Infinite dimensional tilting modules, homological subcategories and recollements

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Finite dimensional tilting theory is well-known.

Let us look at the infinite dimensional titling theory.
There are several groups in this area:

(1) Italy [Angeleri-Huegel, Bazzoni, Colpi, Mantese, Pavarin, Tonolo, ...]

(2) Spain [Herbera, Nicolas, Sanchez, Saorin, ...]

(3) Czech [Stovicek, Trifaj, ...]

(4) Germany [Koenig and his group, ...]

(5) $\cdots$

Main aim

Some developments on **infinitely** generated tilting modules in terms of derived module categories. But restrict to [3] and [4]

[1] Proc. London Math. Soc. 104(2012) 959-996.

[2] arXiv:1107.0444 [Stratifications of derived categories from tilting modules over tame hereditary algebras.]

[3] arXiv:1203.5168v2 [Homological ring epimorphisms and recollements from exact pairs, I.]

[4] arXiv:1206.0522 [Ringel modules and homological subcategories]

These are joint works with Hongxing Chen.

Notations

A :	ring with 1
A-Mod:	cat. of all left R-modules
$\mathrm{add}(M)$:	summands of f. dir. sums of M
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$\mathscr{D}(A)$:	derived cat. of A (or A-Mod)

Definition of tilting modules

Definition

n-tilting module ${}_AT$*:*

(1) $\mathrm{pd}_A(T) \leq n$*:* $\quad P^\bullet \longrightarrow T \to 0,$

(2) $\mathrm{Ext}^i_A(T, T^{(I)}) = 0$ *for all* $i > 0$ *and all set* I*,*

(3) exact seq.: $0 \to {}_AA \to T_0 \to \cdots \to T_n \to 0$, $T_j \in \mathrm{Add}(T)$.

- *good* if $T_i \in \mathrm{add}(T)$.
- *classical* if good and f.g.

Define $B := \mathrm{End}_A(T)$

Happel Theorem

Theorem

${}_AT$: *classical* n*-tilting,* $\Longrightarrow \mathscr{D}(A) \simeq \mathscr{D}(B)$.

Note:

- Derived invaraints
- No new triangulated categories

Natural question:

Is Happel Theorem still true?

$${}_AT\text{: inf. g. tilting} \Longrightarrow \mathscr{D}(A) \simeq \mathscr{D}(B)?$$

Bazzoni's Answer

Theorem (Bazzoni, Bazzoni-Mantese-Tonolo)

${}_AT$: n-tilting $\Longrightarrow$ $\mathscr{D}(A)$: subcategory or quotient of $\mathscr{D}(B)$.

In fact: $\mathscr{D}(B)/\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -) \quad \simeq \quad \mathscr{D}(A)$

Note:

- New triangulated categories
- No derived invariants

What should be
the corresp. Happel Theorem
for inf. g. tilt. mod.s?

Definition of homological ring epimorphisms

Definition

*A ring epimorphism $\lambda : R \longrightarrow S$ is called **homological** if $\mathrm{Tor}_j^R(S,S) = 0$ for $j > 0$.*

Or equivalently, the restriction functor $D(\lambda_*) : \mathscr{D}(S) \longrightarrow \mathscr{D}(R)$ is fully faithful.

Reference: Geigle-Lenzing: J. Algebra 144(1991)273-343.

Theorem

${}_AT$: good tilt., proj.dim ≤ 1, $\Longrightarrow \exists$ homolog. ring epi. $B \longrightarrow C$ and recollement:

$$\mathscr{D}(C) \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} \mathscr{D}(B) \begin{array}{c} \longleftarrow \\ \xrightarrow{j^!} \\ \longleftarrow \end{array} \mathscr{D}(A)$$

- T: classical, $\Rightarrow C = 0$, Happel Theorem.
- $j^! := T \otimes^{\mathbb{L}}_B -, \mathrm{Ker}(j^!) \simeq \mathscr{D}(C)$.
- C: universal localization of B.

Recollements instead of derived equivalences

$n = 1$ indicates:

For inf. g. tilt. mod.s, Happel Theorem should be a recollement of der. module categories.

Immediate question:

Is the above theorem true for n-tilting mod.s ?

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General case

This is a difficult question!

It is related to questions:
(1) When is a universal localization homological?
(2) When is a full triangulated subcategory $\mathcal{T}$ of $\mathscr{D}(B)$ realisable as a der. module cat.?

Question (1) is a very general, old question. Question (2) may be new, but also very general. We shall consider a special case which is related to inf. g. tilting modules.

Definition of exact pairs of ring homomorphisms

$\lambda : R \longrightarrow S$, $\mu : R \longrightarrow T$ are ring hom.s.

The pair (λ, μ) is called exact if

$$0 \to R \to S \oplus T \xrightarrow{\begin{pmatrix} \lambda\otimes 1 \\ -1\otimes\mu \end{pmatrix}} S \otimes_R T \to 0$$

is exact as R-R-bimodules.

- The coproduct $S \sqcup_R T$:

$$\begin{array}{ccc} R & \xrightarrow{\lambda} & S \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\rho} \\ T & \xrightarrow{\varphi} & S \sqcup_R T. \end{array}$$

- Define a universal localization (due to Schofield):

$$\theta: \; B := \begin{pmatrix} S & S \otimes_R T \\ 0 & T \end{pmatrix} \longrightarrow \begin{pmatrix} S \sqcup_R T & S \sqcup_R T \\ S \sqcup_R T & S \sqcup_R T \end{pmatrix} =: C,$$

$$\begin{pmatrix} s_1 & s_2 \otimes t_2 \\ 0 & t_1 \end{pmatrix} \mapsto \begin{pmatrix} (s_1)\rho & (s_2)\rho(t_2)\varphi \\ 0 & (t_1)\varphi \end{pmatrix}$$

This θ is related to inf. g. tilting modules. So our question is:

When is θ homological?

Remark. If θ: homolog., then $\exists$ a recollement

$$\mathscr{D}(S \sqcup_R T) \longrightarrow \mathscr{D}(B) \longrightarrow \mathscr{D}(R)$$

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When is θ homological?

Remark. If θ: homolog., then $\exists$ a recollement

$$\xymatrix{\mathscr{D}(S\sqcup_R T) \ar[r] & \mathscr{D}(B) \ar@/_1.5pc/[l] \ar@/^1.5pc/[l] \ar[r] & \mathscr{D}(R) \ar@/_1.5pc/[l] \ar@/^1.5pc/[l]}$$

Theorem

Given an exact pair (λ, μ) *with* λ *homolog.* $\Rightarrow$

TFAE:

(1) θ *is homological ring epi.*

(2) $\mathrm{Tor}_j^R(T, S) = 0$ *for all* $j > 0$.

Definition

*A full triang. subcat. $\mathscr{T}$ of $\mathscr{D}(B)$ is called **homological** if $\exists$ a homological ring epi $\lambda : B \to C$ such that the restriction is a triangle equivalence from $\mathscr{D}(C)$ to $\mathscr{T}$.*

When is $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ homological in $\mathscr{D}(B)$?

Condition for homological subcategories

Theorem

T : *good* n*-tilting* A*-module,* $B := \mathrm{End}_A(T)$.

TFAE:

(1) $\mathrm{Ker}(T \otimes^{\mathbb{L}}_B -)$*: homological,*

(2) $H^i(\mathrm{Hom}_A(P^\bullet, A) \otimes_A T) = 0$ *for* $i \geq 2$.

$P^\bullet$: proj. resol. of T.

Corollary

A: comm., ${}_AT$: good n-tilt. mod. s.t. $\mathrm{Hom}_A(T_{i+1}, T_i) = 0$ *for* $1 \leq i \leq n-1 \Longrightarrow$ $\mathrm{Ker}(T \otimes^{\mathbb{L}}_B -)$ *is homolog. if and only if proj.dim*$({}_AT) \leq 1$.

Counterexample

For $n \geq 2$, there is an n-tilting A-module T such that $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ is not homological. Thus there is no homol. ring epi $B \to C$ such that the following recollement exists:

$$\mathscr{D}(C) \longrightarrow \mathscr{D}(B) \xrightarrow{\;j^!\;} \mathscr{D}(A)$$

$$j^! := T \otimes_B^{\mathbb{L}} -$$

Open questions:

(1) What should be the replacement of Happel Theorem for inf. g. tilt. modules?

(2) Find more conditions for T, such that $\mathrm{Ker}(T \otimes_B^{\mathbb{L}} -)$ is homological.