Frobenius-Schur theorem for a class of $*$-algebras

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Frobenius-Schur theorem (1)

G: a compact group,

R: $G \to GL_n(\mathbb{C})$ an irrep. of G (cont. over $\mathbb{C}$).

χ: the character of R (i.e. $\chi = \mathrm{Tr} \circ R : G \to \mathbb{C}$).

- χ is **real** if $\exists P \in GL_n(\mathbb{C})$ s.t.
  $$P^{-1} R(g) P \in GL_n(\mathbb{R}) \quad \forall g \in G.$$
  Note: χ is real $\implies$ $\chi(g) \in \mathbb{R}$ for all $g \in G$.
- χ is **complex** if $\chi(g) \notin \mathbb{R}$ for some $g \in G$.
- χ is **quaternionic** (or pseudo-real) if it is not real but $\chi(g) \in \mathbb{R}$ for all $g \in G$.

Frobenius-Schur theorem (2)

Define the **Frobenius-Schur (FS) indicator** $\nu(\chi)$ by

$$\nu(\chi) = \int_G \chi(g^2) d\mu(g),$$

where μ is the Haar measure on G s.t. $\mu(G) = 1$.

Theorem(Frobenius-Schur)

$$\nu(\chi) = \begin{cases} +1 & \text{if } \chi \text{ is real,} \\ 0 & \text{if } \chi \text{ is complex,} \\ -1 & \text{if } \chi \text{ is quaternionic.} \end{cases}$$

There are many generalizations of this theorem...

FS theorem for Hopf algebras

H: fin.-dim'l semisimple Hopf alg.
V: a left H-module with character χ.

$\Rightarrow \nu(\chi)$ is defined by using the Haar integral of H.

Theorem (Linchenko-Montgomery)

If V is simple, then $\nu(\chi) \in \{0, \pm 1\}$. Moreover:

$$\nu(\chi) \neq 0 \Leftrightarrow \left[\begin{array}{c} V \text{ has a non-degenerate } H\text{-invariant} \\ \text{bilinear form } b : V \times V \to \mathbb{C} \end{array} \right].$$

If $\nu(\chi) \neq 0$, then the above b satisfies

$$b(w, v) = \nu(\chi) \cdot b(v, w) \quad \forall v, w \in V,$$

Question on the FS theorem for Hopf alg.

- The FS theorem for Hopf alg. is formulated in terms of invariant bilinear forms.
- Question: *Can we prove the following for Hopf alg.?*

$$\nu(\chi) = \begin{cases} +1 & \text{if } \chi \text{ is real,} \\ 0 & \text{if } \chi \text{ is complex,} \\ -1 & \text{if } \chi \text{ is quaternionic.} \end{cases}$$

- Before asking this question, we need to define real, complex and quaternionic rep. of a Hopf algebra.
- No definitions, since *"we do not have a canonical basis of H which plays the role of the group elements in the group algebra"* [LM].

Today's theme

- We propose a real-complex-quaternionic type FS theorem for a class of $*$-algebras.

We can apply the above results to s.s. Hopf $*$-algebras.

$\Rightarrow$ We can define real, complex and quaternionic representations of such a Hopf algebras.

$\Rightarrow$ For an irreducible $*$-representation of H, we have

$$\nu(\chi) = \begin{cases} +1 & \text{if } \chi \text{ is real,} \\ 0 & \text{if } \chi \text{ is complex,} \\ -1 & \text{if } \chi \text{ is quaternionic.} \end{cases}$$

Our framework

[arXiv:1208.2433]

"FS indicator for categories with duality".

- A category s.t. each object X has the 'dual' $X^\vee$.
- For its object X, the FS indicator $\nu(X)$ is defined.

- Pivotal algebras (Based on *'pivotal Hopf algebras'*).
  - If A is a pivotal algebra, then

  $\Rightarrow$ $\mathbf{Rep}(A)$ is a category with duality.

  $\Rightarrow$ FS indicator $\nu(V)$ is defined $\forall V \in \mathbf{Rep}(A)$.

[arXiv:1208.2435]

"Frobenius-Schur theorem for a class of $*$-algebras".

- Pivotal $*$-algebras $=$ Pivotal algebras with $*$-structure.
- Real, complex and quaternionic reps are defined.
- (Goal) The FS theorem for pivotal $*$-algebras.

Categories with duality

A category with duality is a category $\mathcal{C}$ equipped with

- a contrav. functor $(-)^\vee : \mathcal{C} \to \mathcal{C}$, $X \mapsto X^\vee$.
- a natural trans. $j_X : X \to X^{\vee\vee}$ $(X \in \mathcal{C})$.

satisfying

$$(j_X)^\vee \circ j_{X^\vee} = \mathrm{id}_{X^\vee} \quad (\forall X \in \mathcal{C}).$$

Example:

- $\mathcal{C} = \mathrm{Rep}(G)$, G a group.
- For $V \in \mathrm{Rep}(G)$, $V^\vee$ is the contragradient rep.

FS indicator for categories with duality

Suppose that $\mathcal{C}$ and $(-)^\vee$ are $\mathbb{C}$-linear, and

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{C}}(X, Y) < \infty \quad \forall X, Y \in \mathcal{C}.$$

Define $T_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y^\vee) \to \mathrm{Hom}_{\mathcal{C}}(Y, X^\vee)$ by

$$T_{X,Y}(f) : Y \xrightarrow{j_Y} Y^{\vee\vee} \xrightarrow{f^\vee} X^\vee.$$

Definition (FS indicator)

For $X \in \mathcal{C}$, define $\nu(X) := \mathrm{Tr}(T_{X,X})$.

- Based on the definition of the (higher) FS indicator for pivotal monoidal categories (Ng-Schauenburg).

Pivotal algebras

Pivotal Hopf algebras are defined so that its representation category is a pivotal monoidal category.

$$\begin{pmatrix} \text{pivotal} \\ \text{algebras} \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} \text{pivotal} \\ \text{Hopf algebras} \end{pmatrix} - \begin{pmatrix} \text{coalgebra} \\ \text{structure} \end{pmatrix}.$$

Definition

A pivotal algebra is a triple (A, S, g) consisting of

- an algebra A (over $\mathbb{C}$),
- an anti-algebra map $S : A \longrightarrow A$, and
- an invertible element $g \in A$

s.t. $S(g) = g^{-1}$ and $S^2(a) = gag^{-1}$ $(\forall a \in A)$.

Representation theory of pivotal algebras

Let $A = (A, S, g)$ be a pivotal algebra.

- For $V \in \mathrm{Rep}(V)$, define $V^\vee = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$.
- The action of $a \in A$ on $V^\vee$ is given by
$$(a \cdot f)(v) = f(S(a)v) \quad (f \in V^\vee, v \in V).$$
- Define $j_V : V \to V^{\vee\vee}$ by
$$j_V(v)(f) = f(gv) \quad (v \in V, f \in V^\vee).$$

Then $\mathrm{Rep}(A)$ is a category with duality.

$\Rightarrow$ The FS indicator $\nu(V)$ is defined $\forall V \in \mathrm{Rep}(A)$.

$\Rightarrow$ $\nu(V)$ for simple V has the following property: $\cdots$

FS theorem for pivotal algebras

Theorem

For a simple module $V \in \mathrm{Rep}(A)$, *we have*

$$\nu(V) \in \{0, \pm 1\}.$$

Moreover,

$$\nu(V) \neq 0 \Leftrightarrow \begin{bmatrix} \exists\text{non-degenerate bilinear form} \\ b : V \times V \longrightarrow \mathbb{C} \text{ s.t.} \\ b(av, w) = b(v, S(a)w). \end{bmatrix}$$

If $\nu(V) \neq 0$, *then the above* b *satisfies*

$$b(w, gv) = \nu(V) b(v, w).$$

$\cdots$ an analog of the FS theorem for Hopf algebras.

Formula for the FS indicator

Can we express $\nu(V)$ by the character of V?

- No formula like "$\nu(V) = \int_G \chi(g^2) d\mu(g)$".
- If A is separable, then we can prove:

Theorem

If $A = (A, S, g)$ is separable with separability idempotent $E = \sum E_i' \otimes E_i''$, then we have

$$\nu(V) = \sum \chi_V(S(E_i') g E_i'')$$

for all $V \in \mathrm{Rep}(A)$.

- For our purpose, this formula is not needed.

Pivotal $*$-algebra

A $*$-algebra is a $\mathbb{C}$-algebra with antilinear $* : A \to A$

s.t. $a^{**} = a$, $(ab)^* = b^* a^*$ $\forall a, b \in A$.

- $x \in A$ is positive if $x = a^* a$ for some a.

Definition

A pivotal $*$-algebra is a pivotal algebra (A, S, g) s.t. A is a $*$-algebra, g is positive and

$$(\#) \quad S(S(a)^*)^* = a \quad (\forall a \in A).$$

- Hopf $*$-algebras automatically satisfy $(\#)$ with $S =$ antipode.

Real, complex and quaternionic reps

Set $A_{\mathbb{R}} = \{a \in A \mid S(a)^* = a\}$.
V: a simple left A-module.

- V is **real** if $\exists$ basis of V s.t. the corresponding matrix rep $\rho : A \longrightarrow M_n(\mathbb{C})$ satisfies:
$$\rho(a) \in M_n(\mathbb{R}) \quad \forall a \in A_{\mathbb{R}}.$$
- V is **complex** if $\exists a \in A_{\mathbb{R}}$ s.t. $\chi(a) \notin \mathbb{R}$.
- V is **quaternionic** if it is neither real nor complex.

V: a $*$-representation

$\Leftrightarrow$ $\exists$ a Hermitian inner product $(-,-)$ on V s.t.
$$(v, aw) = (a^* v, w) \quad (a \in A, v, w \in V).$$

FS theorem for pivotal $*$-algebras

Theorem

If V is an irreducible $*$-representation, then

$$\nu(V) = \begin{cases} +1 & \text{if } \chi \text{ is real,} \\ 0 & \text{if } \chi \text{ is complex,} \\ -1 & \text{if } \chi \text{ is quaternionic.} \end{cases}$$

Example. H: fin.-dim'l weak Hopf C^*-algebra.

$\Rightarrow$ $A = (H, S, g)$ is a pivotal $*$-algebra, where

- $S : H \to H$ is the antipode,
- $g \in H$ is the canonical grouplike element.

$\Rightarrow$ FS theorem for fin.-dim'l weak Hopf C^*-algebras.

Further generalizations

Pivotal $*$-algebras arise from:

- Doi's grouplike algebras (a generalization of finite groups and association schemes).

Similar results hold in coalgebraic settings.

- Applications to corepresentation theory for a class of Hopf $*$-algebras, including compact quantum groups (in the sense of Woronowicz).

Thank you for your attention!

For more details on my talk, please refer to:

- [arXiv:1208.2433]
  Frobenius-Schur indicator for categories with duality.
- [arXiv:1208.2435]
  Frobenius-Schur theorem for a class of $*$-algebras.

Appendix. Group algebras (twisted case)

G: a finite group, $\tau \in \mathbf{Aut}(G)$ s.t. $\tau^2 = \mathbf{id}$.
Note: $\mathbb{C}G$ is a $*$-algebra by $g^* = g^{-1}$ $(g \in G)$.
Consider $A_\tau := (\mathbb{C}G, S_\tau, 1)$, where $S_\tau = \tau(g)^{-1}$.

1. A formula of the FS indicator $\nu^\tau(V)$:
$$\nu^\tau(V) = \frac{1}{|G|} \sum_{g \in G} \chi_V(\tau(g) \cdot g).$$
2. The real form $A_\mathbb{R}$:
$$(A_\tau)_\mathbb{R} = \mathrm{span}_\mathbb{R}\{\beta_g^{\mathrm{re}}, \beta_g^{\mathrm{im}} \mid g \in G\},$$
$$\beta_g^{\mathrm{re}} = \frac{1}{2}(g + \tau(g)), \quad \beta_g^{\mathrm{im}} = \frac{1}{2\sqrt{-1}}(g - \tau(g)).$$

By the theorem, we have

$$\nu^{\tau}(V) = \begin{cases} +1 & \text{if } \chi \text{ is `real'}, \\ 0 & \text{if } \chi \text{ is `complex'}, \\ -1 & \text{if } \chi \text{ is `quaternionic'}. \end{cases}$$

The meanings of 'real', 'complex', 'quaternionic' are different from the untwisted case. For example:

$$\begin{aligned} & \rho(a) \in M_n(\mathbb{R}) && \forall a \in (A_{\tau})_{\mathbb{R}} \\ \iff & \rho(\beta_g^{\mathrm{re}}), \rho(\beta_g^{\mathrm{im}}) \in M_n(\mathbb{R}) && \forall g \in G \\ \iff & \rho(\tau(g)) = \overline{\rho(g)} && \forall g \in G \end{aligned}$$

Note: $g = \beta_g^{\mathrm{re}} + \sqrt{-1}\beta_g^{\mathrm{im}}$, $\tau(g) = \beta_g^{\mathrm{re}} - \sqrt{-1}\beta_g^{\mathrm{im}}$.

Theorem (Kawanaka-Matsuyama)

Let χ be an irreducible character of G.
$\nu^\tau(\chi) = +1$ *if and only if* χ *is the character of a matrix rep.* $\rho : G \longrightarrow GL_n(\mathbb{C})$ *s.t.*

$$\rho(\tau(g)) = \overline{\rho(g)} \quad \forall g \in G.$$

$\nu^\tau(\chi) = 0$ *iff ...(omitted; interpret in the same way).*
$\nu^\tau(\chi) = -1$ *iff ...(again omitted).*

- We can also prove a 'twisted' FS theorem for group-like algebras.