Selfinjective algebras of small stable Calabi-Yau dimension

Sergei Ivanov
St. Petersburg State University
(St. Petersburg, Russia)

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Calabi-Yau categories

Definition

A k-linear hom-finite triangulated category $\mathscr{T}$ with shift functor Σ is said to be *Calabi-Yau category* if an iterated shift functor $\Sigma^n = [n]$ is a Serre functor for some $n \geq 0$.

If so then the minimal $n \geq 0$ having this property is called the *Calabi–Yau dimension* of $\mathscr{T}$, and it is denoted by $\mathrm{CYdim}(\mathscr{T})$. If $\mathscr{T}$ is not Calabi-Yau, we set $\mathrm{CYdim}(\mathscr{T}) = \infty$.

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Stable Calabi-Yau dimension of a selfinjective algebra

If A is selfinjective then the stable module category $\underline{\text{mod}}\text{-}A$ is a triangulated category with the shift given by the inverse Ω^{-1} of Heller's syzygy functor.

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The *stable Calabi-Yau dimension* of A is the Calabi-Yau dimension of the stable module category of A.

$$\underline{\text{CY}}\text{dim}(A) = \text{CYdim}(\underline{\text{mod}}\text{-}A)$$

[K. Erdmann, A. Skowroński, "*The stable Calabi-Yau dimension of tame symmetric algebras.*" 2006]

Proposition

Let A be a selfinjective algebra. Then $\underline{\text{CY}}\text{dim}(A) = n$ iff $n \geq 0$ is the least number s.t. $\Omega^{n+1} \cong \underline{\nu}^{-1}$.

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K. Erdmann and A. Skowroński described connected selfinjective algebras of the stable Calabi-Yau dimensions 0 and 1.

$\underline{\mathrm{CY}}\mathrm{dim}(A) = 0 \quad \Leftrightarrow \quad A$ is a Nakayama algebra of Loewy length at most 2.

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Reminder of preprojective algebras

- Let Q be a quiver with an involution $a \mapsto \overline{a}$ of arrows s.t.

  The preprojective algebra of Q is a bound quiver algebra $\mathrm{P}(Q) = kQ/I$ where I is the principal ideal $I = \left(\sum_{a \in Q_1} a\overline{a} \right)$
- The associated graph $\Delta = \Delta_Q$ of the quiver with an involution Q is obtained by replacing each set of arrows $\{a, \overline{a}\}$ by an undirected edge.
- If the involution $a \mapsto \overline{a}$ acts freely then the graph Δ determines the preprojective algebra completely, and hence it is denoted by $\mathrm{P}(\Delta)$.
- It is well-known that the algebra $\mathrm{P}(\Delta)$ is finite dimensional iff Δ is one of the Dynkin graphs $\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_{6,7,8}$. Moreover, in this case, the algebra $\mathrm{P}(\Delta)$ is selfinjective. These algebras are called preprojective algebras of Dynkin type.

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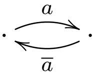
- Let Q be a quiver with an involution $a \mapsto \overline{a}$ of arrows s.t. 

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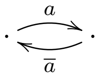
- Let Q be a quiver with an involution $a \mapsto \overline{a}$ of arrows s.t. 

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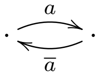
- Let Q be a quiver with an involution $a \mapsto \overline{a}$ of arrows s.t. 

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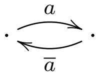
- Let Q be a quiver with an involution $a \mapsto \overline{a}$ of arrows s.t. 

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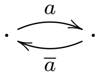
- Let Q be a quiver with an involution $a \mapsto \overline{a}$ of arrows s.t. 

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Preprojective algebra of type $\mathbf{L}_n$

The preprojective algebra of type $\mathbf{L}_n$ is the algebra $\mathrm{P}(Q_{\mathbf{L}_n})$, where $Q_{\mathbf{L}_n}$ is the following quiver with an (not free) involution.

$$\varepsilon{=}\overline{\varepsilon} \circlearrowright 0 \underset{\overline{\alpha}_0}{\overset{\alpha_0}{\rightleftarrows}} 1 \underset{\overline{\alpha}_1}{\overset{\alpha_1}{\rightleftarrows}} \ \cdots \ \underset{\overline{\alpha}_{n-3}}{\overset{\alpha_{n-3}}{\rightleftarrows}} n-2 \underset{\overline{\alpha}_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} n-1$$

This is a finite dimensional selfinjective algebra.

Associated quiver $\mathbf{L}_n := \Delta_{Q_{\mathbf{L}_n}}$:

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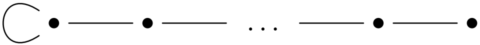

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$\underline{\mathrm{CY}}\dim(A) = 2$

K. Erdmann and A. Skowroński presented the following not fully correct statement.

Proposition (not fully correct)

Let A be a preprojective algebra of Dynkin type or of type $\mathbf{L}_n$ except types $\mathbf{A}_1, \mathbf{A}_2, \mathbf{L}_1$. Then $\underline{\mathrm{CY}}\dim(A) = 2$.

Of course, it is right for algebras of Dynkin type but it does not work for algebras of type $\mathbf{L}_n$.

Proposition

Let $A = \mathrm{P}(\mathbf{L}_n)$ be a preprojective k-algebra of type $\mathbf{L}_n$ for $n \geq 2$. Then A is a symmetric algebra and the following statements hold.

- $\mathrm{char}(k) = 2 \quad \Rightarrow \quad \underline{\mathrm{CY}}\dim(A) = 2$
- $\mathrm{char}(k) \neq 2 \quad \Rightarrow \quad \underline{\mathrm{CY}}\dim(A) = 5$

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Reminder of algebras of quaternion type

- An algebra A is algebra of *quaternion type* if A is connected, symmetric, tame, the Cartan matrix of A is nonsingular, and the stable Auslander-Reiten quiver Γ_A^s of A consists only of tubes of rank at most 2.
- K.Erdmann proved that any algebra of quaternion type is Morita equivalent to one of 12 types of symmetric bound quiver algebras (given by certain quivers with certain relations). We will call the list of these 12 types of algebras *Erdmann's list of algebras of quaternion type*.

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$\underline{\mathrm{CY}}\dim(A) = 3$

K.Erdmann and A.Skowroński proved that the stable Calabi-Yau dimension of any algebra of quaternion type is equal to 3.

I wanted to find a property simple enough to check for selfinjective bound quiver algebras $A = kQ/I$ which satisfies the following conditions:

- it must be formulated in terms of the quiver Q and the ideal I;
- it must follow that $\underline{\mathrm{CY}}\dim(A) = 3$;
- it must be satisfied for all algebras from the Erdmann's list of algebras of quaternion type.

Having so-called *DTI-family of relations* is a property of this kind. Now we will define it.

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Definition of DTI-family of relations

- Let $A = kQ/I$ be a bound quiver algebra. Let α be an arrow in Q. We denote by $\frac{\partial}{\partial\alpha} : kQ \to A \otimes A$ the linear map defined on paths by the formula:

$$\frac{\partial(\alpha_1\alpha_2\ldots\alpha_m)}{\partial\alpha} = \sum_{i\,:\,\alpha_i=\alpha} \alpha_1\ldots\alpha_{i-1}\otimes\alpha_{i+1}\ldots\alpha_n$$

- Let $J = (Q_1) \lhd kQ$. We use the following denotation

$$\mathrm{top}(I) := \frac{I}{JI + IJ}$$

and $\pi : I \to \mathrm{top}(I)$ is the canonical projection.

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Definition

Let $A = kQ/I$ be a bound quiver algebra. A Q_1-indexed family $\mathscr{R} = \{r_\alpha\}_{\alpha \in Q_1}$ of elements in I we call a *DTI-family of relations* (differentially twist invariant) if the following properties are satisfied:

(DTI-1) $r_\alpha \in e_j I e_i$ for $\alpha : i \to j$.

(DTI-2) $\frac{\partial r_\beta}{\partial \alpha} = \mathrm{tw}\left(\frac{\partial r_\alpha}{\partial \beta}\right)$ for any $\alpha, \beta \in Q_1$.

(DTI-3) the family $\pi\mathscr{R} = \{\pi(r_\alpha)\}_{\alpha \in Q_1}$ is a basis in $\mathrm{top}(I)$.

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Let $A = kQ/I$ be a bound quiver algebra. A Q_1-indexed family $\mathscr{R} = \{r_\alpha\}_{\alpha \in Q_1}$ of elements in I we call a *DTI-family of relations* (differentially twist invariant) if the following properties are satisfied:

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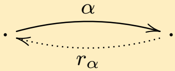

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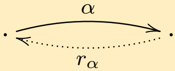

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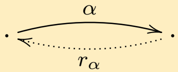

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Example of an algebra with DTI-family of relations

denotation	quiver	relations with denoted DTI-family of relations
$A^{Nak}_{3,2}$		$r_{\alpha_0} = \alpha_1\alpha_2$ $r_{\alpha_1} = \alpha_2\alpha_0$ $r_{\alpha_2} = \alpha_0\alpha_1$

(DTI-1) obvious

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(DTI-3) all paths of length at least 3 lie in $JI + IJ$. It follows that $\mathrm{top}(I) = \langle r_{\alpha_0}, r_{\alpha_1}, r_{\alpha_2} \rangle$

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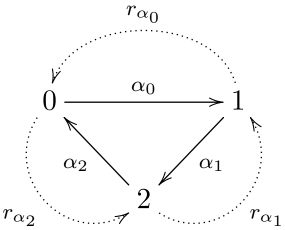
$A_{3,2}^{Nak}$		$r_{\alpha_0} = \alpha_1\alpha_2$ $r_{\alpha_1} = \alpha_2\alpha_0$ $r_{\alpha_2} = \alpha_0\alpha_1$

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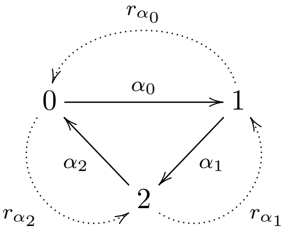
$A_{3,2}^{Nak}$		$r_{\alpha_0} = \alpha_1\alpha_2$ $r_{\alpha_1} = \alpha_2\alpha_0$ $r_{\alpha_2} = \alpha_0\alpha_1$

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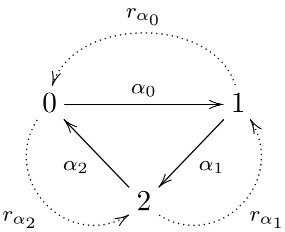
$A^{Nak}_{3,2}$		$r_{\alpha_0} = \alpha_1\alpha_2$ $r_{\alpha_1} = \alpha_2\alpha_0$ $r_{\alpha_2} = \alpha_0\alpha_1$

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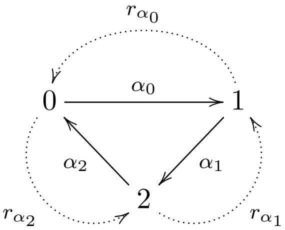
$A_{3,2}^{Nak}$		$r_{\alpha_0} = \alpha_1\alpha_2$ $r_{\alpha_1} = \alpha_2\alpha_0$ $r_{\alpha_2} = \alpha_0\alpha_1$

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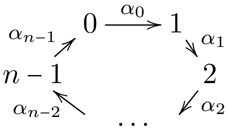
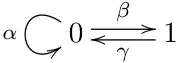
$A_{n,m}^{Nak}$ n divides $m+1$		$r_{\alpha_i} = \alpha_{i+1}\alpha_{i+2}\ldots\alpha_{i+m}$ $0 \leq i < n$
—		$r_\alpha = \beta\gamma - \alpha^2,$ $r_\beta = \gamma\alpha,$ $r_\gamma = \alpha\beta.$

Main results about algebras with DTI-family of relations.

Theorem

Let $A = kQ/I$ be a selfinjective bounded quiver algebra with DTI-family of relations non isomorphic to the algebras $k[x]/(x^n)$ and $A_{3,2}^{Nak}$. Then $\underline{\mathrm{CY}}\mathrm{dim}(A) = 3$.

Theorem

Every algebra from Erdmann's list of quaternion algebras has DTI-family of relations.

$\mathcal{Q}^k(a,b)$

$$r_\alpha = \alpha^2 - (\beta\alpha)^{k-1}\beta - a\alpha^3,$$
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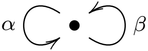

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Let us go back to the preprojective algebra $\mathrm{P}(\mathbf{L}_n)$

$$\mathrm{P}(\mathbf{L}_n) = \mathrm{P}(Q_{\mathbf{L}_n})$$

$Q_{\mathbf{L}_n}$:

$$\varepsilon=\overline{\varepsilon} \circlearrowright 0 \underset{\overline{\alpha}_0}{\overset{\alpha_0}{\rightleftarrows}} 1 \underset{\overline{\alpha}_1}{\overset{\alpha_1}{\rightleftarrows}} \quad \dots \quad \underset{\overline{\alpha}_{n-3}}{\overset{\alpha_{n-3}}{\rightleftarrows}} n-2 \underset{\overline{\alpha}_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} n-1$$

Proposition

Let $A = \mathrm{P}(\mathbf{L}_n)$ be a preprojective k-algebra of type $\mathbf{L}_n$ for $n \geq 2$. Then A is a symmetric algebra and the following statements hold.

- $\mathrm{char}(k) = 2 \quad \Rightarrow \quad \underline{\mathrm{CY}}\mathrm{dim}(A) = 2$
- $\mathrm{char}(k) \neq 2 \quad \Rightarrow \quad \underline{\mathrm{CY}}\mathrm{dim}(A) = 5$

Main ideas of the proof that $\underline{\text{CY}}\text{dim}(\text{P}(\mathbf{L}_n)) = 5$

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- We assume the contrary, that there is an isomorphism of endofunctors $f : \mathrm{Id} \to (-)_\tau$ on the category $\underline{\mathrm{mod}}\text{-}A$.
- We consider the module $M := P_0/\mathrm{rad}^2(P_0)$, where $P_0 = e_0A$, and prove that $\underline{\mathrm{Hom}}_A(M, M_\tau) = \mathrm{Hom}_A(M, M_\tau)$
- We prove that $M \cong M_\tau$ but there is NO an isomorphism $f_M : M \to M_\tau$ s.t. the following diagram is commutative in $\mathrm{mod}\text{-}A$.

$$\begin{array}{ccc} M & \xrightarrow{\varepsilon\cdot} & M \\ {\scriptstyle f_M}\downarrow{\scriptstyle \cong} & & {\scriptstyle f_M}\downarrow{\scriptstyle \cong} \\ M_\tau & \xrightarrow{\varepsilon\cdot} & M_\tau \end{array}$$

- The equality $\underline{\mathrm{Hom}}_A(M, M_\tau) = \mathrm{Hom}_A(M, M_\tau)$ follows that there is no such an isomorphism even in $\underline{\mathrm{mod}}\text{-}A$.
- So $f : \mathrm{Id} \to (-)_\tau$ is NOT a natural transformation. It contradicts to the assumption! $\square$

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- We consider the module $M := P_0/\mathrm{rad}^2(P_0)$, where $P_0 = e_0 A$, and prove that $\underline{\mathrm{Hom}}_A(M, M_\tau) = \mathrm{Hom}_A(M, M_\tau)$
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Thank you for your attention!