Cluster algebras and symmetric matrices

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$\mathbb{T}_n$: n-regular tree
$t_0$: initial vertex
$B_0 = B_{t_0}$: $n \times n$ skew-symmetrizable matrix (initial exchange matrix)
$c_0 = c_{t_0}$: standard basis of $\mathbb{Z}^n$

To each $t$ in $\mathbb{T}_n$ assign $(c_t, B_t) = (c, B)$, a "Y-seed", such that $(c', B') := \mu_k(c, B)$:
The entries of the exchange matrix \( B' = (B'_{ij}) \) are given by

\[
B'_{ij} = \begin{cases} 
-B_{ij} & \text{if } i = k \text{ or } j = k; \\
B_{ij} + [B_{ik}]_+ + [B_{kj}]_+ - [-B_{ik}]_+ + [-B_{kj}]_+ & \text{otherwise.}
\end{cases}
\]  

(1)

The tuple \( c' = (c'_1, \ldots, c'_n) \) is given by

\[
c'_i = \begin{cases} 
-c_i & \text{if } i = k; \\
c_i + [\text{sgn}(c_k) B_{k,i}]_+ c_k & \text{if } i \neq k.
\end{cases}
\]  

(2)
$B$: skew-symmetrizable $n \times n$ matrix

*Diagram* of $B$ is the directed graph such that

- vertices: 1, ..., $n$
- $i \rightarrow j$ if and only if $B_{j,i} > 0$
  - the edge is assigned the weight $|B_{i,j}B_{j,i}|$
  - (if the weight is 1 then we omit it in the picture)

**Quiver notation:**

Diagram of a skew-symmetric matrix = Quiver

- $B_{j,i} > 0$ many arrows from $i$ to $j$
quiver notation

4

diagram notation
$B$: skew-symmetrizable $n \times n$ matrix such that $\Gamma(B)$ is acyclic, i.e.

- $\Gamma(B)$ has no oriented cycles at all.

$A$: the associated generalized Cartan matrix

- $A_{i,i} = 2$
- $A_{i,j} = -|B_{i,j}|$

$\alpha_1, \ldots, \alpha_n$: simple roots

$Q = \text{span}(\alpha_1, \ldots, \alpha_n) \cong \mathbb{Z}^n$: root lattice

$s_i = s_{\alpha_i}: Q \to Q$: reflection

- $s_i(\alpha_j) = \alpha_j - A_{i,j}\alpha_i$

real roots: vectors obtained from the simple roots by a sequence of reflections
$(c_0, B_0)$: initial $Y$-seed with $\Gamma(B_0)$ acyclic

Theorem (Speyer, Thomas)

Each $c$-vector is the coordinate vector of a real root in the basis of simple roots.
$B$: skew-symmetrizable

A **quasi-Cartan companion** of $B$ is a symmetrizable matrix $A$:

- $A_{i,i} = 2$
- $A_{i,j} = \pm B_{i,j}$ for all $i \neq j$.

Diagram of $B$:

```
  4
 /\  \
0 ---- 0 ---- 0
   \  /
    0
```

Diagram of a quasi-Cartan companion of $B$:

```
(+)
/\  \
0 ---- 0 ---- 0
   \  /
    0
```

(-)

(+)

(-)
$B_0$: skew-symmetrizable $n \times n$ matrix such that $\Gamma(B_0)$ is acyclic

$A_0$: the associated generalized Cartan matrix

$B$ is mutation-equivalent to $B_0$

Definition (Barot, Marsh; Parsons)

$\beta_1, \ldots, \beta_n$ real roots form a companion basis for $B$ if

$A = (\langle \beta_i, \beta_j \rangle)$ is a quasi-Cartan companion of $B$

- If $B$ is skew-symmetric, then these form a companion basis if $A = (\beta_i^T A_0 \beta_j)$.

($\beta_i^T$ denotes the transpose of $\beta_i$ viewed as a column vector.)
$B_0$: skew-symmetric matrix such that $\Gamma(B_0)$ is acyclic  
$A_0$: the associated generalized Cartan matrix  
$(c_0, B_0)$: initial $Y$-seed  
$(c, B)$: arbitrary $Y$-seed

Theorem (S.)

$A = (c_i^T A_0 c_j)$ is a quasi-Cartan companion of $B$

Furthermore:

- If $\text{sgn}(B_{j,i}) = \text{sgn}(c_j)$, then $A_{j,i} = c_j^T A_0 c_i = -\text{sgn}(c_j) B_{j,i}$.
- If $\text{sgn}(B_{j,i}) = -\text{sgn}(c_j)$, then $A_{j,i} = c_j^T A_0 c_i = \text{sgn}(c_i) B_{j,i}$.

In particular; if $\text{sgn}(c_j) = -\text{sgn}(c_i)$, then $B_{j,i} = \text{sgn}(c_i) c_j^T A_0 c_i$. 
More properties of the “c-vector companion” $A$:

- Every directed path of the diagram $\Gamma(B)$ has at most one edge $\{i, j\}$ such that $A_{i,j} > 0$.

- Every oriented cycle of the diagram $\Gamma(B)$ has exactly one edge $\{i, j\}$ such that $A_{i,j} > 0$.

- Every non-oriented cycle of the diagram $\Gamma(B)$ has an even number of edges $\{i, j\}$ such that $A_{i,j} > 0$. 
Definition
A set $C$ of edges in $\Gamma(B)$ is called an “admissible cut” if

- every oriented cycle contains exactly one edge in $C$
  (for quivers with potentials, also introduced by Herschend, Iyama; for cluster tilting, introduced by Buan, Reiten, S.)
- every non-oriented cycle contains exactly an even number of edges in $C$.

If $\Gamma(B)$ is mutation-equivalent to an acyclic diagram, then it has an admissible cut of edges: those $\{i, j\}$ such that $A_{i,j} > 0$. 

$B$: skew-symmetric matrix
Equivalently:

if the diagram of a skew-symmetric matrix does not have an admissible cut of edges, then it is not mutation-equivalent to any acyclic diagram.