Bases of acyclic quantum cluster algebras

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Fix $n \leq m \in \mathbb{N}$.

A *compatible pair* $(\widetilde{B}, \Lambda)$:

- $\widetilde{B} = \begin{pmatrix} B \\ B^c \end{pmatrix} \in Mat_{m \times n}(\mathbb{Z})$, full rank;
- $\Lambda \in SkewMat_{m \times m}(\mathbb{Z})$;
- $\Lambda(-\widetilde{B}) = \begin{pmatrix} \delta \cdot \mathbf{1}_n \\ 0 \end{pmatrix}$ for some $\delta \in \mathbb{N}$.

Quantum torus $\mathcal{T} = \mathcal{T}(\Lambda)$:

- Laurent polynomial ring $\mathbb{Z}[q^{\pm\frac{1}{2}}][x_1^{\pm}, \ldots, x_m^{\pm}](+, \cdot)$;
- twisted product $* : x_i * x_j = q^{\frac{1}{2}\Lambda_{ij}} x_i \cdot x_j$.

Quantum cluster algebra (Fomin-Zelevinsky, Berenstein-Zelevinsky) $\mathcal{A}^q = \mathcal{A}^q(\widetilde{B}, \Lambda)$: the $Z[q^{\pm\frac{1}{2}}][x_{n+1}^{\pm}, \ldots, x_m^{\pm}]$-subalgebra of $\mathcal{T}(\Lambda)$.

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Some known results on this direction for $\mathcal{A}^q$ and $\mathcal{A} = \mathcal{A}^q|_{q^{\frac{1}{2}} \mapsto 1}$:

- Hernandez-Leclerc, Lampe: $\mathcal{A}^q$ of type ADE, Kronecker (special coefficient type B^c);
- Geiss-Leclerc-Schröer: dual semi-canonical basis of $\mathcal{A} = \mathcal{A}(w)$ for a given Weyl group element w and special B^c;
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Theorem (Kimura-Qin)

For acyclic principal part B and special coefficient type B^c, we can construct the dual PBW basis $\{M^{\mathcal{A}}(w)\}$, the generic basis $\{\mathbb{L}^{\mathcal{A}}(w)\}$, and the dual canonical basis $\{L^{\mathcal{A}}(w)\}$ of $\mathcal{A}^{q\dagger}$ (as a $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-algebra). The structure constants of $\{L^{\mathcal{A}}(w)\}$ are contained in $\mathbb{N}[q^{\pm}]$.

Question

What happens if we take general B^c?

Quantum cluster monomials are similar for different B^c.

Theorem (Fomin-Zelevinsky, Tran)

Quantum cluster variables (monomials) take the form $x^g F(y)|_{y_i \mapsto x^{\widetilde{B}e_i}}$, where $F(y)$ is a polynomial in $y_1, \ldots, y_n$ such that $F(0) = 1$.

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Theorem (Main result)

For any acyclic $\mathcal{A}^q$, bases and their structure constants are similar for different coefficient types B^c and quantizations Λ.

Remark

Our correction technique in the proof could be used in non-acyclic "good" cases.

In the monoidal category side ($l=1$ case), we have

$$\begin{tikzcd}
\prod_{w\in\mathbb{N}^{2n}}\mathcal{K}^*(w) \arrow[r, "{L(w,v)\mapsto L(w-C_Q v)}"] \arrow[d, "{\sum_{v\in\mathbb{N}^n}\langle\ ,\pi(w,v)\rangle W^w V^v}"] & R_t \arrow[d, "{\chi_{q,t}}^{\leq 0}"] \\
\widehat{\mathcal{Y}}=\mathbb{Z}[t^{\pm}][W^w,V^v]_{w,v} \arrow[r, "{\widehat{\Pi}:W^wV^v\mapsto W^{w-C_Q v}}"'] & \mathcal{Y}=\mathbb{Z}[t^{\pm}][W^w]_w
\end{tikzcd}$$

$K(w)$: Grothendieck group generated by certain perverse sheaves.
$\pi(w,v)\in K(w)$.
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$C_Q:\mathbb{N}^n\to\mathbb{Z}^{2n}$: quantum Cartan matrix.
R^t: deformed Grothendieck ring.
$\widehat{\mathcal{Y}}$, $\mathcal{Y}$: quantum tori (Laurent polynomials with twisted products $*$).
${\chi_{q,t}}^{\leq 0}$: truncated t-analogue of q-character.

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In the quantum cluster algebra side, we have

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\mathcal{DT} = \mathbb{Z}[q^{\pm\frac{1}{2}}][x^g, y^v]_{g,v} \arrow[r, "\widehat{\Pi} : x^g y^v \mapsto x^{g+\widetilde{B}v}"] & \mathcal{T} = \mathbb{Z}[q^{\pm\frac{1}{2}}][x^g]_g \\
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When B^c is special (z-coefficient), we have the following commutative diagram:

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ind: natural linear function on $\mathbb{N}^{2n}$ which depends on $\widetilde{B}$.

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Observation: For general B^{c} and Λ, the failures are:

$$\widehat{\Pi}\mathrm{cor}(W^{w}V^{v})\neq\widehat{\Pi}(\mathrm{cor}W^{w}V^{v}),$$

$$\mathrm{cor}(W^{w^{1}}V^{v^{1}}*W^{w^{2}}V^{v^{2}})\neq\mathrm{cor}(W^{w^{1}}V^{v^{1}})*\mathrm{cor}(W^{w^{2}}V^{v^{2}}),$$

$$\textit{leading term}(\mathrm{cor}\chi_{q,t}{}^{\leq 0}\mathbb{L}(w))\neq x^{\mathrm{ind}\,O(w)},$$

where $x^{\mathrm{ind}\,O(w)}$ is the expected leading term of the generic basis element associated with the generic object $O(w)$ in $\mathcal{C}(\widetilde{B})$.

They become equalities again if we put some explicit correction terms which take values in $x_{n+1},\ldots,x_{\max(m,2n)}$ and $q^{\frac{1}{2}}$-powers.

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Theorem (Bases)

Any acyclic $\mathcal{A}^q$ *have three* $\mathbb{Z}[q^{\pm\frac{1}{2}}][x_{n+1},\ldots,x_m]$*-bases: the generic basis* $\{\mathbb{L}^{\mathcal{A}}(w), w\in\mathcal{J}\}$*, the dual PBW basis* $\{M^{\mathcal{A}}(w), w\in\mathcal{J}\}$*, and the dual canonical basis* $\{L^{\mathcal{A}}(w), w\in\mathcal{J}\}$*, where* $\mathcal{J}=\{w\in\mathbb{N}^{2n}|w_i \text{ or } w_{n+i}=0,\forall 1\leq i\leq n\}$*,*

$$\mathbb{L}^{\mathcal{A}}(w)=\sum_v P_{q^{-\frac{\delta}{2}}}(\mathrm{Gr}_v({}^\sigma W))q^{-\frac{\delta}{2}\dim \mathrm{Gr}_v({}^\sigma W)}x^{\mathrm{ind}\, O(w)+\widetilde{B}v},$$

$$M^{\mathcal{A}}(w)=\sum_v P_{q^{\frac{\delta}{2}}}(\mathcal{L}(v,w))q^{-\frac{\delta}{2}\dim \mathcal{M}^\bullet(v,w)}x^{\mathrm{ind}\, O(w)+\widetilde{B}v},$$

$$L^{\mathcal{A}}(w)=\sum_v a_{v,0;w}(q^{\frac{\delta}{2}})x^{\mathrm{ind}\, O(w)+\widetilde{B}v}.$$

Proof.

We could explicitly compute the transition matrices (depend on B^c, Λ). ☐

In fact, the role of R_t in previous calculation can be replaced by $\mathcal{A}^q$ with special B^c, Λ.

Proposition (Correction technique)

Fix B (not necessarily acyclic). If for some special B^c and quantization Λ, some elements of the quantum torus satisfy an algebraic relation, then for general $(B^c)'$ and Λ', in "good" cases, there exists an explicit similar algebraic relation for similar elements.

Here, "good" means:

1. the original elements take the form $x^g F(y)|_{y^v = x^{\widetilde{B}v}}$;
2. $\widetilde{B}$ and $\widetilde{B}'$ are of full ranks, $\delta' = d \cdot \delta$ for some $d \in \mathbb{N}$.
3. more conditions (eg. $F(0) = 1$, the algebraic relation takes special form, etc.).

In fact, the role of R_t in previous calculation can be replaced by $\mathcal{A}^q$ with special B^c, Λ.

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The algebraic relation might be

1. an exchange relation for quantum cluster variables: "play the role of quantum F-polynomial"
2. a basis element = a polynomial of cluster variables: check that a "basis" is contained in $\mathcal{A}^q$
3. a defining equation for structure constants of a basis:

Theorem (Positive structure constants)

For any acyclic $\mathcal{A}^q$, the structure constants of the dual canonical basis $\{L^{\mathcal{T}}(w), w \in \mathcal{J}\}$ are contained in $\mathbb{N}[q^{\pm\frac{1}{2}}][x_{n+1}^{\pm}, \ldots, x_m^{\pm}]$:

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Here, for any w^1, w^2, w^3 in $\mathcal{J}$, we define an element in $\mathbb{Z}[q^{\pm\frac{1}{2}}][x_{n+1}^{\pm}, \ldots, x_m^{\pm}]$:

$$
\begin{aligned}
{}^{\phi}b^{w^3}_{w^1,w^2;\widetilde{B}} =& x^{\mathrm{ind}\, O(w^1)+\mathrm{ind}\, O(w^2)-\mathrm{ind}\, O(w^3)-\mathrm{ind}(w^1+w^2-w^3)} \\
& \sum_{w:{}^{\phi}w=w^3} b^{w}_{w^1,w^2} x^{\mathrm{ind}\,{}^{f}w} x^{\widetilde{B}v+\mathrm{ind}\, C_Q v},
\end{aligned}
$$

where v is determined by

$$w = w^1 + w^2 - C_Q v.$$