Almost split sequences in the category of complexes of modules

Razieh Vahed

The talk is based on a joint work with Prof. Salarian

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Schedule of the talk

History: AR-theory for complexes

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2. Let R be a commutative noetherian ring which is complete and local and Λ be a finitely generated R-algebra.
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     Moreover, they showed that for any compact object $\mathbf{X} \in \mathbf{K}(\mathrm{Inj}\Lambda)$, there exists an almost split triangle

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Motivation

Let Λ be an artin k-algebra where k is a commutative artin ring.

Question

When does the category of complexes of finitely generated left Λ-modules, $\mathbf{C}(\mathrm{mod}\Lambda)$, have almost split sequence?

In what follows, $\mathrm{Hom}(\mathbf{X}, \mathbf{Y})$ denote the abelian group of chain maps from **X** to **Y**.

A complex **P** is projective if the functor $\mathrm{Hom}(\mathbf{P}, \ )$ is exact.

This is equivalent to say that **P** is exact and $Z_n\mathbf{P} = \mathrm{Ker}(P_n \longrightarrow P_{n-1})$ is projective, for all $n \in \mathbf{Z}$.

So, for any projective module P, the complex

$$\cdots \longrightarrow 0 \longrightarrow P \longrightarrow P \longrightarrow 0 \longrightarrow \cdots$$

is projective. It is known that any projective complex can be written az a coproduct of such complexes.

Projective and Injective Complexes

In what follows, $\mathrm{Hom}(\mathbf{X}, \mathbf{Y})$ denote the abelian group of chain maps from $\mathbf{X}$ to $\mathbf{Y}$.

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is projective. It is known that any projective complex can be written az a coproduct of such complexes.

Dually, a complex $\mathbf{I}$ is injective if the contravariant functor $\mathrm{Hom}(\ ,\mathbf{I})$ is exact.

Again it is known that $\mathbf{I}$ is injective if and only if it is exact and $Z_n\mathbf{I}$ is injective, for all $n \in \mathbb{Z}$.

Therefore, if I is an injective module, the complex

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is injective.

Furthermore, up to isomorphism, any injective complex is a direct product of such complexes.

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Projective and Injective Complexes

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Let $\mathcal{P}(\mathbf{X}, \mathbf{Y})$ denote the subgroup of morphisms belong to $\mathrm{Hom}(\mathbf{X}, \mathbf{Y})$ such that factor through a projective complex.

We take $\underline{\mathrm{Hom}}(\mathbf{X}, \mathbf{Y}) = \mathrm{Hom}(\mathbf{X}, \mathbf{Y})/\mathcal{P}(\mathbf{X}, \mathbf{Y})$.

$\underline{\mathbf{C}}(\mathrm{mod}\Lambda)$ denotes the category with the same objects as $\mathbf{C}(\mathrm{mod}\Lambda)$ and morphism sets $\underline{\mathrm{Hom}}(\mathbf{X}, \mathbf{Y})$.

Dually, we have $\overline{\mathbf{C}}(\mathrm{mod}\Lambda)$.

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Let us fix some notations:

For a finitely generated Λ-module, the " Hom" functor

$$\mathsf{Hom}_\Lambda(\ ,M) : \mathbf{C}(\mathrm{mod}\Lambda) \longrightarrow \mathbf{C}(\mathrm{mod}\Lambda^{\mathrm{op}})$$

$$\mathbf{X} \longmapsto \mathsf{Hom}_\Lambda(\mathbf{X}, M)$$

i-th degree $\Longrightarrow$ $\mathsf{Hom}_\Lambda(X_{-i}, M)$
i-th differential $\Longrightarrow$ $\mathsf{Hom}_\Lambda(\partial_{-i+1}, M)$.
We denote $\mathsf{Hom}_\Lambda(\ ,\Lambda)$ by $(\ )^*$.

When I is the injective envelop $I = E(k/J(k))$,
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Transpose

Let $\mathbf{X} \in \mathbf{C}(\mathrm{mod}\Lambda)$ and

$$\mathbf{Q} \xrightarrow{q} \mathbf{P} \longrightarrow \mathbf{X} \longrightarrow 0$$

be a minimal projective presentation of $\mathbf{X}$. Applying the functor $(\ )^*$, we obtain a map $q^* : \mathbf{P}^* \longrightarrow \mathbf{Q}^*$ in $\mathbf{C}(\mathrm{mod}\Lambda^{\mathrm{op}})$.

We set $\mathrm{Tr}\mathbf{X} := \Sigma^{-1}\mathrm{Coker}q^*$, where Σ^{-1} is the shifting functor to the right.

In fact, $\mathrm{Tr} : \mathbf{C}(\mathrm{mod}\Lambda) \longrightarrow \underline{\mathbf{C}}(\mathrm{mod}\Lambda^{\mathrm{op}})$
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Aslander-Reiten Translation

We denote by τ the composite functor

$$\tau \ : \ \mathbf{C}(\mathrm{mod}\Lambda) \xrightarrow{Tr} \underline{\mathbf{C}}(\mathrm{mod}\Lambda^{\mathrm{op}}) \xrightarrow{\mathbf{D}} \overline{\mathbf{C}}(\mathrm{mod}\Lambda)$$

And τ^{-} denotes the composite functor

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Proposition

On $\mathbf{C}(\mathrm{mod}\Lambda)$

$$\{\text{ Non-projective with local endomorphism}\}$$

$$\Big\updownarrow \tau$$

$$\{\text{ Non-injective with local endomorphism}\}.$$

Proposition

$\{$ Non-projective with local endomorphism of $\mathbf{C}(\mathrm{Gprj}\Lambda)\}$

$$\updownarrow \tau$$

$\{$ Non-injective with local endomorphism of $\mathbf{C}(\mathrm{Ginj}\Lambda)\}$.

Almost split sequence

Let us recall that

a morphism $\beta : \mathbf{Y} \longrightarrow \mathbf{Z}$ is called right almost split if is not a retraction and

$$\begin{tikzcd} & \mathbf{Y}' \arrow[dl, dotted] \arrow[d] \\ \mathbf{Y} \arrow[r, "\beta"] & \mathbf{Z} \end{tikzcd}$$

Dually, a morphism $\alpha : \mathbf{X} \longrightarrow \mathbf{Y}$ is left almost split if α is not a section and

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Definition

An almost split sequence of complexes is an exact sequence

$$0 \longrightarrow \mathsf{X} \xrightarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{Z} \longrightarrow 0$$

in $\mathbf{C}(\mathrm{mod}\Lambda)$ where f is left almost split and g is right almost split.

Proposition

Let $0 \longrightarrow \mathbf{Y} \xrightarrow{g} \mathbf{W} \xrightarrow{f} \mathbf{Z} \longrightarrow 0$ be an exact sequence in $\mathbf{C}(\mathrm{mod}\Lambda)$. The following are equivalent:

- Every chain map $\mathbf{X} \longrightarrow \mathbf{Z}$ factors through f.
- Every chain map $\mathbf{Y} \longrightarrow \tau\mathbf{X}$ factors through g.

Existence theorem

Theorem

1. *Let* $\mathbf{X} \in \mathbf{C}(\mathrm{mod}\Lambda)$ *be a complex of* Λ*-modules with local endomorphism ring and* $0 \longrightarrow \tau\mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{X} \longrightarrow 0$ *be a non-trivial extension which vanishes on the radical* rad$\underline{\mathrm{End}}(\mathbf{X})^{op}$. *Then it is an almost split sequence.*

2. *Let* $\mathbf{X} \in \mathbf{C}(\mathrm{mod}\Lambda)$ *be a complex of* Λ*-modules with local endomorphism ring and* $0 \longrightarrow \mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \tau^{-}\mathbf{X} \longrightarrow 0$ *be a non-trivial extension which vanishes on the radical* rad$\overline{\mathrm{End}}(\mathbf{X})$. *Then it is an almost split sequence.*

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Examples

Example

Let $n \in \mathbb{N}$ and $\mathbf{X} \in \mathbf{C}(\mathrm{mod}\Lambda)$ be non-projective complex such that, for all $i \in \mathbb{Z}$, X_i is indecomposable, $d_i \neq 0$ and $l(X_i) \leq n$.

$\Longrightarrow$ $\mathrm{End}(\mathbf{X})$ is local and $\exists$ $0 \longrightarrow \tau\mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{X} \longrightarrow 0$ a.s.s in $\mathbf{C}(\mathrm{mod}\Lambda)$.

Especially, this happens when $\mathbf{X}$ is a complex of indecomposable projective or injective modules with $d_i \neq 0$.

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Examples

Also, the class of compact complexes of $\mathbf{K}(\mathrm{Inj}\Lambda)$ and $\mathbf{K}(\mathrm{Prj}\Lambda)$ can be considered as examples of unbounded complexes which admit almost split sequence.

Almost Split Triangle

The corresponding Auslander-Reiten theory for triangulated categories has been developed by Happle.

Definition (Happle)

A triangle $\mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{Z} \rightsquigarrow$ in a triangulated category $\mathcal{T}$ is called an almost split triangle, if f is left almost split and g is right almost split.

Given a triangulated category $\mathcal{T}$, we write $\mathcal{T}^c$ for the full subcategory of compact objects in $\mathcal{T}$.

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Theorem (Krause)

Let $\mathcal{T}$ be a triangulated category which is compactly generated. Let $Z \in \mathcal{T}^c$ and $\Gamma = \mathrm{End}_{\mathcal{T}}(Z)$ is local.

$\Longrightarrow \exists$ a.s.t $\Sigma^{-1}\mathbf{t}Z \xrightarrow{\alpha} Y \xrightarrow{\beta} X \xrightarrow{\gamma} \mathbf{t}Z$,

such that

$$\mathrm{Hom}_{\Gamma}(\mathrm{Hom}_{\mathcal{T}}(Z, \Gamma), I) \cong \mathrm{Hom}_{\mathcal{T}}(\Gamma, \mathbf{t}Z),$$

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Theorem

1. (Krause and Le) Let $\mathbf{Z} \in \mathbf{K}^{c}(\mathrm{Inj}\Lambda)$ which is indecomposable.

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2. (Le) Let $\mathbf{Z} \in \mathbf{K}^{c}(\mathrm{Prj}\Lambda)$ which is indecomposable.

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Homotopically minimal

Definition

A complex $\mathbf{X}$ is called *homotopically minimal* if $\varphi \in \mathrm{Hom}_{\mathbf{C}(\mathrm{mod}\Lambda)}(\mathbf{X}, \mathbf{X})$ is an isomorphism provided that φ is an isomorphism in $\mathbf{K}(\mathrm{mod}\Lambda)$.

It was proved that every complex in $\mathbf{C}(\mathrm{Prj}\Lambda)$ or $\mathbf{C}(\mathrm{Inj}\Lambda)$ has a decomposition $\mathbf{X} = \mathbf{X}' \coprod \mathbf{X}''$, such that $\mathbf{X}'$ is homotopically minimal which is unique up to isomorphism and $\mathbf{X}''$ is null homotopic.

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Theorem

1. Let $\mathbf{X} \in \mathbf{C}(\mathrm{prj}\Lambda)$ be a homotopically minimal complex with local endomorphism ring.
   Let $0 \longrightarrow \tau\mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{X} \longrightarrow 0$ be a.s.s in $\mathbf{C}(\mathrm{mod}\Lambda)$.

   $\Longrightarrow \tau\mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{X} \rightsquigarrow$ is a.s.t in $\mathbf{K}(\mathrm{mod}\Lambda)$.

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1. *For any* $\mathbf{X} \in \mathbf{K}^{c}(\mathrm{Prj}\Lambda)$ *with local endomorphism,*

$$\Longrightarrow \exists \text{ a.s.t } \tau\mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{X} \rightsquigarrow \text{ in } \mathbf{K}(\mathrm{mod}\Lambda).$$

2. *For any* $\mathbf{X} \in \mathbf{K}^{c}(\mathrm{Inj}\Lambda)$ *with local endomorphism,*

$$\Longrightarrow \exists \text{ a.s.t } \mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \tau^{-}\mathbf{X} \rightsquigarrow \text{ in } \mathbf{K}(\mathrm{mod}\Lambda).$$

Theorem

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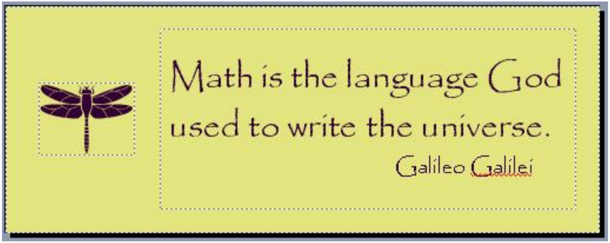
Theorem

The translation τ provides a one-to-one correspondence between the class of objects of $\mathbf{K}^{\mathrm{c}}(\mathrm{Prj}\Lambda)$ with local endomorphism ring and the class of objects of $\mathbf{K}^{\mathrm{c}}(\mathrm{Inj}\Lambda)$ with the same property.


Math is the language God used to write the universe.
Galileo Galilei