Graded maximal Cohen-Macaulay modules over noncommutative graded Gorenstein isolated singularities

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Preliminaries

Throughout this talk,
k : an algebraically closed field of characteristic 0.
A : a finitely generated noetherian connected graded k-algebra, ie,

$$A = k\langle x_1, \dots, x_n\rangle / I \quad \text{where} \quad I : \text{homogeneous}, \deg x_i \in \mathbb{N}^+.$$

$\operatorname{grmod} A$: the category of f.g. graded right A-modules.
$\operatorname{tors} A$: the full subcategory of $\operatorname{grmod} A$ consisting of f.d. modules.
$\operatorname{tails} A := \operatorname{grmod} A / \operatorname{tors} A$.
$\operatorname{tails} A$ is called the noncommutative projective scheme associated to A (cf. [Artin-Zhang]).

Theorem 1 (Serre's theorem)

If $A = k[x_1, \dots, x_n]/I, \deg x_i = 1,$ *then* $\operatorname{tails} A \cong \operatorname{coh}(\operatorname{Proj} A)$.

Noncommutative graded Gorenstein isolated singularities

Definition 2

A is a graded isolated singularity $\overset{\text{def}}{\Longleftrightarrow}$ homological dimension of $\operatorname{tails} A$ is finite, ie,

$$\sup\{i \mid \operatorname{Ext}^i_{\operatorname{tails} A}(\mathcal{M}, \mathcal{N}) \neq 0 \text{ for some } \mathcal{M}, \mathcal{N} \in \operatorname{tails} A\} < \infty.$$

If $A = k[x_1, \dots, x_n]/I, \deg x_i = 1$, then $\operatorname{hdim}(\operatorname{tails} A) < \infty \Leftrightarrow A_{(\mathfrak{p})}$ is regular for any homogeneous prime ideal $\mathfrak{p} \neq \mathfrak{m}$.

Definition 3

A is AS-Gorenstein (AS-regular) of dim d and G-param ℓ $\overset{\text{def}}{\Longleftrightarrow}$

1. $\operatorname{id}_A A = \operatorname{id}_{A^{\mathrm{op}}} A = d < \infty$ ($\operatorname{gldim} A = d < \infty$), and
2. $\operatorname{Ext}^i_A(k, A) \cong \operatorname{Ext}^i_{A^{\mathrm{op}}}(k, A) \cong \begin{cases} k(\ell) & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

The aim of this talk

To study AS-Gorenstein isolated singularities!

Serre functors

Let A be an AS-Gorenstein algebra of dimension d.

1. $\mathrm{H}^i_{\mathfrak{m}}(M) := \lim_{n\to\infty} \mathrm{Ext}^i_A(A/A_{\geq n}, M)$.
2. $\mathrm{CM}^{\mathrm{gr}}(A) := \{M \in \operatorname{grmod} A \mid \mathrm{H}^i_{\mathfrak{m}}(M) = 0\ \forall i \neq d\}$.
3. $\underline{\mathrm{CM}}^{\mathrm{gr}}(A)$: the stable category of $\mathrm{CM}^{\mathrm{gr}}(A)$.
   ($\underline{\mathrm{CM}}^{\mathrm{gr}}(A)$ is a triangulated category w.r.t. $M[-1] = \Omega M$.)

Motivating Theorem (cf. [Iyama-Takahashi])

Let R be a noetherian commutative graded local Gorenstein ring. Assume that R is an isolated singularity. Then $\underline{\mathrm{CM}}^{\mathrm{gr}}(R)$ has the Serre functor.

Let A be an AS-Gorenstein algebra of dimension d and G-param ℓ.

$$\omega_A := \mathrm{H}^d_{\mathfrak{m}}(A)^* \cong A_\nu(-\ell) \quad \exists \nu \in \operatorname{GrAut} A$$

where A_ν is a graded A-A bimodule with action $b * x * a = bx\nu(a)$.
ω_A is called the canonical module of A.
Then the autoequivalence $- \otimes_A \omega_A : \operatorname{grmod} A \to \operatorname{grmod} A$ induces an autoequivalence

$$- \otimes_A \omega_A : \underline{\mathrm{CM}}^{\mathrm{gr}}(A) \to \underline{\mathrm{CM}}^{\mathrm{gr}}(A).$$

Theorem 4

Let A be an AS-Gorenstein algebra of dimension $d \geq 2$. TFAE.

1. *A is a graded isolated singularity.*
2. *$\underline{\mathrm{CM}}^{\mathrm{gr}}(A)$ has the Serre functor $- \otimes_A \omega_A[d-1]$.*

Proposition 5

Let A be an AS-regular algebra of dimension 2, and let G be a finite subgroup of $\operatorname{GrAut} A$ such that $\operatorname{hdet} \sigma = 1$ for all $\sigma \in G$. Then the fixed subring A^G is CM-representation-finite. Moreover, A^G is an AS-Gorenstein isolated singularity.

Example

Let

$$A = k\langle x, y\rangle/(xy + yx), \quad \deg x = \deg y = 1,$$

and let $G = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \leq \operatorname{GrAut} A$. Then A^G is an AS-Gorenstein isolated singularity. Hence $\underline{\mathrm{CM}}^{\mathrm{gr}}(A^G)$ has the Serre functor.

cluster tilting modules

Definition 6

Let A be an AS-Gorenstein algebra. $X \in \mathrm{CM}^{\mathrm{gr}}(A)$ is called an *n-cluster tilting module* if

$$\begin{aligned}\mathrm{add}\{X(s) | s \in \mathbb{Z}\} &= \{M \in \mathrm{CM}^{\mathrm{gr}}(A) \mid \mathrm{Ext}^i_A(M, X) = 0 \ (0 < i < n)\} \\ &= \{M \in \mathrm{CM}^{\mathrm{gr}}(A) \mid \mathrm{Ext}^i_A(X, M) = 0 \ (0 < i < n)\}.\end{aligned}$$

A is CM-representation-finite $\Leftrightarrow$ A has a 1-cluster tilting module.

The existence of n-cluster tilting module is a generalization of the notion of CM-representation finiteness.

Motivating Theorem (cf. [Iyama-Takahashi])

Let $S = k[x_1, \ldots, x_d], \deg x_i = 1$, G a finite subgroup of $\mathrm{SL}_d(k)$, and S^G the fixed subring of S. Then

1. $S * G \cong \mathrm{End}_{S^G}(S)$ as graded algebras.
2. Assume that S^G is an isolated singularity. Then $S \in \mathrm{CM}^{\mathrm{gr}}(S^G)$ is a $(d-1)$-cluster tilting module.

Theorem 7

Let $A = k\langle x_1, \ldots, x_n \rangle / I$ *be an AS-regular domain of dimension* $d \geq 2$ *and G-param* ℓ, $\deg x_i = 1$. *Take* $r \in \mathbb{N}^+$ *such that* $r \mid \ell$.

$$G = \left\langle \begin{pmatrix} \xi & 0 & \cdots & 0 \\ 0 & \xi & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \xi \end{pmatrix} \right\rangle \leq \operatorname{GrAut} A$$

where ξ *is a primitive* r*-th root of unity. Then*

1. $A * G \cong \operatorname{End}_{A^G}(A)$ *as graded algebras.*
2. A^G *is an AS-Gorenstein isolated singularity, and* $A \in \operatorname{CM}^{\mathrm{gr}}(A^G)$ *is a* $(d-1)$*-cluster tilting module.*
3. $\operatorname{gldim} \operatorname{End}_{A^G}(A) = d$.

We can obtain examples of cluster tilting modules over non-orders.

Example

Let $A =$

$$k\langle x, y\rangle/(\alpha xy^2 + \beta yxy + \alpha y^2x + \gamma x^3, \alpha yx^2 + \beta xyx + \alpha x^2y + \gamma y^3), \quad \deg x = \deg y = 1$$

where $\alpha, \beta, \gamma \in k$ are generic scalars. Then A is an AS-regular algebra of dimension 3 and G-param 4. Let

$$G = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} \right\rangle \leq \operatorname{GrAut} A$$

where ξ is a primitive 4-th root of unity. Then A^G is an AS-Gorenstein isolated singularity, and $A \in \mathrm{CM}^{\mathrm{gr}}(A^G)$ is a 2-cluster tilting module. Moreover, we have $\operatorname{gldim} \operatorname{End}_{A^G}(A) = 3$.