Pullback of finite dimensional algebras

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Notation

- k an algebraically closed field
- $A = kQ_A/I_A$ where Q_A is a finite quiver and I_A is an admissible ideal of kQ_A
- e_A the identity element of A
- $\mathrm{ind}A$ the category whose objects are isomorphism classes of indecomposable right A-modules

Pullback of algebras

Let A, B and C be algebras. Given two epimorphisms $f_A\colon A \twoheadrightarrow B$ and $f_C\colon C \twoheadrightarrow B$ the **pullback** of f_A and f_C is the subalgebra of $A \times C$ defined by

$$R = \{(a, c) \in A \times C \mid f_A(a) = f_C(c)\}$$

Purpose

Let A, B and C be algebras and $f_A\colon A \twoheadrightarrow B$ and $f_C\colon C \twoheadrightarrow B$ be epimorphisms. Let R be the pullback of f_A and f_C. We want to find relations between properties of A, B and C and properties of R.

(1) bounded quiver
(2) category of modules
(3) classes of algebras (hereditary, shod, quasitilted, etc)

The case in question

- $A = kQ_A/I_A$ and $C = kQ_C/I_C$
- Q_B full and convex subquiver of Q_A and Q_C
- $I_A \cap kQ_B = I_C \cap kQ_B := I_B$

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- $B := kQ_B/I_B \cong e_B A e_B \cong e_B C e_B$ is a common quotient of A and C
- $f_A \colon a \mapsto e_B a e_B$ and $f_C \colon c \mapsto e_B c e_B$ are epimorphisms

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- $B := kQ_B/I_B \cong e_B A e_B \cong e_B C e_B$ is a common quotient of A and C
- $f_A \colon a \mapsto e_B a e_B$ and $f_C \colon c \mapsto e_B c e_B$ are epimorphisms
- The pullback is $R = \{(a, c) \in A \times C \mid e_B a e_B = e_B c e_B\}$

The bounded quiver

Theorem (Igusa-Platzeck-Todorov-Zacharia - 1987)

Let R be the pullback of $f_A\colon A \twoheadrightarrow B$ and $f_C\colon C \twoheadrightarrow B$. Let

- $Q_R = Q_A \coprod_{Q_B} Q_C$ be the pushout of the inclusion $Q_B \to Q_A$ and $Q_B \to Q_C$.
- I_R be the ideal of kQ_R generated by I_A, I_C and the paths linking $(Q_A)_0 \setminus (Q_B)_0$ and $(Q_C)_0 \setminus (Q_B)_0$.

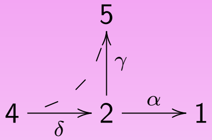
Then $R \cong kQ_R/I_R$.

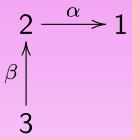
Example

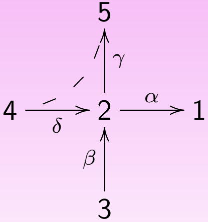
$$Q_A = \begin{array}{ccccc} & & 5 & & \\ & & \uparrow{\scriptstyle \gamma} & & \\ 4 & \xrightarrow[\delta]{} & 2 & \xrightarrow{\alpha} & 1 \end{array}, \quad Q_B = 2 \xrightarrow{\alpha} 1 \text{ and } Q_C = \begin{array}{ccc} 2 & \xrightarrow{\alpha} & 1 \\ {\scriptstyle \beta}\uparrow & & \\ 3 & & \end{array}$$

The pullback: $$Q_R = \begin{array}{ccccc} & & 5 & & \\ & & \uparrow{\scriptstyle \gamma} & & \\ 4 & \xrightarrow[\delta]{} & 2 & \xrightarrow{\alpha} & 1 \\ & & {\scriptstyle \beta}\uparrow & & \\ & & 3 & & \end{array}$$

Example

$Q_A =$ 

, $Q_B = 2 \xrightarrow{\alpha} 1$ and $Q_C =$ 

The pullback: $Q_R =$ 

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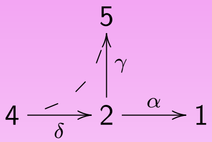
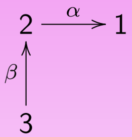
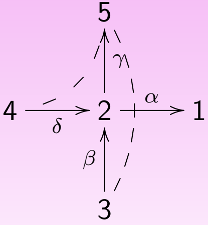
$Q_A =$ 

, $Q_B = 2 \xrightarrow{\alpha} 1$ and $Q_C =$ 

The pullback: $Q_R =$ 

Question

Is the same result true for the Auslander-Reiten quiver? That is, $\Gamma_R = \Gamma_A \coprod_{\Gamma_B} \Gamma_C$?

Remark:

- $(\Gamma_R)_0$: objects of $\mathrm{ind} R$

Question

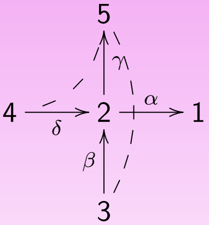
Is the same result true for the Auslander-Reiten quiver? That is, $\Gamma_R = \Gamma_A \coprod_{\Gamma_B} \Gamma_C$?

Remark:

- $(\Gamma_R)_0$: objects of $\mathrm{ind} R$

$$\mathrm{ind} R = \mathrm{ind} A \cup \mathrm{ind} C?$$

In the last example, the module $I(2) = {}^{4}{}_{2}{}^{3}$ is neither an A-module nor a C-module.

$Q_R =$ 

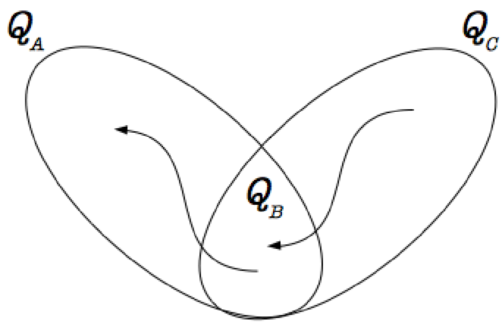
In this case $\mathrm{ind} R \neq \mathrm{ind} A \cup \mathrm{ind} C$

Definition

Let R be the pullback of $f_A \colon A \twoheadrightarrow B$ and $f_C \colon C \twoheadrightarrow B$. We say that R is an **oriented pullback** if its bounded quiver satisfies

- there is no path from $(Q_B)_0$ to $(Q_C)_0 \setminus (Q_B)_0$ and neither from $(Q_A)_0 \setminus (Q_B)_0$ to $(Q_C)_0$

The oriented pullback



The oriented pullback

In this case,

- $P(x) \cong P_A(x)$ for any $x \in (Q_A)_0$
- $P(x) \cong P_C(x)$ for any $x \in (Q_C)_0 \setminus (Q_B)_0$
- $I(x) \cong I_C(x)$ for any $x \in (Q_C)_0$
- $I(x) \cong I_A(x)$ for any $x \in (Q_A)_0 \setminus (Q_B)_0$

Special kinds of oriented pullback

Condition $*$(IPTZ87)

(1) Q_B: $\bullet \leftarrow \bullet \leftarrow \cdots \leftarrow \bullet$, with no relations;

(2) for each arrow $\alpha : x \to y$ in $(Q_B)_1$ and each path $\phi\colon z \rightsquigarrow y$ from $z \in (Q_C)_0 \setminus (Q_B)_0$ to y there is a path $\psi\colon z \rightsquigarrow x$ from z to x such that $\psi\alpha - \phi \in I_C$.

Theorem (IPTZ87)

Let R be the pullback of $f_A\colon A \twoheadrightarrow B$ and $f_C\colon C \twoheadrightarrow B$ such that Q_R satisfies Condition $*$. Then

$$\mathrm{ind} R = \mathrm{ind} A \cup \mathrm{ind} C$$

Definition

Let R be the pullback of $f_A\colon A \twoheadrightarrow B$ and $f_C\colon C \twoheadrightarrow B$. We say that R is a **Dynkin oriented pullback** if it is an oriented pullback and its bounded quiver satisfies the following conditions:

1. B is an hereditary algebra and each connected component is of Dynkin type with an unique sink;
2. for each arrow $\alpha : x \to y$ in $(Q_B)_1$ and each path $\phi\colon z \rightsquigarrow y$ from $z \in (Q_C)_0 \setminus (Q_B)_0$ to y then there is a path $\psi\colon z \rightsquigarrow x$ from z to x such that $\psi\alpha - \phi \in I_C$.

Theorem 1

Let R be the Dynkin oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$. Then

$$\mathrm{ind} R = \mathrm{ind} A \cup \mathrm{ind} C$$

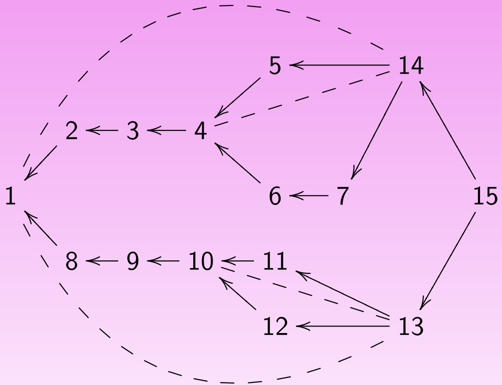
Example - condition $*$

$$Q_A = \begin{array}{ccc} 1 & \leftarrow & 3 \\ & & \downarrow \\ & & 2 \end{array} \qquad Q_B = \begin{array}{c} 3 \\ \downarrow \\ 2 \end{array} \qquad Q_C = \begin{array}{ccc} 3 & \leftarrow & 4 \\ \downarrow & & \\ 2 & & \end{array}$$

The pullback R is

$$\begin{array}{ccccc} 1 & \longleftarrow & 3 & \longleftarrow & 4 \\ & & \downarrow & & \\ & & 2 & & \end{array}$$

Example



where $(Q_A)_0 \setminus (Q_B)_0 = \{1\}$ and $(Q_C)_0 \setminus (Q_B)_0 = \{13, 14, 15\}$. In this case Q_B has two components: $\mathbb{D}_5$ and $\mathbb{E}_6$

Oriented pullback and classes of algebras

- Λ is **shod** $\Leftrightarrow$ pd$M \leq 1$ or id$M \leq 1$ for any $M \in \mathrm{ind}\Lambda$
- Λ is **quasitilted** $\Leftrightarrow$ Λ is shod and gl.dim$\Lambda \leq 2$

Oriented pullback and classes of algebras

Lemma

Let R be the oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$. Then

- if $M \in \mathrm{ind} A$ then $\mathrm{pd} M = \mathrm{pd}_A M$
- if $M \in \mathrm{ind} C$ then $\mathrm{id} M = \mathrm{id}_C M$

Lemma

Let R be the oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$ such that A and C are hereditary algebras. If $\mathrm{ind} R = \mathrm{ind} A \cup \mathrm{ind} C$ then R is a *shod* algebra.

Oriented pullback and classes of algebras

Lemma (Wiseman - 1985)

Let R be the oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$. Then $\text{gl.dim} R \leq \max\{\text{gl.dim} A, \text{gl.dim} C\} + \text{pd}_A B$

Lemma

Let R be the oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$ such that A and C are hereditary algebras. Then $\text{gl.dim} R \leq 2$.

Oriented pullback and classes of algebras

Theorem 2

Let R be the oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$ such that A and C are hereditary algebras. If $\mathrm{ind} R = \mathrm{ind} A \cup \mathrm{ind} C$ then R is a *quasitilted* algebra.

Oriented pullback and classes of algebras

Theorem 2

Let R be the oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$ such that A and C are hereditary algebras. If $\mathrm{ind} R = \mathrm{ind} A \cup \mathrm{ind} C$ then R is a *quasitilted* algebra.

Corollary

Let R be the Dynkin oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$ such that A and C are hereditary algebras.Then R is a *quasitilted* algebra.

M **convex**: given M' and M'' indecomposable in add$M$:
$M' \to X_1 \to X_2 \to \cdots \to X_t \to M'' \Rightarrow X_i \in \text{add}M \ \forall i = 1, ..., t$.

Theorem 4

Let R be the oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$ such that DA' is a convex R-module. If A and C are hereditary algebras then R is a tilted algebra.

Conjectures

Conjecture

Let R be the oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$. If A is hereditary then DA' is a convex R-module.

Consequence

Let R be the oriented pullback of $A \twoheadrightarrow B$ and $C \twoheadrightarrow B$. If A and C are hereditary algebras then R is a tilted algebra.

References

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- A. N. Wiseman. *Projective modules over pullback rings*, Math. Proc. Cambridge Philos. Soc., **97**(3): 399-406, 1985.

Thank you!