Pullback of finite dimensional algebras

Heily Wagner

Universidade de São Paulo - Brazil

This work is a part of the author’s PhD thesis under supervision of Prof. Flávio Ulhoa Coelho

Supported by CNPq

August 2012
Notation

- $k$ an algebraically closed field
- $A = kQ_A/I_A$ where $Q_A$ is a finite quiver and $I_A$ is an admissible ideal of $kQ_A$
- $e_A$ the identity element of $A$
- $\text{ind}A$ the category whose objects are isomorphism classes of indecomposable right $A$-modules
Let $A$, $B$ and $C$ be algebras. Given two epimorphisms $f_A: A \twoheadrightarrow B$ and $f_C: C \twoheadrightarrow B$ the pullback of $f_A$ and $f_C$ is the subalgebra of $A \times C$ defined by

$$R = \{(a, c) \in A \times C \mid f_A(a) = f_C(c)\}$$
Let $A$, $B$ and $C$ be algebras and $f_A : A \to B$ and $f_C : C \to B$ be epimorphisms. Let $R$ be the pullback of $f_A$ and $f_C$. We want to find relations between properties of $A$, $B$ and $C$ and properties of $R$.

(1) bounded quiver

(2) category of modules

(3) classes of algebras (hereditary, shod, quasitilted, etc)
The case in question

- $A = kQ_A/I_A$ and $C = kQ_C/I_C$
- $Q_B$ full and convex subquiver of $Q_A$ and $Q_C$
- $I_A \cap kQ_B = I_C \cap kQ_B := I_B$
The case in question

- $A = kQ_A/I_A$ and $C = kQ_C/I_C$
- $Q_B$ full and convex subquiver of $Q_A$ and $Q_C$
- $I_A \cap kQ_B = I_C \cap kQ_B := I_B$
- $B := kQ_B/I_B \cong e_B Ae_B \cong e_B Ce_B$ is a common quotient of $A$ and $C$
- $f_A: a \mapsto e_B ae_B$ and $f_C: c \mapsto e_B ce_B$ are epimorphisms
The case in question

- $A = kQ_A/I_A$ and $C = kQ_C/I_C$
- $Q_B$ full and convex subquiver of $Q_A$ and $Q_C$
- $I_A \cap kQ_B = I_C \cap kQ_B := I_B$
- $B := kQ_B/I_B \cong e_B Ae_B \cong e_B Ce_B$ is a common quotient of $A$ and $C$
- $f_A: a \mapsto e_B ae_B$ and $f_C: c \mapsto e_B ce_B$ are epimorphisms
- The pullback is $R = \{(a, c) \in A \times C \mid e_B ae_B = e_B ce_B\}$
The bounded quiver

Theorem (Igusa-Platzeck-Todorov-Zacharia - 1987)

Let $R$ be the pullback of $f_A: A \rightarrow B$ and $f_C: C \rightarrow B$. Let

- $Q_R = Q_A \bigsqcup_{Q_B} Q_C$ be the pushout of the inclusion $Q_B \rightarrow Q_A$ and $Q_B \rightarrow Q_C$.
- $I_R$ be the ideal of $kQ_R$ generated by $I_A$, $I_C$ and the paths linking $(Q_A)_0 \setminus (Q_B)_0$ and $(Q_C)_0 \setminus (Q_B)_0$.

Then $R \cong kQ_R/I_R$. 
Example

\[ Q_A = \begin{array}{c}\downarrow \gamma \\
4 \overset{\delta}{\rightarrow} 2 \overset{\alpha}{\rightarrow} 1 \end{array} \text{, } Q_B = \begin{array}{c}\downarrow \beta \\
2 \overset{\alpha}{\rightarrow} 1 \end{array} \text{ and } Q_C = \begin{array}{c}\uparrow \gamma \\
2 \overset{\alpha}{\rightarrow} 1 \end{array} \]

The pullback: \[ Q_R = \begin{array}{c}\downarrow \gamma \\
4 \overset{\delta}{\rightarrow} 2 \overset{\alpha}{\rightarrow} 1 \end{array} \]
Example

\[ Q_A = 4 \xrightarrow{\delta} 2 \xrightarrow{\alpha} 1 \] , \[ Q_B = 2 \xrightarrow{\alpha} 1 \] and \[ Q_C = 3 \xrightarrow{\beta} 1 \] 

The pullback: \[ Q_R = 4 \xrightarrow{\delta} 2 \xrightarrow{\alpha} 1 \]
Example

\[ Q_A = \begin{array}{ccc}
4 & \xrightarrow{\delta} & 2 \\
\gamma & \downarrow & \\
5 & \rightarrow & 1
\end{array} , \quad Q_B = \begin{array}{ccc}
2 & \xrightarrow{\alpha} & 1
\end{array} \quad \text{and} \quad Q_C = \begin{array}{ccc}
2 & \xrightarrow{\alpha} & 1 \\
\beta & \downarrow & \\
3 & \rightarrow & \\
\gamma & \downarrow & \\
5 & \rightarrow & 1
\end{array}

The pullback: \[ Q_R = \begin{array}{ccc}
4 & \xrightarrow{\delta} & 2 \\
\gamma & \downarrow & \\
5 & \rightarrow & 1
\end{array} \quad \text{and} \quad Q_C = \begin{array}{ccc}
2 & \xrightarrow{\alpha} & 1 \\
\beta & \downarrow & \\
3 & \rightarrow & \\
\gamma & \downarrow & \\
5 & \rightarrow & 1
\end{array} \]
Question

Is the same result true for the Auslander-Reiten quiver? That is, $\Gamma_R = \Gamma_A \bigsqcup_{\Gamma_B} \Gamma_C$?

Remark:

- $(\Gamma_R)_0$: objects of $\text{ind} R$
Question

Is the same result true for the Auslander-Reiten quiver? That is, $\Gamma_R = \Gamma_A \bigsqcup_{\Gamma_B} \Gamma_C$?

Remark:

- $(\Gamma_R)_0$: objects of $\text{ind}R$

$\text{ind}R = \text{ind}A \cup \text{ind}C$?
In the last example, the module $I(2) = 4^3_2$ is neither an $A$-module nor a $C$-module.

$$Q_R = \begin{array}{c}
5 \\
\downarrow \\
\gamma \\
\downarrow \\
2 \\
\alpha \\
\downarrow \\
1 \\
\downarrow \\
\beta \\
\downarrow \\
3
\end{array}$$

In this case $\text{ind} R \neq \text{ind} A \cup \text{ind} C$
Definition

Let $R$ be the pullback of $f_A: A \rightarrow B$ and $f_C: C \rightarrow B$. We say that $R$ is an **oriented pullback** if its bounded quiver satisfies

- there is no path from $(Q_B)_0$ to $(Q_C)_0 \setminus (Q_B)_0$ and neither from $(Q_A)_0 \setminus (Q_B)_0$ to $(Q_C)_0$
The oriented pullback

\[ Q_A \quad Q_B \quad Q_C \]
The oriented pullback

In this case,

- $P(x) \cong P_A(x)$ for any $x \in (Q_A)_0$
- $P(x) \cong P_C(x)$ for any $x \in (Q_C)_0 \setminus (Q_B)_0$
- $l(x) \cong l_C(x)$ for any $x \in (Q_C)_0$
- $l(x) \cong l_A(x)$ for any $x \in (Q_A)_0 \setminus (Q_B)_0$
Special kinds of oriented pullback

**Condition * (IPTZ87)**

1. $Q_B: \bullet \leftarrow \bullet \leftarrow \cdots \leftarrow \bullet$, with no relations;
2. for each arrow $\alpha: x \rightarrow y$ in $(Q_B)_1$ and each path $\phi: z \leadsto y$ from $z \in (Q_C)_0 \setminus (Q_B)_0$ to $y$ there is a path $\psi: z \leadsto x$ from $z$ to $x$ such that $\psi\alpha - \phi \in I_C$.

**Theorem (IPTZ87)**

Let $R$ be the pullback of $f_A: A \twoheadrightarrow B$ and $f_C: C \twoheadrightarrow B$ such that $Q_R$ satisfies Condition *. Then

$$\text{ind} R = \text{ind} A \cup \text{ind} C$$
Definition

Let $R$ be the pullback of $f_A: A \to B$ and $f_C: C \to B$. We say that $R$ is a **Dynkin oriented pullback** if it is an oriented pullback and its bounded quiver satisfies the following conditions:

1. $B$ is an hereditary algebra and each connected component is of Dynkin type with an unique sink;
2. for each arrow $\alpha : x \to y$ in $(Q_B)_1$ and each path $\phi : z \rightsquigarrow y$ from $z \in (Q_C)_0 \setminus (Q_B)_0$ to $y$ then there is a path $\psi : z \rightsquigarrow x$ from $z$ to $x$ such that $\psi \alpha - \phi \in I_C$.
Theorem 1

Let $R$ be the Dynkin oriented pullback of $A ightarrow B$ and $C ightarrow B$. Then

$$\text{ind} R = \text{ind} A \cup \text{ind} C$$
Example - condition *

\[ Q_A = \begin{array}{c}
1 \\
2 \\
\end{array} 
\begin{array}{c}
3 \\
\end{array} \quad Q_B = \begin{array}{c}
3 \\
2 \\
\end{array} \quad Q_C = \begin{array}{c}
3 \\
2 \\
\end{array} \]

The pullback \( R \) is

\[ \begin{array}{c}
1 \\
2 \\
\end{array} 
\begin{array}{c}
3 \\
\end{array} 
\begin{array}{c}
4 \\
\end{array} \]
Example

where \((Q_A)_0 \setminus (Q_B)_0 = \{1\}\) and \((Q_C)_0 \setminus (Q_B)_0 = \{13, 14, 15\}\). In this case \(Q_B\) has two components: \(D_5\) and \(E_6\).
Oriented pullback and classes of algebras

- $\Lambda$ is shod $\iff$ $\text{pd}M \leq 1$ or $\text{id}M \leq 1$ for any $M \in \text{ind}\Lambda$
- $\Lambda$ is quasitilted $\iff$ $\Lambda$ is shod and $\text{gl.dim}\Lambda \leq 2$
Oriented pullback and classes of algebras

Lemma
Let $R$ be the oriented pullback of $A \rightarrow B$ and $C \rightarrow B$. Then
- if $M \in \text{ind}A$ then $\text{pd}M = \text{pd}_AM$
- if $M \in \text{ind}C$ then $\text{id}M = \text{id}_CM$

Lemma
Let $R$ be the oriented pullback of $A \rightarrow B$ and $C \rightarrow B$ such that $A$ and $C$ are hereditary algebras. If $\text{ind}R = \text{ind}A \cup \text{ind}C$ then $R$ is a shod algebra.
Oriented pullback and classes of algebras

Lemma (Wiseman - 1985)
Let $R$ be the oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$. Then $\text{gl.dim} R \leq \max\{\text{gl.dim} A, \text{gl.dim} C\} + \text{pd}_A B$

Lemma
Let $R$ be the oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$ such that $A$ and $C$ are hereditary algebras. Then $\text{gl.dim} R \leq 2$. 
Oriented pullback and classes of algebras

**Theorem 2**

Let $R$ be the oriented pullback of $A \rightarrow B$ and $C \rightarrow B$ such that $A$ and $C$ are hereditary algebras. If $\text{ind} R = \text{ind} A \cup \text{ind} C$ then $R$ is a *quasitilted* algebra.
Oriented pullback and classes of algebras

**Theorem 2**

Let $R$ be the oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$ such that $A$ and $C$ are hereditary algebras. If $\text{ind} R = \text{ind} A \cup \text{ind} C$ then $R$ is a *quasitilted* algebra.

**Corollary**

Let $R$ be the Dynkin oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$ such that $A$ and $C$ are hereditary algebras. Then $R$ is a *quasitilted* algebra.
**M convex:** given $M'$ and $M''$ indecomposable in $\text{add} M$:

$$M' \to X_1 \to X_2 \to \cdots \to X_t \to M'' \Rightarrow X_i \in \text{add} M \ \forall i = 1, \ldots, t.$$

**Theorem 4**

Let $R$ be the oriented pullback of $A \to B$ and $C \to B$ such that $DA'$ is a convex $R$-module. If $A$ and $C$ are hereditary algebras then $R$ is a tilted algebra.
Conjectures

**Conjecture**

Let $R$ be the oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$. If $A$ is hereditary then $DA'$ is a convex $R$-module.

**Consequence**

Let $R$ be the oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$. If $A$ and $C$ are hereditary algebras then $R$ is a tilted algebra.
References


Thank you!