TREE MODULES - THORSTEN WEIST

Let $k$ be an algebraically closed field and $Q = (Q_0, Q_1)$ be a quiver without oriented cycles. For a fixed representation $X$ of the quiver $Q$ we choose a basis $\mathcal{B}$ of each vector space $X_i$, $i \in Q_0$.

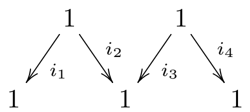
**Definition 1.** *The coefficient quiver $\Gamma(X, \mathcal{B})$ of a representation $X$ has vertex set $\mathcal{B}$ and arrows between vertices are defined by the condition: if $(X_{\alpha,\mathcal{B}})_{b,b'} \neq 0$, there exists an arrow $(\alpha, b, b') : b \mapsto b'$.*

*A representation $X$ is called a tree module if there exists a basis $\mathcal{B}$ for $X$ such that the corresponding coefficient quiver is a tree.*

This leads us to the following problem stated by Ringel, see [5]: Does there exist an indecomposable tree module for every root $d \in \mathbb{N}Q_0$? In particular, Ringel conjectured that there should be more than one isomorphism class for imaginary roots.

**Example 2.**

- Let $K(m)$ be the generalized Kronecker quiver. We consider the dimension vector $(d, e) = (2, 3)$ with $m \geq 3$. The coefficient quiver obtained from




  by colouring the arrows in the colours $\{1, \ldots, m\}$ such that we get a subquiver of the universal covering quiver of $K(m)$, gives rise to an indecomposable tree module.

Analogously, we can construct indecomposable tree modules for every root $(d, e)$ of $K(m)$ such that $d \leq e \leq (m-1)d$. They are exceptional if understood as representations of the universal cover. Since exceptional representations are tree modules, see [4], we can apply the reflection functor in order to obtain indecomposable tree modules for arbitrary roots. We obtain the following result, where the map $r : \mathbb{Z}^2 \to \mathbb{Z}^2$ is defined by $r(d, e) := (e, me - d)$, see [7]:

**Theorem 3.**
(1) *For every root $(d, e)$ of the generalized Kronecker quiver there exists an indecomposable tree module.*
(2) *Let $k, l, n \in \mathbb{N}_0$. For each root $(d, e) \neq r^l(n, kn)$ there exists a Schurian tree module.*

Following the results of [8], we sketch how to construct indecomposable tree modules for every imaginary Schur root of a quiver $Q$ without oriented cycles. Therefore, fixed a pair of representations $X$, $Y$ we always choose a tree-shaped basis of $\mathrm{Ext}(X, Y)$, i.e. the corresponding matrices are of type $E(s,t)_{ij} = \delta_{si}\delta_{tj}$. Based on [6], the algorithm of [1] leads us to the following statement where we also use the notation of [6]:

**Proposition 4.** *Let $\alpha$ be an imaginary Schur root. Then at least one the following cases holds:*

(1) *There exist a real Schur root $\beta$ and $t \in \mathbb{N}_+$ such that $\gamma = \alpha - t\beta$ is an imaginary Schur root. Moreover, we have $\beta \in \gamma^\perp$ and $\hom(\beta, \gamma) = 0$ or $\beta \in^\perp \gamma$ and $\hom(\gamma, \beta) = 0$.*
(2) *There exist a real Schur root $\beta$ and a real or isotropic Schur root $\gamma$ and $d, e \in \mathbb{N}_+$ such that $\alpha = \beta^d + \gamma^e$. Moreover, we have $\beta \in \gamma^\perp$ and $\hom(\beta, \gamma) = 0$ or $\beta \in^\perp \gamma$ and $\hom(\gamma, \beta) = 0$ and $(d, e)$ is a root of $K(\mathrm{ext}(\beta, \gamma))$ or $K(\mathrm{ext}(\gamma, \beta))$.*

(3) *There exist two imaginary Schur roots* $\gamma$ *and* $\delta$ *such that* $\gamma+\delta=\alpha$. *Moreover, we have* $\delta\in\gamma^{\perp}$ *and* $\hom(\delta,\gamma)=0$.

This Proposition gives us a recipe how to decompose Schur roots in order to construct an indecomposable tree module of such a root. In the first two cases we may restrict to one of the two possible cases. In the first case let $X_\beta$ and $X_\gamma$ be the corresponding indecomposable representations. Since they are exceptional, by [4] it follows that $X_\beta$ and $X_\gamma$ are tree modules. Since we also have $\mathrm{Ext}(X_\beta,X_\gamma)=0$, see [6], it follows that the subcategory consisting of middle terms of sequences of the form

$$0\to X_\gamma^d\to X_\alpha\to X_\beta^e\to 0$$

is equivalent to the category $R_{e,d}(K(\mathrm{ext}(\beta,\gamma))$. Thus by applying Theorem 3 we get that there exists an indecomposable tree module of dimension $\alpha$. In the second case by applying Ringel's reflection functor, see [3], we obtain the following diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & \bigoplus_{i=1}^{n-k} S & = & \bigoplus_{i=1}^{n-k} S & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & X & \longrightarrow & X^S & \longrightarrow & \bigoplus_{i=1}^{n} S & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y^S & \longrightarrow & \bigoplus_{i=1}^{k} S & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Now, since $X^S$ is indecomposable, one checks that $Y^S$ is indecomposable as well. In the last case, we first construct indecomposable tree modules of dimension $\gamma$ and $\delta$. By [2] it follows that $\mathrm{Hom}(X_\gamma,X_\delta)=0$. Thus the middle terms of non-splitting exact sequences of the form

$$0\to X_\delta\to X_\alpha\to X_\gamma\to 0$$

are indecomposable. Thus in summary we get the following result, see [8]:

**Theorem 5.** *For every imaginary Schur root there exists an indecomposable tree module.*

## References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