Coloured quivers for higher clusters via Ext-quivers of hearts

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Cluster algebras and cateogries

- Cluster algebras were introduced by Fomin and Zelevinsky in 2000.
- The combinatorial ingredient of cluster theory, the quiver mutation, has been categorified by Buan-Marsh-Reineke-Reiten-Todorov in 2005.sjf
- The cluster category $\mathcal{C}_2(Q)$ can be realized as orbit category (Keller) or quotient category (Amiot) and can be generalized to higher cluster categories $\mathcal{C}_m(Q)$.

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Aim and method

- We aim to show that Buan-Thomas' coloured quivers for higher clusters can be interpreted as Ext-quivers of hearts in certain derived categories.
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T-structure and its heart

- A *t-structure* $\mathcal{P}$ on a triangulated category $\mathcal{D}$ is the torsion part of some torsion pair (w.r.t. triangles) $\langle \mathcal{P}^{\perp}, \mathcal{P} \rangle$ on $\mathcal{D}$ such that $\mathcal{P}[1] \subset \mathcal{P}$.
- A t-structure $\mathcal{P}$ is *bounded* if for every object M, the shifts $M[k]$ are in $\mathcal{P}$ for $k \gg 0$ and in $\mathcal{P}^{\perp}$ for $k \ll 0$.
- The *heart* of a (bounded) t-structure $\mathcal{P}$ is the full subcategory $\mathcal{H} = \mathcal{P}^{\perp}[1] \cap \mathcal{P}$.
- We define an order relation $\mathcal{H}_1 \leq \mathcal{H}_2$ by $\mathcal{P}_2 \subset \mathcal{P}_1$, or $\mathcal{P}_1^{\perp} \subset \mathcal{P}_2^{\perp}$.

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Simple HRS-tilting

Proposition [Happel, Reiten, Smalø]

Let $\mathcal{H}$ be a heart in a triangulated category $\mathcal{D}$ with torsion pair $\mathcal{H} = \langle \mathcal{F}, \mathcal{T} \rangle$. Then there is a heart $\mathcal{H}^\sharp$ with torsion pair $\mathcal{H}^\sharp = \langle \mathcal{T}, \mathcal{F}[1] \rangle$, called the *forward tilts* of $\mathcal{H}$.

We say a forward tilt is *simple* if $\mathcal{F} = \langle S \rangle$ for a rigid simple S, and write it as $\mathcal{H}^\sharp_S$.

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Remark:$\text{EG}(\mathcal{D})$ is the Hasse quiver of the order relation $\leq$.

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For any heart $\mathcal{H}$ in $\mathcal{D}$ and $N \geq 2$, define the interval

$$\mathrm{EG}_N(\mathcal{D}, \mathcal{H}) = \{\mathcal{H}' \in \mathrm{EG}(\mathcal{D}) \mid \mathcal{H} \leq \mathcal{H}' \leq \mathcal{H}[N-2]\},$$

and $\mathrm{EG}^{\circ}_N(\mathcal{D}, \mathcal{H})$ its 'principal' component containing $\mathcal{H}$.

Outline

The bounded derived category $\mathcal{D}(Q)$

- Let Q be an acyclic quiver.
- We denote by $\mathcal{H}_Q = \operatorname{mod} \mathbf{k}Q$ the module category of the path algebra $\mathbf{k}Q$.
- Let $\mathcal{D}(Q) = \mathcal{D}^b(\operatorname{mod} \mathbf{k}Q)$ be its bounded derived category.
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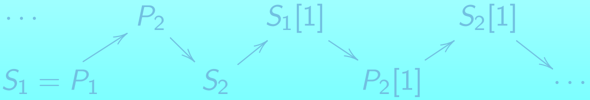
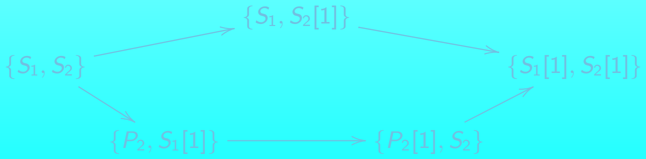
Example: A_2

Let $Q := 2 \to 1$ be a quiver of type A_2. A piece of AR-quiver of $\mathcal{D}(Q)$ is:



The interval $\mathrm{EG}_3(Q, \mathcal{H}_Q)$ is as follows:



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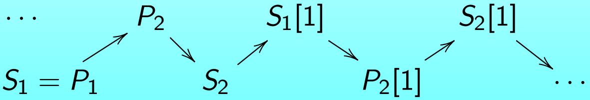

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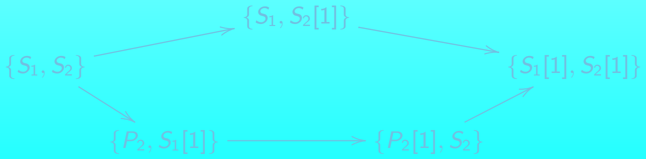

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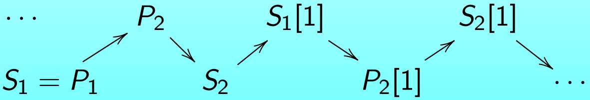

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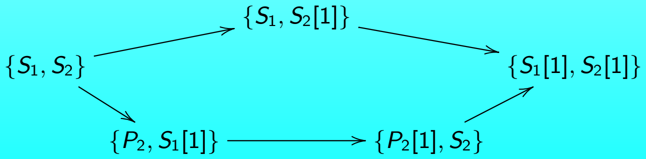

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Projectives of a heart

Definition

We say that an object $P \in \mathcal{D}$ is a *projective* of $\mathcal{H}$ if

- $\mathrm{Hom}^k(P, M) = 0, \forall k \neq 0, M \in \mathcal{H}$;

$\mathrm{Proj}\,\mathcal{H}$ = the set of indecomposable projectives of $\mathcal{H}$.

Note that a projective *of* a heart is not necessary *in* the heart.

Lemma

$\mathrm{Proj}\,\mathcal{H}$ is a silting set, i.e. $\mathrm{Ext}^m(P_1, P_2) = 0$ for any $P_1, P_2 \in \mathrm{Proj}\,\mathcal{H}$ and integer $m > 0$.

The converse is true in general, cf. Koenig-Yang.

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Cluster tilting sets

- The *m-cluster category* $\mathcal{C}_m(Q)$ is the orbit category $\mathcal{D}(Q)/\tau^{-1} \circ [m-1]$.
- An *m-cluster tilting set* $\{P_j\}_{j=1}^n$ in $\mathcal{C}_m(Q)$ is an Ext-configuration, i.e. a maximal collection of non-isomorphic indecomposables such that $\mathrm{Ext}^k_{\mathcal{C}_m(Q)}(P_i, P_j) = 0$, for any $1 \le k \le m-1$.

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Mutation

The *forward mutation* μ_i acts on an *m*-cluster tilting set $\{P_j\}_{j=1}^n$, by replacing P_i by

$$P_i^\sharp = \mathrm{Cone}(P_i \to \bigoplus_{j \neq i} \mathrm{Irr}(P_i, P_j)^* \otimes P_j),$$

where $\mathrm{Irr}(P_i, P_j)$ is the space of irreducible maps $P_i \to P_j$, in the additive subcategory $\mathrm{Add} \bigoplus_{i=1}^n P_i$ of $\mathcal{C}_m(Q)$.

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Cluster exchange graphs

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The *exchange graph* $\mathrm{CEG}_m(Q)$ of m-clusters is the oriented graph whose vertices are m-cluster tilting sets and whose edges are the forward mutations.

Theorem [King-Qiu, cf. Ingalls-Thomas, Buan-Reiten-Thomas]

There is a bijection on vertex sets:

$$\begin{aligned} \mathcal{J} : \mathrm{EG}^\circ_N(Q, \mathcal{H}_Q) &\to \mathrm{CEG}_{N-1}(Q), \\ \mathcal{H} &\mapsto \mathrm{Proj}\,\mathcal{H}\,. \end{aligned}$$

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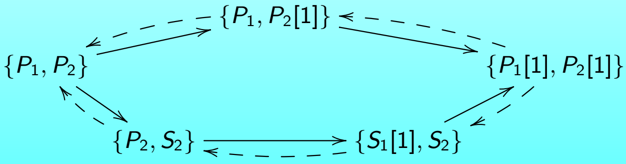
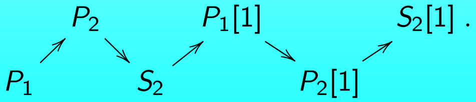


recall that we have a piece of the AR-quiver



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Coloured quivers

For any cluster tilting set $\mathbf{T} = \{T_1, \ldots, T_n\}$ in $\mathrm{CEG}_m(Q)$, Buan-Thomas defined a coloured quiver $\mathcal{Q}^{\mathrm{c}}(\mathbf{T})$ with coloured quiver mutation.

Note that

- The multiplicity of the colour-zero arrows in $\mathcal{Q}(\mathbf{T})$ is given by $\mathrm{Irr}(T_i, T_j)$.
- $\mathcal{Q}^{\mathrm{c}}(\mathbf{T})$ is monochromatic and skew-symmetric:

$$T_i \underset{m-1-c}{\overset{c}{\rightleftarrows}} T_j, \quad 0 \le c \le m-1 \text{ is the colour.}$$

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We can turn a coloured quiver $\mathcal{Q}^{c}(\mathbf{T})$ into the *augmented graded quiver* $\mathcal{Q}^{+}(\mathbf{T})$ by

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Ext-quivers of hearts

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Let $\mathcal{H}$ be a finite heart in a triangulated category $\mathcal{D}$. The Ext-quiver $\mathcal{Q}(\operatorname{Sim}\mathcal{H})$ is the graded quiver whose vertices are $\operatorname{Sim}\mathcal{H}$ and whose graded edges correspond to a basis of $\operatorname{End}^\bullet(\mathbf{S},\mathbf{S})$, where $\mathbf{S} = \bigoplus_{S\in\operatorname{Sim}\mathcal{H}} S$.

Further, define the *CY-N double* of a graded quiver $\mathcal{Q}$, denoted by ${}^N\overline{\mathcal{Q}}$.

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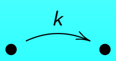

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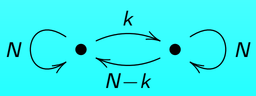

First link

Recall $\mathcal{J} : \mathrm{EG}^{\circ}_{N}(Q, \mathcal{H}_Q) \hookrightarrow \mathrm{CEG}_{N-1}(Q), \mathcal{H} \mapsto \mathrm{Proj}\,\mathcal{H}$.

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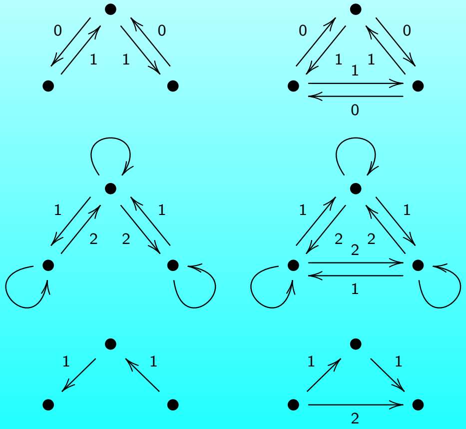
Examples: Quivers of A_3-type and $N = 3$

Coloured

Augmented

Ext-quiver



Outline

The Ginzburg dg algebra

Let $\Gamma_N Q$ be the *N-Calabi-Yau Ginzburg (dg) algebra* associated to Q.

- $\mathcal{D}(\Gamma_N Q) = \mathcal{D}_{fd}(\operatorname{mod} \Gamma_N Q)$, which is N-Calabi-Yau.
- $\mathcal{D}(\Gamma_N Q)$ admits a standard heart $\mathcal{H}_\Gamma \cong \mathcal{H}_Q = \operatorname{mod} \mathbf{k}Q$.
- The spherical twists of simples in $\mathcal{H}_\Gamma$ generate the Seidel-Thomas braid group $\operatorname{Br}(\Gamma_N Q)$.
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- $\mathcal{D}(\Gamma_N Q) = \mathcal{D}_{fd}(\operatorname{mod} \Gamma_N Q)$, which is N-Calabi-Yau.
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(Keller) The quotient $\Gamma_N Q \to Q$ induces a functor

$$\mathcal{I} : \mathcal{D}(Q) \to \mathcal{D}(\Gamma_N Q),$$

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Proposition [King-Qiu]

There is an isomorphism $\mathcal{I}_* : \mathrm{EG}^{\circ}_N(Q, \mathcal{H}_Q) \to \mathrm{EG}^{\circ}_N(\Gamma_N Q, \mathcal{H}_\Gamma)$, $\widehat{\mathcal{H}} \mapsto \mathcal{H}$, such that $\mathcal{H}$ is determined by $\mathrm{Sim}\,\mathcal{H} = \mathcal{I}(\mathrm{Sim}\,\widehat{\mathcal{H}})$ and satisfying

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Outline

ICRA2012

Yu Qiu

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Introduction

From hearts to clusters

Exchange graph of hearts

The exchange graph of $\mathcal{D}(Q)$

Exchange graph of clusters

Coloured quivers and Ext-quivers

Calabi-Yau double

The Calabi-Yau category

The result

The maps

Amiot constructed a short exact sequence of triangulated categories:

$$0 \to \mathcal{D}(\Gamma_N Q) \to \mathrm{per}(\Gamma_N Q) \to \mathcal{C}_{N-1}(Q) \to 0,$$

which induces a $\mathrm{Br}(\Gamma_N Q)$-invariant map

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Isomorphisms between exchange graphs

Theorem [King-Qiu]

We have bijections (as vertex set):

$$\begin{array}{ccccc} \widehat{\mathcal{H}} \in & \mathrm{EG}^{\circ}_N(Q, \mathcal{H}_Q) & \xrightarrow[\cong]{\mathcal{J}} & \mathrm{CEG}_{N-1}(Q) & \ni \operatorname{Proj} \widehat{\mathcal{H}} \\ & \cong \big\downarrow \mathcal{I}_* & & \upsilon \big\uparrow \cong & \\ \mathcal{H} \in & \mathrm{EG}^{\circ}_N(\Gamma_N Q, \mathcal{H}_\Gamma) & \xrightarrow[\cong]{} & \mathrm{EG}^{\circ}(\Gamma_N Q)/\operatorname{Br} & \end{array}$$

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Theorem [King-Qiu]

For any heart $\mathcal{H} \in \mathrm{EG}^{\circ}(\Gamma_N\, Q)$, the Ext-quiver $\mathcal{Q}(\mathcal{H})$ is equal to the augmented graded quiver $\mathcal{Q}^{+}(v(\mathcal{H}))$ of the corresponding cluster tilting set.

Duality

Projectives	Simples
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END

Thanks for your attention!!