Cluster algebras and symmetric matrices

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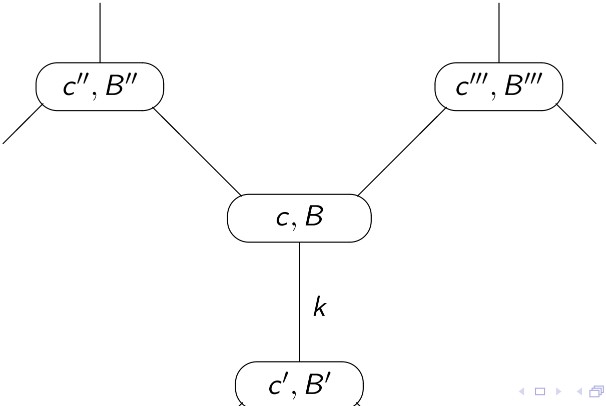
$\mathbb{T}_n$: n-regular tree
t_0: initial vertex
$B_0 = B_{t_0}$: $n \times n$ skew-symmetrizable matrix (initial exchange matrix)
$\mathbf{c}_0 = \mathbf{c}_{t_0}$: standard basis of $\mathbb{Z}^n$
To each t in $\mathbb{T}_n$ assign $(\mathbf{c}_t, B_t) = (\mathbf{c}, B)$, a "$Y$-seed", such that $(\mathbf{c}', B') := \mu_k(\mathbf{c}, B)$:



- The entries of the exchange matrix $B' = (B'_{ij})$ are given by

$$B'_{ij} = \begin{cases} -B_{ij} & \text{if } i = k \text{ or } j = k; \\ B_{ij} + [B_{ik}]_+[B_{kj}]_+ - [-B_{ik}]_+[-B_{kj}]_+ & \text{otherwise.} \end{cases} \tag{1}$$

- The tuple $\mathbf{c}' = (\mathbf{c}'_1, \ldots, \mathbf{c}'_n)$ is given by

$$\mathbf{c}'_i = \begin{cases} -\mathbf{c}_i & \text{if } i = k; \\ \mathbf{c}_i + [sgn(\mathbf{c}_k)B_{k,i}]_+\mathbf{c}_k & \text{if } i \neq k. \end{cases} \tag{2}$$

B: skew-symmetrizable $n \times n$ matrix

Diagram of B is the directed graph such that

- vertices: $1, \ldots, n$
- $i \longrightarrow j$ if and only if $B_{j,i} > 0$
    - the edge is assigned the weight $|B_{i,j} B_{j,i}|$
    - (if the weight is 1 then we omit it in the picture)

Quiver notation:

Diagram of a skew-symmetric matrix = Quiver

- $B_{j,i} > 0$ many arrows from i to j

quiver notation

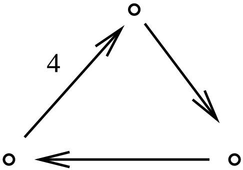


diagram notation

B: skew-symmetrizable $n \times n$ matrix such that $\Gamma(B)$ is *acyclic*, i.e.

- $\Gamma(B)$ has no oriented cycles at all.

A: the associated generalized Cartan matrix

- $A_{i,i} = 2$
- $A_{i,j} = -|B_{i,j}|$

$\alpha_1, \ldots, \alpha_n$: simple roots
$Q = span(\alpha_1, \ldots, \alpha_n) \cong \mathbb{Z}^n$: root lattice
$s_i = s_{\alpha_i}: Q \to Q$: reflection

- $s_i(\alpha_j) = \alpha_j - A_{i,j}\alpha_i$

real roots: vectors obtained from the simple roots by a sequence of reflections

$(\mathbf{c}_0, B_0)$: initial Y-seed with $\Gamma(B_0)$ acyclic

Theorem (Speyer, Thomas)

Each $\mathbf{c}$-vector is the coordinate vector of a real root in the basis of simple roots.

B: skew-symmetrizable

A *quasi-Cartan companion* of B is a symmetrizable matrix A:

- $A_{i,i} = 2$
- $A_{i,j} = \pm B_{i,j}$ for all $i \neq j$.

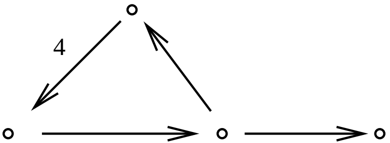
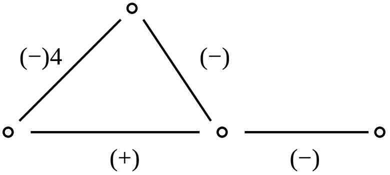


diagram of B



a quasi-Cartan companion of B

B_0: skew-symmetrizable $n \times n$ matrix such that $\Gamma(B_0)$ is acyclic
A_0: the associated generalized Cartan matrix
B is mutation-equivalent to B_0

Definition (Barot, Marsh; Parsons)

$\beta_1, \ldots, \beta_n$ real roots form a *companion basis* for B if
$A = (<\beta_i, \beta_j^\vee>)$ is a quasi-Cartan companion of B

- If B is skew-symmetric, then these form a companion basis if $A = (\beta_i^T A_0 \beta_j)$.

(β_i^T denotes the transpose of β_i viewed as a column vector.)

B_0: *skew-symmetric* matrix such that $\Gamma(B_0)$ is acyclic
A_0: the associated generalized Cartan matrix
$(\mathbf{c}_0, B_0)$: initial Y-seed
$(\mathbf{c}, B)$: arbitrary Y-seed

Theorem (S.)

$A = (\mathbf{c}_i^T A_0 \mathbf{c}_j)$ is a quasi-Cartan companion of B

Furthermore:

- If $sgn(B_{j,i}) = sgn(\mathbf{c}_j)$, then $A_{j,i} = \mathbf{c}_j^T A_0 \mathbf{c}_i = -sgn(\mathbf{c}_j) B_{j,i}$.
- If $sgn(B_{j,i}) = -sgn(\mathbf{c}_j)$, then $A_{j,i} = \mathbf{c}_j^T A_0 \mathbf{c}_i = sgn(\mathbf{c}_i) B_{j,i}$.

In particular; if $sgn(\mathbf{c}_j) = -sgn(\mathbf{c}_i)$, then $B_{j,i} = sgn(\mathbf{c}_i)\mathbf{c}_j^T A_0 \mathbf{c}_i$.

More properties of the "**c**-vector companion" A :

- Every *directed path* of the diagram $\Gamma(B)$ has *at most one* edge $\{i, j\}$ such that $A_{i,j} > 0$.
- Every *oriented cycle* of the diagram $\Gamma(B)$ has *exactly one* edge $\{i, j\}$ such that $A_{i,j} > 0$.
- Every *non-oriented cycle* of the diagram $\Gamma(B)$ has an *even number* of edges $\{i, j\}$ such that $A_{i,j} > 0$.

B: skew-symmetric matrix

Definition

A set C of edges in $\Gamma(B)$ is called an "admissible cut" if

- every oriented cycle contains exactly one edge in C

  (for quivers with potentials, also introduced by Herschend, Iyama; for cluster tilting, introduced by Buan, Reiten, S.)
- every non-oriented cycle contains exactly an even number of edges in C.

If $\Gamma(B)$ is mutation-equivalent to an acyclic diagram, then it has an admissible cut of edges: those $\{i, j\}$ such that $A_{i,j} > 0$.

Equivalently:

if the diagram of a skew-symmetric matrix does not have an admissible cut of edges, then it is not mutation-equivalent to any acyclic diagram.