On Ringel-Hall products and extensions in tame cases

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In general, it is difficult to give formulas for Ringel-Hall numbers. In the noncyclic tame quiver cases, certain Ringel-Hall numbers for special classes of modules have been given in the literature. More precisely in the Kronecker case we already have an almost complete description of these numbers (due to Baumann-Kassel, Pu Zhang and Szántó). For other tame cases there are some partial results involving special objects (due to Pu Zhang and Hubery).

Our aim is to obtain more information on Ringel-Hall numbers in tame cases leading us to results on cardinalities of quiver Grassmannians and extensions.

Let k be a finite field with q elements and consider the path algebra kQ where Q is a quiver of tame type, i.e. of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. When Q is of type $\tilde{A}_n$ we exclude the cyclic orientation.

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So kQ is a finite dimensional tame hereditary algebra where the category of finite dimensional (hence finite) right modules is denoted by $\mathrm{mod}\text{-}kQ$. Let $[M]$ be the isomorphism class of $M \in \mathrm{mod}\text{-}kQ$ and $\alpha_M = |\,\mathrm{Aut}(M)|$.

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The category $\mathrm{mod}\text{-}kQ$ can and will be identified with the category $\mathrm{rep}\text{-}kQ$ of the finite dimensional k-representations of the quiver Q.

The dimension vector of M is denoted by $\underline{\dim}\ M$.

We consider the Euler form $\langle , \rangle$ of kQ and the corresponding quadratic form $\mathbf{q}$.

Then $\mathbf{q}$ is positive semi-definite with radical $\mathbb{Z}\delta$.

Here the minimal radical vector δ is known for each type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ (see Dlab-Ringel).

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The defect of a module M is $\partial M = \langle \delta, \underline{\dim} M \rangle = -\langle \underline{\dim} M, \delta \rangle$

Consider the Auslander-Reiten translates $\tau = D\,\mathrm{Ext}^1(-, kQ)$ and $\tau^{-1} = \mathrm{Ext}^1(D(kQ), -)$, where $D = \mathrm{Hom}_k(-, k)$.
An indecomposable module M is preprojective (preinjective) if exists a positive integer m such that $\tau^m(M) = 0$ ($\tau^{-m}(M) = 0$).
Otherwise M is said to be regular.
Observe that an indecomposable module M is preprojective (preinjective, regular) iff $\partial M < 0$ ($\partial M > 0$, $\partial M = 0$).

The Auslander-Reiten quiver of kQ has a preprojective component (with all the isoclasses of indecomposable preprojectives), a preinjective component (with all the isoclasses of indecomposable preinjectives).
All the other components (containing the isoclasses of indecomposable regulars) are "tubes" of the form $\mathbb{Z}A_\infty/m$, where m is the rank of the tube.

The tubes are indexed by the points of the scheme $\mathbb{P}^1_k$, the degree of a point $x \in \mathbb{P}^1_k$ being denoted by $\deg x$.
A tube of rank 1 is called homogeneous, otherwise is called non-homogeneous. We have at most 3 non-homogeneous tubes indexed by points x of degree $\deg x = 1$. All the other tubes are homogeneous.

In case of a homogeneous tube $\mathcal{T}_x$ we will denote by $R_x[t]$ the indecomposable regular with quasi-socle $R_x[1]$ and quasi-length t.

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We consider now the rational Ringel-Hall algebra $\mathcal{H}(kQ)$ of the algebra kQ.

Its $\mathbb{Q}$-basis is formed by the isomorphism classes $[M]$ from $\mathrm{mod}\text{-}kQ$.

The multiplication is defined by

$$[N_1][N_2] = \sum_{[M]} F^M_{N_1 N_2}[M].$$

The structure constants $F^M_{N_1 N_2} = |\{U \subseteq M|\ U \cong N_2,\ M/U \cong N_1\}|$ are called Ringel-Hall numbers.

Lemma

Let P be an indecomposable preprojective with defect $\partial P = -1$.

a) *If $\mathcal{T}_x$ is a homogeneous tube then*
$\dim_k \mathrm{Hom}(P, R_x[1]) = \deg x \neq 0$.

b) *If $\mathcal{T}_x$ is a non-homogeneous tube of rank m then there is a unique $i_0 \in \{1, \ldots, m\}$ such that* $\dim_k \mathrm{Hom}(P, R_x^{i_0}[1]) = 1 \neq 0$ *and* $\dim_k \mathrm{Hom}(P, R_x^{i}[1]) = 0$ *for $i \neq i_0$.*

With the previous notations let $R_x^P[1] = R_x[1]$ for $\mathcal{T}_x$ homogeneous and $R_x^P[1] = R_x^{i_0}[1]$ for $\mathcal{T}_x$ non-homogeneous.

Theorem (Szántó)

Let $P \ncong P'$ be indecomposable preprojectives with defect -1.
If $\mathrm{Hom}(P, P') \neq 0$ *then* $F_{XP}^{P'} = 1$ *for* X *satisfying conditions*
i) X *is a regular module with* $\underline{\dim} X = \underline{\dim} P' - \underline{\dim} P$,
ii) *if* X *has an indecomposable component from a tube* $\mathcal{T}_x$, *then the quasi-top of this component is the quasi-simple regular* $R_x^{P'}[1]$,
iii) *the indecomposable components of* X *are taken from pairwise different tubes.*
Otherwise $F_{XP}^{P'} = 0$.

Theorem (Szántó)

For an indecomposable preprojective P of defect -1 and $\mathcal{T}_x$ a homogeneous tube we have

$$[R_x[\lambda]][P] = \sum q^{\deg x|\mu|} \cdot g^{\lambda}_{\mu(|\lambda-\mu|)}(q^{\deg x}) \cdot \frac{\alpha_{R_x[|\lambda-\mu|]}\alpha_{R_x[\mu]}}{\alpha_{R_x[\lambda]}} [P_\mu \oplus R_x[\mu]],$$

the summation going over all partitions μ such that $\lambda - \mu$ is a horizontal strip. Here P_μ denotes the indecomposable preprojective of dimension $\underline{\dim} P_\mu = \underline{\dim} P + \deg x(|\lambda| - |\mu|)\delta$. Also $g^{\lambda}_{\mu(|\lambda-\mu|)}$ is the classical Hall polynomial.

Theorem (Szántó)

Let $R_x^s = \bigoplus_{i=1}^m t_i R_x^i[1]$ *be a quasi-semisimple from the non-homogeneous tube* $\mathcal{T}_x$ *and* P *an indecomposable preprojective of defect -1. Suppose that* $\tau^{-1} R_x^P[1] = R_x^{i_0}[1]$.

a) *If* $t_{i_0} = 0$ *then*

$$[R_x^s][P] = q^{t_{i_0}-1}[P \oplus R_x^s].$$

b) *If* $t_{i_0} > 0$ *then*

$$[R_x^s][P] = q^{t_{i_0}-1}[P \oplus R_x^s] + [P' \oplus R_x'^s],$$

where P' *denotes the unique indecomposable preprojective of dimension* $\underline{\dim} P' = \underline{\dim} P + \underline{\dim} R_x^{i_0}[1]$*, up to isomorphism, and* $R_x'^s = t_1 R_x^1[1] \oplus \ldots \oplus (t_{i_0} - 1) R_x^{i_0}[1] \oplus \ldots \oplus t_m R_x^m[1]$.

Theorem (Hubery, Szántó)

Let I an indecomposable preinjective of defect 1 and P an indecomposable preprojective of defect -1.

a) *If* $\mathrm{Ext}^1(I,P)=0$ *then*
$[I][P]=q^{\dim_k \mathrm{Hom}(P,I)}[P\oplus I]=q^{\langle \underline{\dim} P, \underline{\dim} I\rangle}[P\oplus I]$.

b) *If* $\mathrm{Ext}^1(I,P)\neq 0$ *then* $[I][P]=q^{\dim_k \mathrm{Hom}(P,I)}[P\oplus I]+\frac{1}{q-1}\sum_{[X]}\alpha_X[X]$ *where the (nonempty) sum is taken over all modules X satisfying the following conditions*

i) *X is a regular module with* $\underline{\dim} X=\underline{\dim} I+\underline{\dim} P$,

ii) *if X has an indecomposable component from a tube $\mathcal{T}_x$, then the quasi-top of this component is the quasi-simple regular $R_x^P[1]$*

iii) *the indecomposable components of X are taken from pairwise different tubes.*

Let K be the Kronecker quiver and k a finite field with q elements. For any module $M \in \mathrm{mod}\text{-}kK$, and any $\underline{e} = (a, b)$ in $\mathbb{N}^2$, we denote by $Gr_{\underline{e}}(M)_k$ the Grassmannian of submodules of M with dimension vector $\underline{e}$:

$$Gr_{\underline{e}}(M)_k = \{N \in \mathrm{mod}\text{-}kK | N \leq M, \underline{dim}(N) = \underline{e}\}.$$

Then we have that

$$|Gr_{\underline{e}}(M)_k| = \sum_{\substack{[X],[Y] \\ \underline{dim} Y = \underline{e}}} F^M_{XY}$$

These cardinalities appear in the theory of quantized cluster algebras.

We denote by P_n the indecomposable preprojective of dimension $(n+1, n)$, by I_n the indecomposable preinjective of dimension $(n, n+1)$ and by $R_x[t]$ the indecomposable regular with quasi-socle $R_x[1]$ and quasi-length t from the tube $\mathcal{T}_x$ (every tube being homogeneous in the Kronecker case).

For $\ell, a \in \mathbb{Z}$, $\ell > 0$ we will denote by $\binom{a}{\ell}_q = \frac{(q^a-1)\dots(q^{a-\ell+1}-1)}{(q^\ell-1)\dots(q-1)}$ the Gaussian (q-binomial) coefficients.

Theorem (Szántó)

a) $$|Gr_{(a,b)}(P_n)_k| = \begin{cases} 0 & \textit{for } a<0 \textit{ or } b<0 \\ 1 & \textit{for } a=b=0 \\ \binom{n+1-b}{n+1-a}_q \binom{a-1}{a-b-1}_q & \textit{otherwise} \end{cases}$$

b) $$|Gr_{(a,b)}(I_n)_k| = \begin{cases} 0 & \textit{for } a>n \textit{ or } b>n+1 \\ 1 & \textit{for } a=n,\ b=n+1 \\ \binom{n-b}{a-b}_q \binom{a+1}{b}_q & \textit{otherwise} \end{cases}$$

c) *Suppose that* $\deg x = 1$. *Then*

$$|Gr_{(a,b)}(R_x[t])_k| = \begin{cases} 0 & \textit{for } a<0 \textit{ or } b<0 \\ \binom{t-b}{t-a}_q \binom{a}{a-b}_q & \textit{otherwise} \end{cases}$$

Let K be the Kronecker quiver and k an arbitrary field. Let I, I' be preinjective (decomposable) modules.

There is a nice combinatorial rule due to Szántó and Szöllősi describing the extensions of I by I'.

Thank you for your attention!