## Bases of acyclic quantum cluster algebras

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Introduction	Quantum cluster algebra
Explicit construction	
Correction technique	Main result

Fix  $n \leq m \in \mathbb{N}$ . A compatible pair  $(\widetilde{B}, \Lambda)$ : •  $\widetilde{B} = \begin{pmatrix} B \\ B^c \end{pmatrix} \in Mat_{m \times n}(\mathbb{Z})$ , full rank; •  $\Lambda \in SkewMat_{m \times m}(\mathbb{Z})$ ; •  $\Lambda(-\widetilde{B}) = \begin{pmatrix} \delta \cdot \mathbf{1}_n \\ 0 \end{pmatrix}$  for some  $\delta \in \mathbb{N}$ . Quantum torus  $\mathcal{T} = \mathcal{T}(\Lambda)$ :

- Laurent polynomial ring  $\mathbb{Z}[q^{\pm \frac{1}{2}}][x_1^{\pm}, \dots, x_m^{\pm}](+, \cdot);$
- twisted product  $*: x_i * x_j = q^{\frac{1}{2}\Lambda_{ij}} x_i \cdot x_j$ .

- Quantum cluster variables : generators defined recursively by mutations.
- Quantum cluster monomials: certain monomials of cluster variables.

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# $\mathcal{A}^q$ should have a "dual canonical basis" which contains all quantum cluster monomials.

Some known results on this direction for  $\mathcal{A}^q$  and  $\mathcal{A} = \mathcal{A}^q|_{q^{\frac{1}{2}} \mapsto 1}$ :

- Hernandez-Leclerc, Lampe: A<sup>q</sup> of type ADE, Kronecker (special coefficient type B<sup>c</sup>);
- Geiss-Leclerc-Schröer: dual semi-canonical basis of  $\mathcal{A} = \mathcal{A}(w)$  for a given Weyl group element w and special  $B^c$ ;
- Musiker-Schiffler-Williams: two bases of A arising from unpunctured surfaces (with conjecturally positive structure constants);
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#### Theorem (Kimura-Qin)

For acyclic principal part B and special coefficient type  $B^c$ , we can construct the dual PBW basis  $\{M^{\mathcal{A}}(w)\}$ , the generic basis  $\{\mathbb{L}^{\mathcal{A}}(w)\}$ , and the dual canonical basis  $\{L^{\mathcal{A}}(w)\}$  of  $\mathcal{A}^{q\dagger}$  (as a  $\mathbb{Z}[q^{\pm \frac{1}{2}}]$ -algebra). The structure constants of  $\{L^{\mathcal{A}}(w)\}$  are contained in  $\mathbb{N}[q^{\pm}]$ .

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What happens if we take general B<sup>c</sup>?

Quantum cluster monomials are similar for different  $B^c$ .

Theorem (Fomin-Zelevinsky, Tran)

Quantum cluster variables (monomials) take the form  $x^{g}F(y)|_{y_{i}\mapsto x^{\widetilde{B}e_{i}}}$ , where F(y) is a polynomial in  $y_{1}, \ldots, y_{n}$  such that F(0) = 1.

It would be nice if similar results hold for basis-elements: A B A C

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## Theorem (Main result)

For any acyclic  $\mathcal{A}^q$ , bases and their structure constants are similar for different coefficient types  $B^c$  and quantizations  $\Lambda$ .

#### Remark

*Our correction technique in the proof could be used in non-acyclic "good" cases.* 

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Monoidal categorification Pseudo-monoidal categorification

In the monoidal category side (l = 1 case), we have  $\Pi_{w \in \mathbb{N}^{2n}} \mathcal{K}^*(w) \xrightarrow{L(w, v) \mapsto L(w - C_Q v)} \mathcal{R}_t$   $\downarrow \sum_{v \in \mathbb{N}^n} \langle \ , \pi(w, v) \rangle W^w V^v \qquad \qquad \qquad \downarrow \chi_{q,t} \leq 0$   $\widehat{\mathcal{Y}} = \mathbb{Z}[t^{\pm}][W^w, V^v]_{w,v} \xrightarrow{\widehat{\Pi} : W^w V^v \mapsto W^{w - C_Q v}} \mathcal{Y} = \mathbb{Z}[t^{\pm}][W^w]_w$ 

K(w): Grothendieck group generated by certain perverse sheaves.  $\pi(w, v) \in K(w)$ .  $\mathcal{K}^*(w)$ : dual of K(w).

 $\{L(w, v)\}$ : basis dual to simple perverse sheaves.

 $C_Q: \mathbb{N}^n \to \mathbb{Z}^{2n}$ : quantum Cartan matrix.

*R<sup>t</sup>*: deformed Grothendieck ring.

 $\hat{\mathcal{Y}}, \mathcal{Y}$ : quantum tori (Laurent polynomials with twisted products \*).  $\chi_{q,t} \leq 0$ : truncated *t*-analogue of *q*-character.

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## In the quantum cluster algebra side, we have

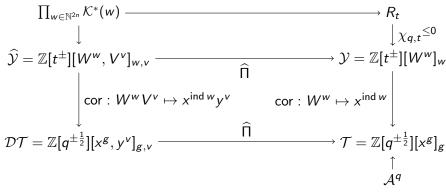
$$\mathcal{DT} = \mathbb{Z}[q^{\pm \frac{1}{2}}][x^g, y^v]_{g, v} \xrightarrow{\widehat{\Pi} : x^g y^v \mapsto x^{g + \widetilde{B}_v}} \mathcal{T} = \mathbb{Z}[q^{\pm \frac{1}{2}}][x^g]_g$$

$$\uparrow$$

$$\mathcal{A}^q$$

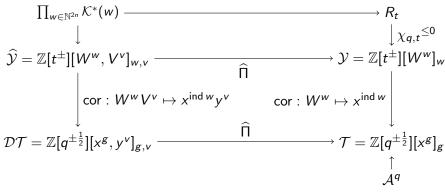
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When  $B^c$  is special (*z*-coefficient), we have the following commutative diagram:



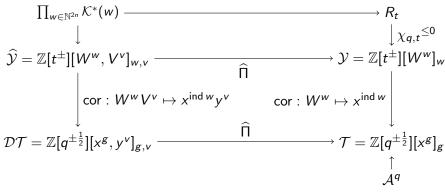
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**Observation**: For general  $B^c$  and  $\Lambda$ , the failures are:

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where  $x^{\text{ind }O(w)}$  is the expected leading term of the generic basis element associated with the generic object O(w) in  $C(\tilde{B})$ . They become equalities again if we put some explicit correction terms which take values in  $x_{n+1}, \dots, x_{\max(m,2n)}$  and  $q^{\frac{1}{2}}$ -powers.

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## Theorem (Bases)

Any acyclic  $\mathcal{A}^q$  have three  $\mathbb{Z}[q^{\pm \frac{1}{2}}][x_{n+1}, \ldots, x_m]$ -bases: the generic basis  $\{\mathbb{L}^{\mathcal{A}}(w), w \in \mathcal{J}\}$ , the dual PBW basis  $\{M^{\mathcal{A}}(w), w \in \mathcal{J}\}$ , and the dual canonical basis  $\{L^{\mathcal{A}}(w), w \in \mathcal{J}\}$ , where  $\mathcal{J} = \{w \in \mathbb{N}^{2n} | w_i \text{ or } w_{n+i} = 0, \forall 1 \leq i \leq n\}$ ,

$$L^{\mathcal{A}}(w) = \sum_{v} P_{q^{-\frac{\delta}{2}}}(\operatorname{Gr}_{v}(^{\sigma}W))q^{-\frac{\delta}{2}\dim\operatorname{Gr}_{v}(^{\sigma}W)}x^{\operatorname{ind}O(w)+\widetilde{B}v},$$
$$M^{\mathcal{A}}(w) = \sum_{v} P_{q^{\frac{\delta}{2}}}(\mathcal{L}(v,w))q^{-\frac{\delta}{2}\dim\mathcal{M}^{\bullet}(v,w)}x^{\operatorname{ind}O(w)+\widetilde{B}v},$$
$$L^{\mathcal{A}}(w) = \sum_{v} a_{v,0;w}(q^{\frac{\delta}{2}})x^{\operatorname{ind}O(w)+\widetilde{B}v}.$$

#### Proof.

We could explicitly compute the transition matrices (depend on  $B^c$ ,  $\Lambda$ ).

## In fact, the role of $R_t$ in previous calculation can be replaced by $\mathcal{A}^q$ with special $B^c$ , $\Lambda$ .

#### Proposition (Correction technique)

Fix B (not necessarily acyclic). If for some special  $B^c$  and quantization  $\Lambda$ , some elements of the quantum torus satisfy an algebraic relation, then for general ( $B^c$ )' and  $\Lambda$ ', in "good" cases, there exists an explicit similar algebraic relation for similar elements.

Here, "good" means:

- the original elements take the form  $x^g F(y)|_{y^v = x^{\widetilde{B}v}}$ ;
- (2)  $\widetilde{B}$  and  $\widetilde{B}'$  are of full ranks,  $\delta' = d \cdot \delta$  for some  $d \in \mathbb{N}$ .
- (a) more conditions (eg. F(0) = 1, the algebraic relation takes special form, etc.).

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## Proposition (Correction technique)

Fix B (not necessarily acyclic). If for some special  $B^c$  and quantization  $\Lambda$ , some elements of the quantum torus satisfy an algebraic relation, then for general ( $B^c$ )' and  $\Lambda$ ', in "good" cases, there exists an explicit similar algebraic relation for similar elements.

## Here, "good" means:

- the original elements take the form  $x^{g}F(y)|_{y^{v}=x^{\widetilde{B}v}}$ ;
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Here, "good" means:

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an exchange relation for quantum cluster variables:

"play the role of quantum F-polynomial"

- (2) a basis element = a polynomial of cluster variables: check that a "basis" is contained in  $\mathcal{A}^q$
- a defining equation for structure constants of a basis:

#### Theorem (Positive structure constants)

$$L^{\mathcal{A}}(w^{1}) * L^{\mathcal{A}}(w^{2}) = q^{\frac{1}{2}\wedge(\operatorname{ind}(w^{1}),\operatorname{ind}(w^{2})) - \frac{\delta}{2}\wedge^{z}(\operatorname{ind}^{z}(w^{1}),\operatorname{ind}^{z}(w^{2}))}$$

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$$L^{\mathcal{A}}(w^{1}) * L^{\mathcal{A}}(w^{2}) = q^{\frac{1}{2}\Lambda(\operatorname{ind}(w^{1}),\operatorname{ind}(w^{2})) - \frac{\delta}{2}\Lambda^{z}(\operatorname{ind}^{z}(w^{1}),\operatorname{ind}^{z}(w^{2}))} \\ \cdot \sum_{w^{3} \in \mathcal{J}} {}^{\phi} b^{w^{3}}_{w^{1},w^{2};\widetilde{B}}(q^{\frac{\delta}{2}}) L^{\mathcal{A}}(w^{3}).$$

Here, for any 
$$w^1$$
,  $w^2$ ,  $w^3$  in  $\mathcal{J}$ , we define an element in  
 $\mathbb{Z}[q^{\pm \frac{1}{2}}][x_{n+1}^{\pm}, \dots, x_m^{\pm}]:$ 

$${}^{\phi}b_{w^1,w^2;\widetilde{B}}^{w^3} = x^{\operatorname{ind} O(w^1) + \operatorname{ind} O(w^2) - \operatorname{ind} O(w^3) - \operatorname{ind}(w^1 + w^2 - w^3)}$$

$$\sum_{w:{}^{\phi}w = w^3} b_{w^1,w^2}^w x^{\operatorname{ind} {}^{f}w} x^{\widetilde{B}v + \operatorname{ind} C_Qv},$$

where v is determined by

$$w = w^1 + w^2 - C_Q v.$$

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