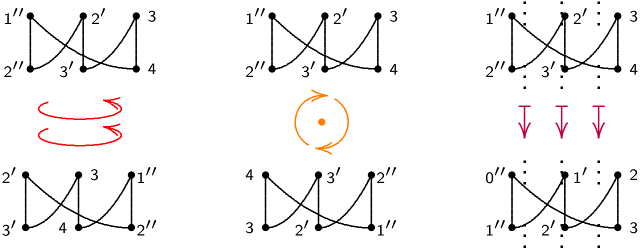
International Conference on Representations of Algebras

Bielefeld, August 3-17, 2012



Markus Schmidmeier (Florida Atlantic University):

ADE Posets

Contents

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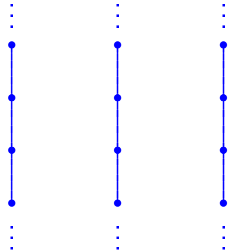

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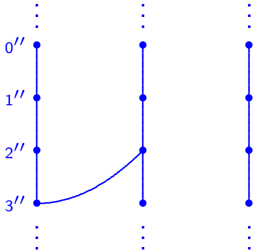

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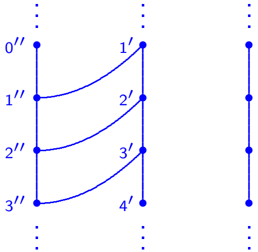

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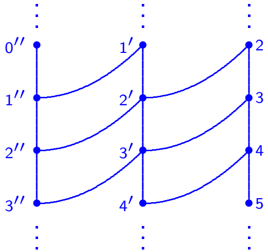

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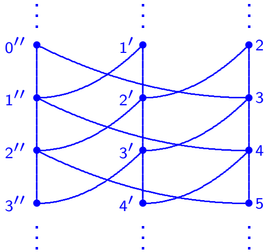

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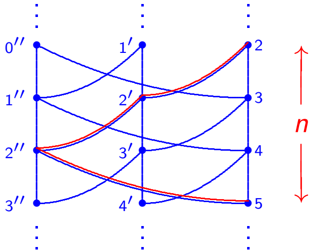

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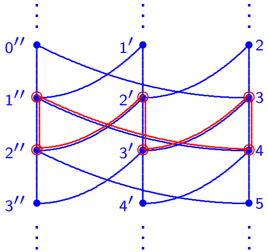

Poset representations

Let $\mathcal{P}$ be one of the above posets.

Poset representations: By $\operatorname{rep}_K \mathcal{P}$ we denote the category of all systems $\big(V_*, (V_x)_{x\in\mathcal{P}}\big)$ which satisfy:

- V_* is a finite dimensional K-vector space,
- $V_x \subset V_*$ is a subspace for each $x \in \mathcal{P}$,
- $V_x \subset V_y$ holds whenever $x < y$ in $\mathcal{P}$, and
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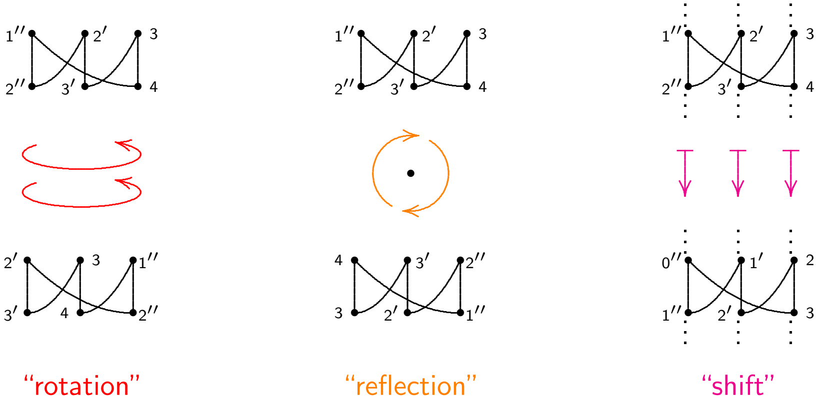
A chain of categories: The relations for $\mathcal{P}(n+1)$ are satisfied in $\mathcal{P}(n)$, hence the categories $\operatorname{rep}_K \mathcal{P}(n), n \in \mathbb{N}$, form a chain:

$$\operatorname{rep}_K \mathcal{P}(1) \subset \operatorname{rep}_K \mathcal{P}(2) \subset \operatorname{rep}_K \mathcal{P}(3) \subset \cdots$$

Symmetries of the poset

We consider three symmetry operations of the posets, pictured here for the poset $\mathcal{P}(3)$, and the endofunctors which they induce on its category of representations.



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- In turn, the operations on $\operatorname{rep}\mathcal{P}(n)$ motivate invariants:

$$\begin{aligned} \text{graded shift} &\rightsquigarrow \text{slope } \sigma \\ \text{duality} &\rightsquigarrow \text{drift } \delta = \frac{d\,\sigma}{d\,n} \\ \text{AR-translation} &\rightsquigarrow \text{curvature } \kappa = \frac{d\,\sigma}{d\,\tau} \end{aligned}$$

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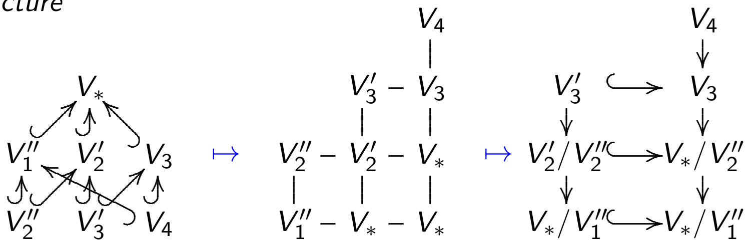
- Using the invariants we study the chain of categories:

$$\operatorname{rep}_K \mathcal{P}(1) \subset \operatorname{rep}_K \mathcal{P}(2) \subset \operatorname{rep}_K \mathcal{P}(3) \subset \cdots$$

Related categories

The category $\mathcal{S}(n)$ of invariant subspaces (C.M. Ringel, D. Simson, P. Zhang e.a.) occurs as a factor (M. Kleiner):

Proposition. *The functor* $G : \operatorname{rep}_K \mathcal{P}(n) \to \mathcal{S}(n)$ *given by the picture*



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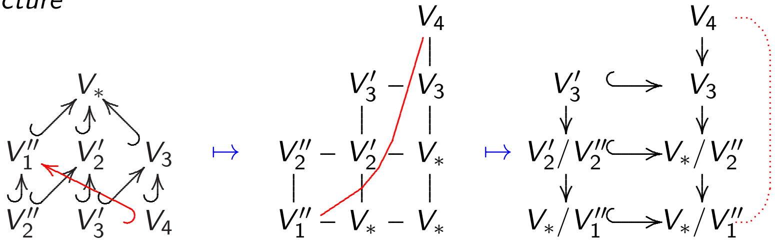

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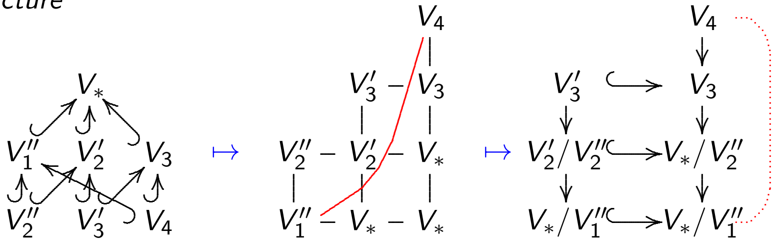
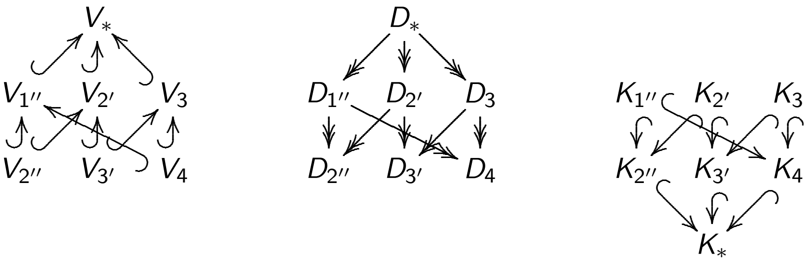


is full and has kernel the ideal $\mathcal{Z}$ consisting of all maps which factor through a sum of projective poset representations of type P_i''.

Corollary. The stable category $\underline{\mathrm{rep}}_K \mathcal{P}(n)$ is equivalent to the stable category of vector bundles over weighted projective lines $\mathbb{X}(2, 3, n)$ (X.-W. Chen, D. Kussin, H. Lenzing, H. Meltzer). Moreover, $\mathrm{rep}_K \mathcal{P}(n)$ is equivalent to the category of graded lattices over tiled orders (W. Rump).

The reflection

Each of the posets $\mathcal{P}(n)$ is symmetric with respect to reflection at one of the centers. On the module category, this operation induces the reflection duality:



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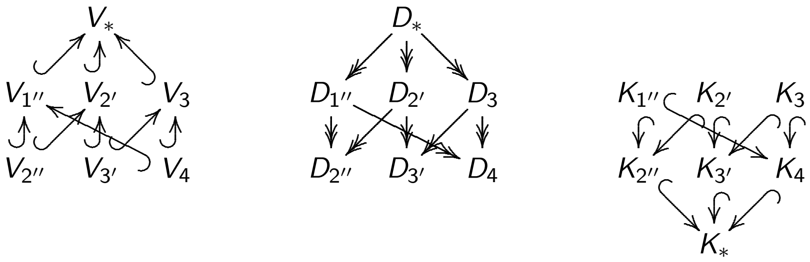

Lemma. *The reflection duality* $D : \operatorname{rep}_K \mathcal{P}(n) \to \operatorname{rep}_K \mathcal{P}(n)$ *preserves the ideal* $\mathcal{Z}$ *and induces the duality on the quotient* $\mathcal{S}(n) = \operatorname{rep}_K \mathcal{P}(n)/\mathcal{Z}$ *given by*

$$D\big(U \subset V\big) = \big(D(V/U) \subset DV\big).$$

The rotation

The rotation of the poset gives rise to a autoequivalence R of order three on its category of representations.

We pick the orientation such that $V \mapsto V' \mapsto V'' \mapsto V$.

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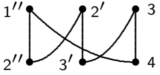
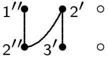
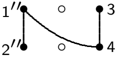
Remark: For a representation $M = (M_i' \subset M_i)_{i \in \mathbb{Z}} \in \mathcal{S}(n)$, the corresponding poset representation encodes the subspaces M_i', the ambient spaces M_i and the factor spaces M_i/M_i' in the following subposets of $\mathcal{P}(n)$.



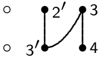
poset $\mathcal{P}$



subspace



ambient space



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poset $\mathcal{P}$ subspace ambient space factor space

Proposition. *For the unbounded posets of discrete representation type, the rotation is just the square of the Auslander-Reiten translation, up to a shift.*

$$R = \tau^2 \left[\frac{6-n}{3} \right]$$

The shift

The shift in the poset in vertical direction gives rise to the graded shift on the category of representations.

The position with respect to the shift is measured by the slope.

Roughly, the slope is just the barycenter of the generators in the aligned poset, where generators for multiple columns are averaged out.

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Roughly, the slope is just the barycenter of the generators in the aligned poset, where generators for multiple columns are averaged out.

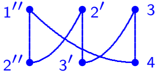
Slope formula: $\sigma(V) = \frac{1}{g} \sum_{x \in \mathcal{P}(n)} \mu(x) \sigma_x(V)$ for $V \in \operatorname{rep}_K \mathcal{P}(n)$,

where g is the number of generators,

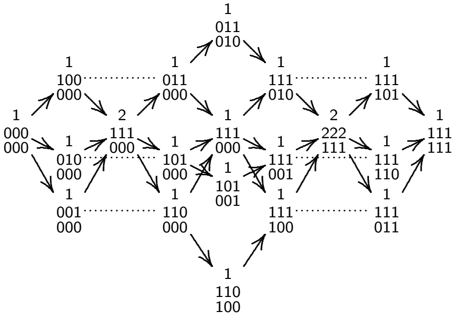
$$\mu(x) = \begin{cases} y + \frac{n}{3} & \text{if } x = y'' \\ y & \text{if } x = y' \\ y - \frac{n}{3} & \text{if } x = y \end{cases} \quad \text{and}$$

$$\begin{aligned} \sigma_x(V) &= \dim \frac{V_x}{V_{x+1}} - \frac{1}{2}\big(\dim \frac{V_{x+1}+V'_x}{V_{x+1}} + \dim \frac{V^-_{x+1}+V_x}{V^-_{x+1}} \big) \\ &\quad - \frac{1}{6}\big(\dim \frac{V_{x+1}+V''_x}{V_{x+1}} + \dim \frac{V^-_{x+1}+V'_x}{V^-_{x+1}} + \dim \frac{V^=_{x+1}+V_x}{V^=_{x+1}} \big). \end{aligned}$$

Example

The category $\mathrm{rep}_k\,\mathcal{P}(3)$ for $\mathcal{P}(3) =$ 

has the following partial Auslander-Reiten quiver:



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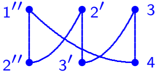
The category $\text{rep}_k\,\mathcal{P}(3)$ for $\mathcal{P}(3) =$ 

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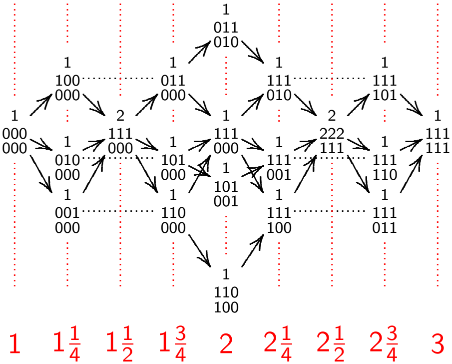
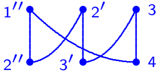


- The slope increases with the graded shift.
- It is invariant under rotation.
- There is constant d such that $\sigma(DX) = d - \sigma(X)$ holds.

The drift

For $X \in \operatorname{rep}_K \mathcal{P}(n_0)$, we consider the sequence $\sigma_n(X)$ of slopes when X is considered an object in $\operatorname{rep}_K \mathcal{P}(n)$ for $n \geq n_0$.

Four Examples: Consider the poset $\mathcal{P}(3) =$ 

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$\dim X$	$\sigma_n(X)$ for $n > 3$	$\sigma_3(X)$	drift
$\begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{smallmatrix}$	$\frac{1}{2}\left(1 + \frac{n}{3} + \frac{1}{2}(3 + 3 - \frac{n}{3})\right) = 2 + \frac{n}{12}$	$2\frac{1}{4}$	$\frac{1}{12}$

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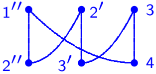
Four Examples: Consider the poset $\mathcal{P}(3) =$ .

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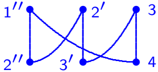
Four Examples: Consider the poset $\mathcal{P}(3) =$ 

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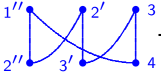
Four Examples: Consider the poset $\mathcal{P}(3) =$ 

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The curvature

Theorem. *If* $X \to Y$ *is an irreducible morphism in* $\mathrm{rep}_K\,\mathcal{P}(n)$ *then*

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Definition: The curvature in $\text{rep}_K\,\mathcal{P}(n)$ is defined as

$$\kappa(n) = \frac{d\,\sigma}{d\,\tau} = \frac{6-n}{6}$$

n	1	2	3	4	5	6	7	8	9	$\cdots$	12
$\underline{\Gamma}$	$\emptyset$	$\mathbb{Z}\mathbb{A}_2$	$\mathbb{Z}\mathbb{D}_4$	$\mathbb{Z}\mathbb{E}_6$	$\mathbb{Z}\mathbb{E}_8$	$\textcircled{\mathbb{E}_8}$	$\mathbb{Z}\mathbb{A}_\infty$	$\cdots$			
$[1]$	$-$	$\tau^{\frac{3}{2}}\varphi$	$\tau^2\rho$	$\tau^3\varphi$	τ^6	$-$	τ^{-6}				
$\kappa(n)$	$\emptyset$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$\cdots$	-1

Summary

Symmetry properties of the posets $\mathcal{P}(n)$ lead to invariants for the study of $\operatorname{rep}_K \mathcal{P}(n)$.

- For each object, the slope determines its position within $\operatorname{rep}_K \mathcal{P}(n)$.
- The drift is the change of the slope as the object moves along the chain

$$\operatorname{rep}_K \mathcal{P}(1) \subset \operatorname{rep}_K \mathcal{P}(2) \subset \operatorname{rep}_K \mathcal{P}(3) \subset \cdots$$

- The curvature is positive, zero or negative, depending on whether $\operatorname{rep}_K \mathcal{P}(n)$ is of discrete, tame or wild representation type.

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Thank You!