Trees and Locally Free Modules

or

from decomposability to non-deconstructibility via infinite dimensional tilting theory

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Decomposable classes

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Some classic examples

- [Kaplansky'58] The class $\mathcal{P}_0$ is decomposable (and $\kappa = \aleph_1$ is sufficient for all R).
- [Faith-Walker'67] The class $\mathcal{I}_0$ of all injective modules is decomposable, iff R is right noetherian.
- [Gruson-Jensen'73, Huisgen-Zimmermann'79] Mod-R is decomposable, iff R is right pure-semisimple ($\kappa = \aleph_0$ is sufficient in this case).

Transfinite extensions and deconstructible classes

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Let $\mathcal{C}$ be a class of modules. A module M is a transfinite extension of the modules in $\mathcal{C}$, provided that there exists an increasing sequence $(M_\alpha \mid \alpha \leq \sigma)$ consisting of submodules of M such that $M_0 = 0$, $M_\sigma = M$, and

- $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for each limit ordinal $\alpha \leq \sigma$, and
- for each $\alpha < \sigma$, $M_{\alpha+1}/M_\alpha$ is isomorphic to an element of $\mathcal{C}$.

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Definition (Eklof)

A class of modules $\mathcal{A}$ is deconstructible, provided there is a cardinal κ such that $\mathcal{A} = \mathrm{Trans}(\mathcal{A}_\kappa)$, where $\mathcal{A}_\kappa$ denotes the class of all $< \kappa$-presented modules from $\mathcal{A}$.

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For each $n < \omega$, the classes $\mathcal{P}_n$, $\mathcal{I}_n$, and $\mathcal{F}_n$ of all modules of projective, injective, and flat dimension $\leq n$ are deconstructible.

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For each set of modules $\mathcal{S}$, the class ${}^\perp(\mathcal{S}^\perp)$ is deconstructible.
Here, $\mathcal{S}^\perp = \mathrm{KerExt}^1_R(\mathcal{S}, -)$, and ${}^\perp\mathcal{B} = \mathrm{KerExt}^1_R(-, \mathcal{B})$.

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The ubiquity of approximations [Saorín-Šťovíček'11], [Enochs'12]

All deconstructible classes are precovering; the ones closed under products are also preenveloping.

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Assume that R is not right perfect. Then the class of all flat Mittag-Leffler modules is not deconstructible.

A module M is flat Mittag-Leffler provided the functor $M \otimes_R -$ is exact, and for each system of left R-modules $(N_i \mid i \in I)$, the canonical map $M \otimes_R \prod_{i \in I} N_i \to \prod_{i \in I} M \otimes_R N_i$ is monic.

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A module M is locally $\mathcal{F}$-free, if M possesses a subset $\mathcal{S}$ consisting of countable direct sums of modules from $\mathcal{F}$, such that

- each countable subset of M is contained in an element of $\mathcal{S}$, and
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Note: If M is countably generated and locally $\mathcal{F}$-free, then M is a countable direct sum of modules from $\mathcal{F}$.

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In particular, the Baer-Specker group $\mathbb{Z}^\kappa$ is $\aleph_1$-projective for each κ, but not free.

Theorem (Herbera-T.'12)

flat Mittag-Leffler = $\aleph_1$-*projective.*

Trees for locally $\mathcal{F}$-free modules

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Let $\mathrm{Br}(T_\kappa)$ denote the set of all branches of T_κ. Each $\nu \in \mathrm{Br}(T_\kappa)$ can be identified with an ω-sequence of ordinals $< \kappa$, so

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So card $T_\kappa = \kappa$, and card $\mathrm{Br}(T_\kappa) = \kappa^\omega$.

Notation: $\ell(\tau)$ denotes the length of τ for each $\tau \in T_\kappa$.

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$\varinjlim_\omega \mathcal{F}$ denotes the class of all the modules N that are countable direct limits of the modules from $\mathcal{F}$. W.l.o.g., N is the direct limit of a chain

$$F_0 \xrightarrow{g_0} F_1 \xrightarrow{g_1} \dots \xrightarrow{g_{i-1}} F_i \xrightarrow{g_i} F_{i+1} \xrightarrow{g_{i+1}} \dots$$

with $F_i \in \mathcal{F}$ and $g_i \in \mathrm{Hom}_R(F_i, F_{i+1})$ for all $i < \omega$.

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Example

Let $\mathcal{F}$ be the class of all countably generated projective modules. Then $\varinjlim_\omega \mathcal{F}$ is the class of all countably presented flat modules.

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$$\pi_{\nu\restriction i}(x_{\nu i}) = x,$$

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Let $Y_{\nu i} = \{x_{\nu i} \mid x \in F_i\}$. Then $Y_{\nu i}$ is a submodule of P isomorphic to F_i.

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Put $X_{\nu i} = \sum_{j \leq i} Y_{\nu j}$. Then $X_{\nu i} \subseteq X_{\nu, i+1}$ for each $i < \omega$.

Let $X_\nu = \bigcup_{i<\omega} X_{\nu i}$, and $L = \sum_{\nu \in \mathrm{Br}(T_\kappa)} X_\nu$.

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Lemma

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Lemma (Griffith's trick)

Let M be a module such that $\mathrm{Ext}^1_R(A, M) = 0$ for each locally $\mathcal{F}$-free module A. Then $\mathrm{Ext}^1_R(N, M) = 0$ for each module $N \in \varinjlim_\omega \mathcal{F}$.

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- $\mathcal{F}$ a class of modules,
- $\mathcal{L}$ the class of all locally $\mathcal{F}$-free modules,
- $\mathcal{D}$ the class of all direct summands of the modules M that fit into an exact sequence

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where F' is a free module and C' is a countable direct sum of modules from $\mathcal{F}$.

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Theorem

Assume that $\mathcal{L}$ is closed under transfinite extensions, but there exists a countably presented module $C \in (\varinjlim_{\omega} \mathcal{F}) \setminus \mathcal{D}$. Then $\mathcal{L}$ is not deconstructible.

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T is a tilting module provided that

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A characterization of tilting classes

The tilting classes are exactly the classes of the form $\mathcal{S}^\perp$, where $\mathcal{S}$ is a set of strongly finitely presented modules of bounded projective dimension.

Non-deconstructibility via tilting

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Corollary

Let T be a tilting module which is a direct sum of countably presented modules, $T = \bigoplus_{i \in I} T_i$. Let $\mathcal{F}_T$ be a representative set of $\{T_i \mid i \in I\}$. Assume $\varinjlim_\omega \mathcal{F}_T \nsubseteq \mathcal{D}_T$. Then $\mathcal{L}_T$ is not deconstructible.

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Warning: The tilting module T above cannot be $\sum$-pure-injective.

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Then $N \in (\varinjlim_\omega \mathcal{F}_T) \setminus \mathcal{D}_T$, and $\mathcal{L}_T =$ the class of all $\aleph_1$-projective modules, so we recover

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Corollary (Herbera-T.'12)

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Lemma (Bazzoni-Eklof-T.'05)

The module $T_P = \bigcap_{p \in P} R_p \oplus \bigoplus_{q \in mspec(R) \setminus P} E(R/q)$ is tilting, and each tilting module is equivalent to one like this.

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Assume that $P \neq \emptyset$ and $mspec(R) \setminus P$ is countable. Then the class $\mathcal{L}_{T_P}$ is not deconstructible.

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- For $P = \mathrm{mspec}(R)$, we recover the flat Mittag-Leffler case in this setting.
- Warning: for $P = \emptyset$, we have $T_P = Q \oplus \bigoplus_{q \in \mathrm{mspec}(R)} E(R/q)$, and $\mathcal{L}_{T_P} = \mathcal{I}_0$ is deconstructible.

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Then $p^{\perp}$ is a tilting class (and $M \in p^{\perp}$, iff M has no non-zero finitely generated preprojective direct summands).

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- p a representative set of all indecomposable preprojective modules.

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Corollary (Slávik-T.)

If R is countable, then the generic module $G \in (\varinjlim_\omega \mathcal{F}_L) \setminus \mathcal{D}_L$. Thus the class $\mathcal{L}_L$ is not deconstructible.