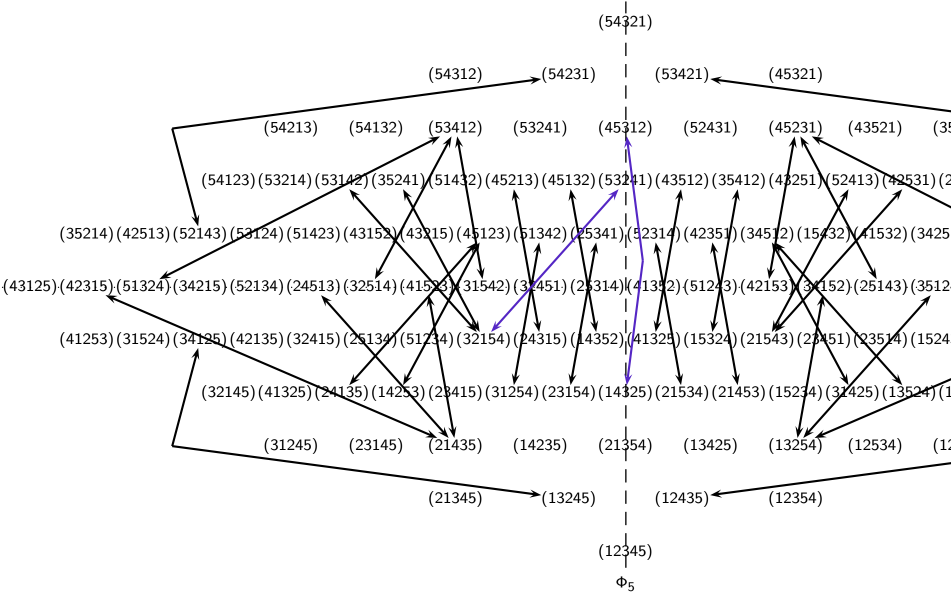
Example. The quiver $Q(5)$ of $A(5)$ from $Q(4)$



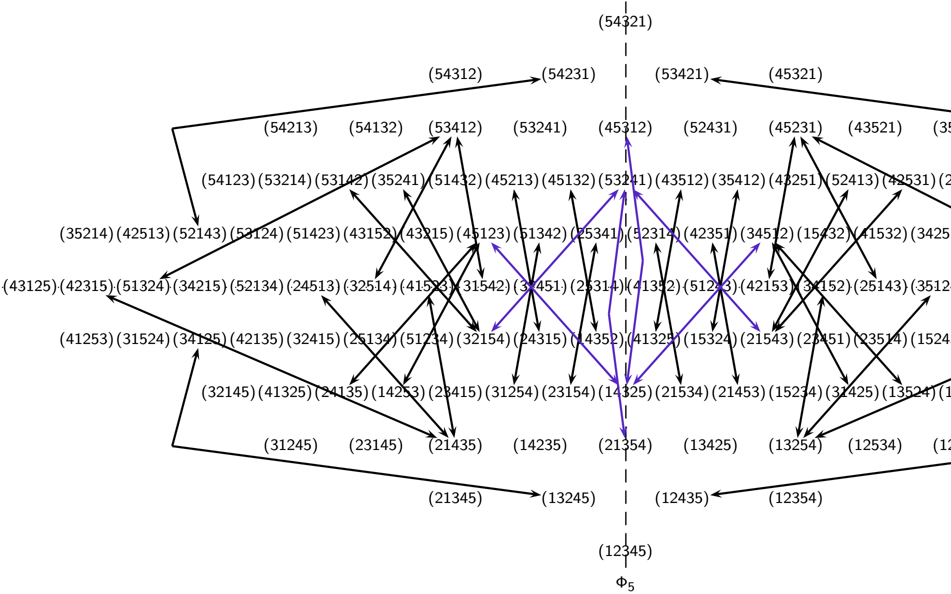
Example. The quiver $Q(5)$ of $A(5)$ from $Q(4)$



Example. The quiver $Q(5)$ of $A(5)$ from $Q(4)$



Example. The quiver $Q(5)$ of $A(5)$ from $Q(4)$



Relations of $A(n)$

Relations of $A(n)$

Relations of $A(n)$

$$\boxed{A(n) \text{ is quadratic } \;\Rightarrow\; \mathcal{I}(n) = \langle \{\rho \mid \rho = \sum_i c_i(\sigma \to \nu_i \to \tau)\} \rangle}$$

Relations of $A(n)$

$$\boxed{A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle \{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\} \rangle}$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$

Relations of $A(n)$

$$\boxed{A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle \{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\} \rangle}$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\}$

Relations of $A(n)$

$$\boxed{A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle\{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\}\rangle}$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

Relations of $A(n)$

$$\boxed{A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle \{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\} \rangle}$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

- ν_i is a neighbour of σ and τ for any i

Relations of $A(n)$

$$\boxed{A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle \{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\}\rangle}$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

- ν_i is a neighbour of σ and τ for any i

- ν_i not a neighbour of σ or τ for some i

Relations of $A(n)$

$$\boxed{A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle \{\rho \mid \rho = \sum_i c_i(\sigma \to \nu_i \to \tau)\} \rangle}$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

- ν_i is a neighbour of σ and τ for any i

    1. $|l(\sigma) - l(\tau)| = 0$

- ν_i not a neighbour of σ or τ for some i

Relations of $A(n)$

$$\boxed{A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle \{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\}\rangle}$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

- ν_i is a neighbour of σ and τ for any i

  1. $|I(\sigma) - I(\tau)| = 0$
  2. $|I(\sigma) - I(\tau)| = 2$

- ν_i not a neighbour of σ or τ for some i

Relations of $A(n)$

$$A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle \{\rho \mid \rho = \sum_i c_i(\sigma \to \nu_i \to \tau)\} \rangle$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

- ν_i is a neighbour of σ and τ for any i

  1. $|l(\sigma) - l(\tau)| = 0$

  2. $|l(\sigma) - l(\tau)| = 2$

  1.1. $\sigma = \tau$

- ν_i not a neighbour of σ or τ for some i

Relations of $A(n)$

$$\boxed{A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle \{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\} \rangle}$$

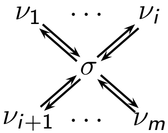
Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

- ν_i is a neighbour of σ and τ for any i

  1. $|I(\sigma) - I(\tau)| = 0$

  2. $|I(\sigma) - I(\tau)| = 2$

  1.1. $\sigma = \tau$



- ν_i not a neighbour of σ or τ for some i

Relations of $A(n)$

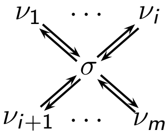
$$\boxed{A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle \{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\} \rangle}$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

- ν_i is a neighbour of σ and τ for any i

  1. $|l(\sigma) - l(\tau)| = 0$ **2.** $|l(\sigma) - l(\tau)| = 2$

  1.1. $\sigma = \tau$



  1.2. $\sigma \neq \tau$

- ν_i not a neighbour of σ or τ for some i

Relations of $A(n)$

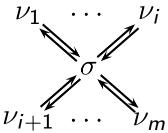
$$\boxed{A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle \{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\} \rangle}$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

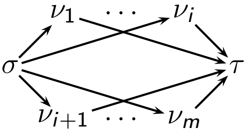
- ν_i is a neighbour of σ and τ for any i

1. $|l(\sigma) - l(\tau)| = 0$

1.1. $\sigma = \tau$



1.2. $\sigma \neq \tau$



2. $|l(\sigma) - l(\tau)| = 2$

- ν_i not a neighbour of σ or τ for some i

Relations of $A(n)$

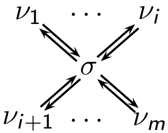
$$\boxed{A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle \{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\} \rangle}$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

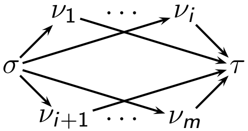
- ν_i is a neighbour of σ and τ for any i

1. $|l(\sigma) - l(\tau)| = 0$

1.1. $\sigma = \tau$

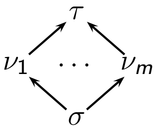


1.2. $\sigma \neq \tau$



2. $|l(\sigma) - l(\tau)| = 2$

$\sigma < \tau$



- ν_i not a neighbour of σ or τ for some i

Relations of $A(n)$

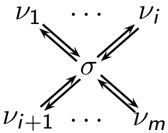
$$\boxed{A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle\{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\}\rangle}$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

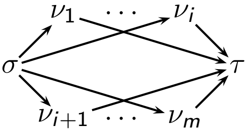
- ν_i is a neighbour of σ and τ for any i

  1. $|l(\sigma) - l(\tau)| = 0$

  1.1. $\sigma = \tau$

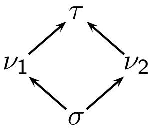


  1.2. $\sigma \neq \tau$



  2. $|l(\sigma) - l(\tau)| = 2$

  $\sigma < \tau$



- ν_i not a neighbour of σ or τ for some i

Relations of $A(n)$

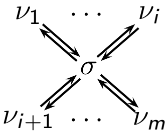
$$\boxed{A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle \{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\}\rangle}$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

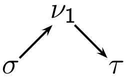
- ν_i is a neighbour of σ and τ for any i

1. $|l(\sigma) - l(\tau)| = 0$

1.1. $\sigma = \tau$

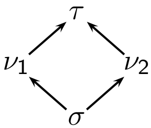


1.2.



2. $|l(\sigma) - l(\tau)| = 2$

$\sigma < \tau$



- ν_i not a neighbour of σ or τ for some i

Relations of $A(n)$

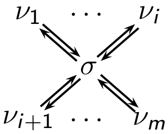
$$\boxed{A(n) \text{ is quadratic } \Rightarrow \ \mathcal{I}(n) = \langle \{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\} \rangle}$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

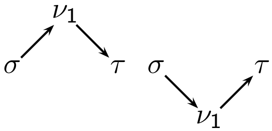
- ν_i is a neighbour of σ and τ for any i

1. $|l(\sigma) - l(\tau)| = 0$

1.1. $\sigma = \tau$

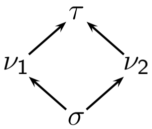


1.2.



2. $|l(\sigma) - l(\tau)| = 2$

$\sigma < \tau$



- ν_i not a neighbour of σ or τ for some i

Relations of $A(n)$

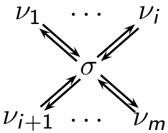
$$A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle \{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\}\rangle$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \dots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

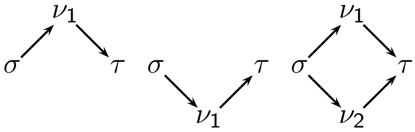
- ν_i is a neighbour of σ and τ for any i

1. $|l(\sigma) - l(\tau)| = 0$

1.1. $\sigma = \tau$

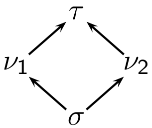


1.2.



2. $|l(\sigma) - l(\tau)| = 2$

$\sigma < \tau$



- ν_i not a neighbour of σ or τ for some i

Relations of $A(n)$

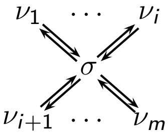
$$\boxed{A(n) \text{ is quadratic } \Rightarrow \mathcal{I}(n) = \langle \{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\}\rangle}$$

Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \ldots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

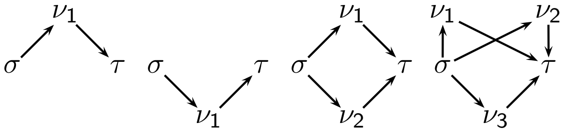
- ν_i is a neighbour of σ and τ for any i

1. $|l(\sigma) - l(\tau)| = 0$

1.1. $\sigma = \tau$

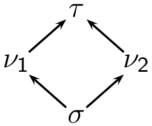


1.2.



2. $|l(\sigma) - l(\tau)| = 2$

$\sigma < \tau$



- ν_i not a neighbour of σ or τ for some i

Relations of $A(n)$

$$\boxed{A(n) \text{ is quadratic} \;\Rightarrow\; \mathcal{I}(n) = \langle \{\rho \mid \rho = \textstyle\sum_i c_i(\sigma \to \nu_i \to \tau)\}\rangle}$$

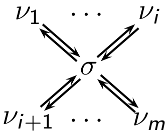
Let $\sigma, \tau \in \mathrm{Sym}(n)$ and $\{\nu_1, \dots, \nu_m\} = \{\nu \mid \sigma \to \nu \to \tau \in Q(n)\} \neq \emptyset$

- ν_i is a neighbour of σ and τ for any i

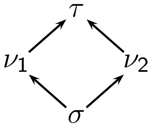
1. $|l(\sigma) - l(\tau)| = 0$

2. $|l(\sigma) - l(\tau)| = 2$

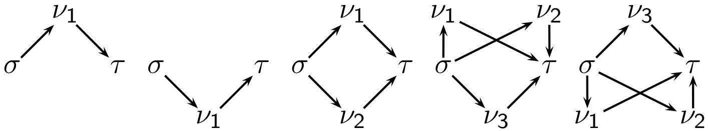
1.1. $\sigma = \tau$



$\sigma < \tau$



1.2.

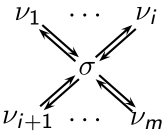


- ν_i not a neighbour of σ or τ for some i

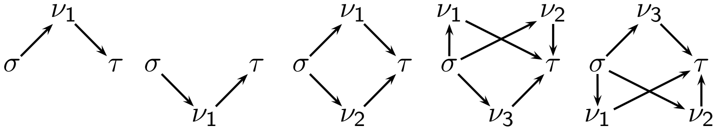
Relations of $A(n)$

1. $|l(\sigma) - l(\tau)| = 0$

1.1. $\sigma = \tau$



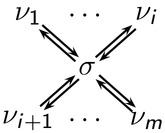
1.2.



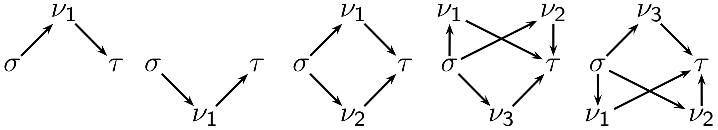
Relations of $A(n)$

1. $|l(\sigma) - l(\tau)| = 0$

1.1. $\sigma = \tau$



1.2.

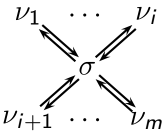


Lemma

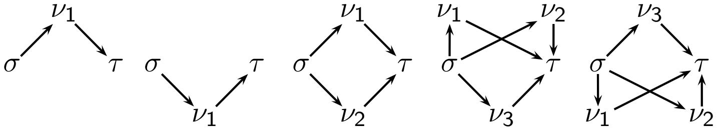
Relations of $A(n)$

1. $|l(\sigma) - l(\tau)| = 0$

1.1. $\sigma = \tau$



1.2.



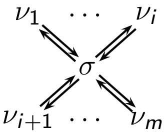
Lemma

The paths $\{\sigma \to \upsilon \to \tau \mid \upsilon \lhd \sigma, \tau\}$ *are linearly independent in* $A(n)$.

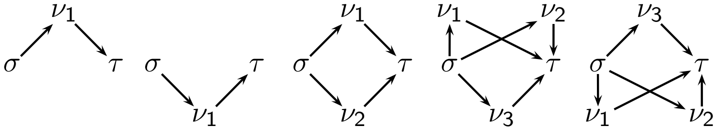
Relations of $A(n)$

1. $|l(\sigma) - l(\tau)| = 0$

1.1. $\sigma = \tau$



1.2.



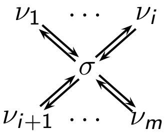
Lemma

The paths $\{\sigma \to \upsilon \to \tau \mid \upsilon \lhd \sigma, \tau\}$ *are linearly independent in* $A(n)$*.*
Let $\nu \in \mathrm{Sym}(n)$ *with* $\sigma, \tau \lhd \nu$

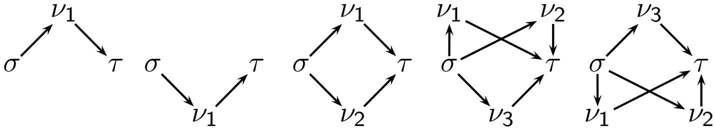
Relations of $A(n)$

1. $|l(\sigma) - l(\tau)| = 0$

1.1. $\sigma = \tau$



1.2.



Lemma

The paths $\{\sigma \to \upsilon \to \tau \mid \upsilon \lhd \sigma, \tau\}$ *are linearly independent in* $A(n)$*.* *Let* $\nu \in \mathrm{Sym}(n)$ *with* $\sigma, \tau \lhd \nu$*, then*

$$(\sigma \to \nu \to \tau) - \sum_{\sigma,\tau \rhd \upsilon} c_\upsilon (\sigma \to \upsilon \to \tau) \in \mathcal{I}(n)$$

Relations of $A(n)$

Lemma

Relations of $A(n)$

Lemma

Let $\sigma, \nu_i, \tau \in \mathrm{Sym}(n)$ *and* $\sum c_i(\sigma \to \nu_i \to \tau) \in \mathcal{I}(n)$.

Relations of $A(n)$

Lemma

Let $\sigma, \nu_i, \tau \in \mathrm{Sym}(n)$ *and* $\sum c_i(\sigma \to \nu_i \to \tau) \in \mathcal{I}(n)$. *Then*

- $\sum c_i((\sigma, n+1) \to (\nu_i, n+1) \to (\tau, n+1)) \in \mathcal{I}(n+1)$

Relations of $A(n)$

Lemma

Let $\sigma, \nu_i, \tau \in \mathrm{Sym}(n)$ *and* $\sum c_i(\sigma \to \nu_i \to \tau) \in \mathcal{I}(n)$. *Then*

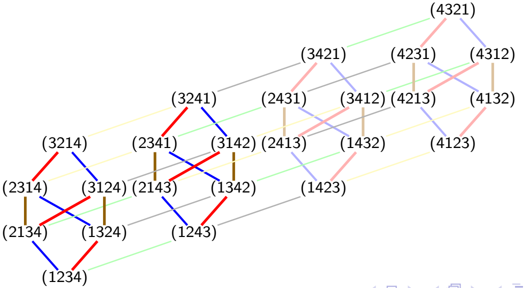
- $\sum c_i((\sigma, n+1) \to (\nu_i, n+1) \to (\tau, n+1)) \in \mathcal{I}(n+1)$
- $\sum c_i(\sigma \cdot (n, n+1) \to \nu_i \cdot (n, n+1) \to \tau \cdot (n, n+1)) \in \mathcal{I}(n+1)$

Relations of $A(n)$

Lemma

Let $\sigma, \nu_i, \tau \in \mathrm{Sym}(n)$ *and* $\sum c_i(\sigma \to \nu_i \to \tau) \in \mathcal{I}(n)$*. Then*

- $\sum c_i((\sigma, n+1) \to (\nu_i, n+1) \to (\tau, n+1)) \in \mathcal{I}(n+1)$
- $\sum c_i(\sigma \cdot (n, n+1) \to \nu_i \cdot (n, n+1) \to \tau \cdot (n, n+1)) \in \mathcal{I}(n+1)$

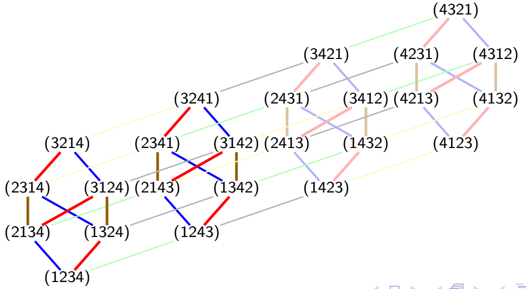


Relations of $A(n)$

Lemma

Let $\sigma, \nu_i, \tau \in \mathrm{Sym}(n)$ *and* $\sum c_i(\sigma \to \nu_i \to \tau) \in \mathcal{I}(n)$. *Then*

- $\sum c_i((\sigma, n+1) \to (\nu_i, n+1) \to (\tau, n+1)) \in \mathcal{I}(n+1)$
- $\sum c_i(\sigma\cdot(n, n+1) \to \nu_i\cdot(n, n+1) \to \tau\cdot(n, n+1)) \in \mathcal{I}(n+1)$
- $\sigma \to (n, n+1)\cdot\sigma \to \sigma \in \mathcal{I}(n+1)$



Relations of $A(n)$

Lemma

Let $\sigma, \nu_i, \tau \in \mathrm{Sym}(n)$ *and* $\sum c_i(\sigma \to \nu_i \to \tau) \in \mathcal{I}(n)$. *Then*

- $\sum c_i((\sigma, n+1) \to (\nu_i, n+1) \to (\tau, n+1)) \in \mathcal{I}(n+1)$
- $\sum c_i(\sigma\cdot(n, n+1) \to \nu_i\cdot(n, n+1) \to \tau\cdot(n, n+1)) \in \mathcal{I}(n+1)$
- $\sigma \to (n, n+1)\cdot\sigma \to \sigma \in \mathcal{I}(n+1)$