Classifying resolving subcategories by grade consistent functions

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Joint work with Hailong Dao

Building a module out of another module

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Question

Let $\mathbf{M}, \mathbf{N}$ be finitely generated modules. When can $\mathbf{M}$ be built out of $\mathbf{N}$ by taking

- direct summands,
- extensions, and
- syzygies?

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- direct summands,
- extensions, and
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$\langle \mathbf{X} \rangle =$ the smallest subcategory of $\operatorname{mod} \mathbf{R}$ containing $\mathbf{X}$
and closed under the above three operations

When $\mathbf{M} \in \langle \mathbf{N} \rangle$?

When $\mathbf{M} \notin \langle \mathbf{N} \rangle$?

Example

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1. $R/(xy) \in \langle R/(x) \oplus R/(y) \oplus k \rangle$
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1. There is an exact sequence
$0 \to R/(xy) \to R/(x) \oplus R/(y) \to k \to 0,$
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   which induces an exact sequence
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2. If $R/(x) \in \langle R/(y) \rangle$, then
   $(R/(x))_{(x)} \in \langle (R/(y))_{(x)} \rangle = \operatorname{proj} R_{(x)}$, a contradiction.

Motivation

Want to establish a criterion for $\mathbf{M}$ to be in $\langle \mathbf{N} \rangle$.

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Since $\langle \mathbf{N} \rangle$ is resolving, we should be done if we can:

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Classify resolving subcategories.

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Classify resolving subcategories.

Example

Using our classification, we will see (without making exact sequences):

$$R = k[[x, y]]/(xy) \ \Rightarrow\ R/(x) \in \langle R/(x - y) \oplus (x, y) \rangle$$

Resolving subcategories in PD(R)

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$\mathrm{PD}(R) = \{ M \in \operatorname{mod} R \mid \operatorname{pd} M < \infty \}$

Definition

An $\mathbb{N}$-valued function f on $\operatorname{Spec} R$ is **grade consistent** if:

- $f(\mathfrak{p}) \leq \operatorname{grade}(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$,
- If $\mathfrak{p} \subseteq \mathfrak{q}$ in $\operatorname{Spec} R$, then $f(\mathfrak{p}) \leq f(\mathfrak{q})$.

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Theorem 1

R = commutative Noetherian ring

$$\left\{ \begin{array}{c} \text{Resolving subcategories of} \\ \text{mod R contained in PD(R)} \end{array} \right\} \underset{\psi}{\overset{\phi}{\underset{\longleftarrow}{\overset{\longrightarrow}{1-1}}}} \left\{ \begin{array}{c} \text{Grade consistent} \\ \text{functions on Spec R} \end{array} \right\}$$

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$$\phi(\mathcal{X}) = [\, \mathfrak{p} \mapsto \max_{X \in \mathcal{X}} \{\operatorname{pd} X_{\mathfrak{p}}\} \,]$$
$$\psi(f) = \{\, M \in \operatorname{mod} R \mid \operatorname{pd} M_{\mathfrak{p}} \le f(\mathfrak{p}) \text{ for all } \mathfrak{p} \,\}$$

Remark

Angeleri Hügel, Pospíšil, Šťovíček and Trlifaj have recently given essentially the same classification as Theorem 1 independently.

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Corollary (Auslander (1962))

$\mathbf{R}$ = regular ring   TFAE for $\mathbf{M}, \mathbf{N} \in \mathbf{PD(R)}$:

1. $\mathrm{pd}\, M_{\mathfrak{p}} \leq \mathrm{pd}\, N_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}\, R$.
2. $\mathrm{Supp}\, \mathrm{Tor}_i^R(M, X) \subseteq \mathrm{Supp}\, \mathrm{Tor}_i^R(N, X)$ for all $i > 0$ and all $X \in \mathrm{mod}\, R$.

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In fact, these two conditions are equivalent to $\mathsf{M} \in \langle \mathsf{N} \rangle$.

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$R =$ Cohen-Macaulay ring

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$\phi(\mathcal{X}) = [\,\mathfrak{p} \mapsto \operatorname{ht} \mathfrak{p} - \min_{X \in \mathcal{X}}\{\operatorname{depth} X_\mathfrak{p}\}\,]$

$\psi(f) = \{\, M \in \operatorname{mod} R \mid \operatorname{depth} M_\mathfrak{p} \geq \operatorname{ht} \mathfrak{p} - f(\mathfrak{p}) \text{ for all } \mathfrak{p} \,\}$

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Corollary

$(R, \mathfrak{m}, k)$ = d-dimensional Cohen-Macaulay local ring
If R is an isolated singularity, then:

$$\left\{ \begin{matrix} \text{Resolving subcategories} \\ \text{of } \mathrm{mod}\, R \text{ containing } \Omega^d k \end{matrix} \right\} \underset{\underset{\psi}{\longleftarrow}}{\overset{\phi}{\underset{1-1}{\longrightarrow}}} \left\{ \begin{matrix} \text{Grade consistent} \\ \text{functions on } \mathrm{Spec}\, R \end{matrix} \right\}$$

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Example

$(R, \mathfrak{m}) = 1$-dimensional Cohen-Macaulay reduced local ring

$L =$ nonzero R-module of finite length

$\Rightarrow \phi(\langle L \oplus \mathfrak{m} \rangle) = \mathrm{ht}$

$\Rightarrow \langle L \oplus \mathfrak{m} \rangle = \psi(\mathrm{ht}) = \operatorname{mod} R$

In particular:

$$R = k[[x, y]]/(xy) \Rightarrow R/(x) \in \langle R/(x - y) \oplus (x, y) \rangle$$

Resolving subcategories over complete intersections

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Theorem 3

$\mathbf{R}$ = locally complete intersection ring

$$\left\{\begin{array}{c}\text{Resolving}\\ \text{subcategories}\\ \text{of } \operatorname{mod} \mathbf{R}\end{array}\right\} \underset{\Psi}{\overset{\Phi}{\underset{\longleftarrow}{\overset{\longrightarrow}{1-1}}}} \left\{\begin{array}{c}\text{Resolving}\\ \text{subcategories}\\ \text{contained}\\ \text{in } \mathbf{CM}(\mathbf{R})\end{array}\right\} \times \left\{\begin{array}{c}\text{Resolving}\\ \text{subcategories}\\ \text{contained}\\ \text{in } \mathbf{PD}(\mathbf{R})\end{array}\right\}$$

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$\mathrm{CM}(R) = \{\, M \in \operatorname{mod} R \mid M \text{ is maximal Cohen-Macaulay} \,\}$
$\Phi(\mathcal{X}) = (\mathcal{X} \cap \mathrm{CM}(R), \mathcal{X} \cap \mathrm{PD}(R))$
$\Psi(\mathcal{Y}, \mathcal{Z}) = \langle \mathcal{Y} \cup \mathcal{Z} \rangle$

Corollary

$\mathbf{R}$ = locally hypersurface ring

$$\left\{\begin{array}{c}\text{Resolving}\\ \text{subcategories}\\ \text{of } \mathbf{mod}\,\mathbf{R}\end{array}\right\} \underset{\Psi}{\overset{\Phi}{\underset{\longleftarrow}{\overset{\longrightarrow}{1-1}}}} \left\{\begin{array}{c}\text{Specialization}\\ \text{closed subsets}\\ \text{of } \mathbf{Sing}\,\mathbf{R}\end{array}\right\} \times \left\{\begin{array}{c}\text{Grade consistent}\\ \text{functions}\\ \text{on } \mathbf{Spec}\,\mathbf{R}\end{array}\right\}$$

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$\Phi(\mathcal{X}) = (\mathrm{IPD}(\mathcal{X}), \phi(\mathcal{X}))$
$\mathrm{IPD}(\mathcal{X}) = \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{pd} X_{\mathfrak{p}} = \infty \text{ for some } X \in \mathcal{X} \,\}$
$\Psi(W, f) = \{\, M \in \psi(f) \mid \mathrm{IPD}(M) \subseteq W \,\}$

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Theorem 4

Let $\mathbf{R}$ be a local complete intersection. Then the resolving subcategories contained in $\mathbf{CM(R)}$ are precisely the thick subcategories of $\mathbf{CM(R)}$ containing $\mathbf{R}$.

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Corollary

Let $\mathbf{R} = \mathbf{S}/(x_1, \ldots, x_n)$ be a local complete intersection. Using Therems 3,4 and Stevenson's classification theorem, one can classify all the resolving subcategories of $\mathbf{mod}\,\mathbf{R}$.

History of the classification of subcategories

Since 1960s

- Ring Theory
- Stable Homotopy Theory
- Algebraic Geometry
- Modular Representation Theory

Classifications of subcategories of module categories

1. Gabriel (1962), Garkusha–Prest (2008)
   Serre and localizing subcategories
2. Hovey (2000), T (2008), Krause (2008)
   Wide subcategories and full subcategories closed under submodules and extensions
3. Stanley–Wang (2011)
   Narrow subcategories and torsion classes
4. Krause–Stevenson (preprint)
   Thick subcategories containing projectives

Classifications of thick and localizing subcategories of triangulated categories

1. Devinatz–Hopkins–Smith (1988), Hopkins–Smith (1998)
   Stable homotopy categories
2. Hopkins (1988), Neeman (1992), Thomason (1997)
   Derived categories of commutative rings and schemes
3. Benson–Carlson–Rickard (1997), Friedlander–Pevtsova (2007), Benson–Iyengar–Krause (2011)
   Stable and derived categories of finite groups and group schemes
4. T (2010), Stevenson (preprint)
   Singularity categories of complete intersections

Classifications of resolving subcategories

1. Auslander–Reiten (1991)
   Contravariantly finite resolving subcategories over an Artin algebra of finite global dimension
2. T (2011)
   Contravariantly finite resolving subcategories over a Gorenstein Henselian local ring
3. T (2010)
   Resolving subcategories contained in $\mathsf{CM}(R)$ over a local hypersurface R

Examples of a grade consistent function

$f : \operatorname{Spec} R \longrightarrow \mathbb{N}$, $\mathfrak{p} \in \operatorname{Spec} R$

1. $f(\mathfrak{p}) = 0$
2. $f(\mathfrak{p}) = \operatorname{grade}(\mathfrak{p})$
3. $R = \text{domain} \quad f(\mathfrak{p}) = \begin{cases} 1 & (\mathfrak{p} \neq 0) \\ 0 & (\mathfrak{p} = 0) \end{cases}$

By the map ϕ in Theorem 1:

1. $\mapsto \operatorname{proj} R$
2. $\mapsto \operatorname{PD}(R)$
3. $\mapsto \{ M \in \operatorname{mod} R \mid \operatorname{pd} M \leq 1 \}$

By the map ϕ in Theorem 2:

1. $\mapsto \operatorname{CM}(R)$
2. $\mapsto \operatorname{mod} R$
3. $\mapsto \{ M \in \operatorname{mod} R \mid \operatorname{CMdim} M \leq 1 \}$

Resolving subcategories in $\mathsf{PD}_0(R)$

Corollary

$(R, \mathfrak{m}, k)$ = commutative Noetherian local ring of depth t
All the resolving subcategories contained in $\mathsf{PD}_0(R)$ are:

$$\operatorname{proj} R = \mathsf{PD}_0^0(R) \subsetneq \mathsf{PD}_0^1(R) \subsetneq \cdots \subsetneq \mathsf{PD}_0^t(R) = \mathsf{PD}_0(R).$$

Moreover, for each integer $1 \leq n \leq t$:

$$\mathsf{PD}_0^n(R) = \langle \operatorname{Tr}\Omega^{n-1}k \rangle.$$

$\mathsf{PD}_0(R) = \{\, M \in \mathsf{PD}(R) \mid M_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-free for all $\mathfrak{p} \neq \mathfrak{m} \,\}$
$\mathsf{PD}_0^n(R) = \{\, M \in \mathsf{PD}_0(R) \mid \operatorname{pd} M \leq n \,\}$

Criteria for a module to be in a resolving subcategory

Corollary

$\mathcal{X}$ = resolving subcategory of $\operatorname{mod} R$ contained in $\mathrm{PD}(R)$
TFAE for $M \in \operatorname{mod} R$:

1. $M \in \mathcal{X}$.
2. $\operatorname{pd} M_{\mathfrak{p}} \le \max_{X \in \mathcal{X}}\{\operatorname{pd} X_{\mathfrak{p}}\}$ for all $\mathfrak{p}$.

Corollary

R = Cohen-Macaulay ring
$\mathcal{X}$ = dominant resolving subcategory of $\operatorname{mod} R$
TFAE for $M \in \operatorname{mod} R$:

1. $M \in \mathcal{X}$.
2. $\operatorname{depth} M_{\mathfrak{p}} \ge \min_{X \in \mathcal{X}}\{\operatorname{depth} X_{\mathfrak{p}}\}$ for all $\mathfrak{p}$.

1-dimensional Cohen-Macaulay local rings

Corollary

R = Cohen-Macaulay local ring of dimension 1

Then $\{\text{Grade consistent functions}\} = \{0, \mathbf{grade}\}$. Hence:

$$\left\{\begin{matrix}\text{Resolving subcategories} \\ \text{contained in } \mathsf{PD}(R)\end{matrix}\right\} = \{\mathsf{proj}\, R, \mathsf{PD}(R)\},$$

$$\left\{\begin{matrix}\text{Dominant resolving} \\ \text{subcategories}\end{matrix}\right\} = \{\mathsf{CM}(R), \mathsf{mod}\, R\}.$$

Basic definitions

1. An extension of modules M and N is a module E such that there is an exact sequence
$$0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0.$$
2. The grade of an ideal I is defined as:
$$\operatorname{grade}(I) = \inf\{\, i \geq 0 \mid \operatorname{Ext}^i_R(R/I, R) \neq 0 \,\}.$$
This is equal to the maximum of the lengths of regular sequences in I.
3. A local ring $(R, \mathfrak{m})$ is an isolated singularity if $R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$.

4. A subset W of $\operatorname{Spec} R$ is specialization closed if:

$$\mathfrak{p} \in W,\ \ \mathfrak{q} \in \operatorname{Spec} R,\ \ \mathfrak{p} \subseteq \mathfrak{q} \ \ \Rightarrow \ \ \mathfrak{q} \in W.$$

This is nothing but a union of Zariski-closed subsets.

5. The singular locus of R is defined as:

$$\operatorname{Sing} R = \{\, \mathfrak{p} \in \operatorname{Spec} R \mid R_{\mathfrak{p}} \text{ is not regular} \,\}.$$

6. A thick subcategory $\mathcal{X}$ of $\mathrm{CM}(R)$ is a full subcategory closed under direct summands and satisfying the "2 out of 3 property": for an exact sequence

$$0 \to L \to M \to N \to 0$$

of modules in $\mathrm{CM}(R)$, if two of L, M, N are in $\mathcal{X}$, then so is the third.