Hall type algebras associated to triangulated categories

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Outline

1. Derived Hall algebras
2. The derived Riedtmann-Peng formula
3. Motivic Hall algebras

Definition of derived Hall algebras

- k: finite field, $q = |k|$
- $\mathcal{C}$: a k-additive finitary, Krull-Schmidt triangulated category with (left) homologically finite condition, i.e., $\forall X, Y \in \mathcal{C}$, $\{X, Y\} := |\prod_{i>0} |\mathrm{Hom}(X[i], Y)|^{(-1)^i}| < \infty$.

Example: derived categories; counter-example: the cluster category.

Set $\mathcal{H}(\mathcal{C}) = \bigoplus_{[X];X\in\mathcal{C}} \mathbb{Q}u_{[X]}$ with the multiplication defined by

$$u_{[X]} * u_{[Y]} = \sum_{[L]} F_{XY}^{L} u_{[L]},$$

where

$$F_{XY}^{L} = \frac{|\mathrm{Hom}(L, Y)_{X[1]}|}{|\mathrm{Aut}Y|} \cdot \frac{\{L, Y\}}{\{Y, Y\}} = \frac{|\mathrm{Hom}(X, L)_{Y}|}{|\mathrm{Aut}X|} \cdot \frac{\{X, L\}}{\{X, X\}}.$$

The associativity of derived Hall algebras

Theorem (Toën, Xiao-Xu)

$\mathcal{H}(\mathcal{C})$ is an associative algebra with the unit $u_{[0]}$.

To prove $u_{[Z]} * (u_{[X]} * u_{[Y]}) = (u_{[Z]} * u_{[X]}) * u_{[Y]}$ is equivalent to prove

$$\sum_{[L]} F_{XY}^{L} F_{ZL}^{M} = \sum_{[L']} F_{ZX}^{L'} F_{L'Y}^{M}.$$

$$\mathrm{LHS} = \frac{1}{|\mathrm{Aut}X| \cdot \{X,X\}} \sum_{[L]} \sum_{[L']} \frac{|\mathrm{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]}|}{|\mathrm{Aut}L|} \cdot \frac{\{M \oplus X, L\}}{\{L,L\}}$$

$$\mathrm{RHS} = \frac{1}{|\mathrm{Aut}X| \cdot \{X,X\}} \sum_{[L']} \sum_{[L]} \frac{|\mathrm{Hom}(L', M \oplus X)_{L}^{Y,Z[1]}|}{|\mathrm{Aut}L'|} \cdot \frac{\{L', M \oplus X\}}{\{L',L'\}}$$

The relation between objects is encoded in the following diagram:

$$
\begin{array}{ccccccc}
Z & = & Z & & & & \\
{\scriptstyle l'}\downarrow & & \downarrow{\scriptstyle l} & & & & \\
L' & \xrightarrow{f'} & M & \overset{g'}{\dashrightarrow} & Y & \overset{h'}{\dashrightarrow} & L'[1] \\
{\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} & & \| & & \downarrow{\scriptstyle m'[1]} \\
X & \xrightarrow{f} & L & \xrightarrow{g} & Y & \xrightarrow{h} & X[1] \\
{\scriptstyle n'}\downarrow & & \downarrow{\scriptstyle n} & & & & \\
Z[1] & = & Z[1] & & & &
\end{array}
\qquad (1)
$$

$\Longleftrightarrow$ a distinguished triangle

$$
L' \xrightarrow{\begin{pmatrix} f' & -m' \end{pmatrix}} M \oplus X \xrightarrow{\begin{pmatrix} m \\ f \end{pmatrix}} L \xrightarrow{\theta} L'[1] \qquad (2)
$$

The triangle induces two sets

$$\mathrm{Hom}(M\oplus X, L)_{L'[1]}^{Y,Z[1]} := \{(m,f)\in \mathrm{Hom}(M\oplus X, L) \mid$$

$$Cone(f)\simeq Y, Cone(m)\simeq Z[1] \text{ and } Cone(m,f)\simeq L'[1]\}$$

and

$$\mathrm{Hom}(L', M\oplus X)_{L}^{Y,Z[1]} := \{(f',-m')\in \mathrm{Hom}(L', M\oplus X) \mid$$

$$Cone(f')\simeq Y, Cone(m')\simeq Z[1] \text{ and } Cone(f',-m')\simeq L\}$$

The symmetry-I: The orbit spaces of $\mathrm{Hom}(M\oplus X, L)_{L'[1]}^{Y,Z[1]}$ and $\mathrm{Hom}(L', M\oplus X)_{L}^{Y,Z[1]}$ under the action of $\mathrm{Aut}L$ and $\mathrm{Aut}L'$ coincide.

More explicitly, the symmetry is the identity:

$$\frac{|\mathrm{Hom}(M\oplus X,L)^{Y,Z[1]}_{L'[1]}|}{|\mathrm{Aut}L|}\frac{\{M\oplus X,L\}}{\{L',L\}\{L,L\}}$$

$$=\frac{|\mathrm{Hom}(L',M\oplus X)^{Y,Z[1]}_{L}|}{|\mathrm{Aut}L'|}\frac{\{L',M\oplus X\}}{\{L',L\}\{L',L'\}}.$$

$\Longrightarrow$ LHS=RHS.

Outline

The derived Riedtmann-Peng formula

Assume $\mathcal{C}$ is (left) homologically finite over a finite field k.

Theorem

For any X, Y *and* L *in* $\mathcal{C}$, *we have*

$$\frac{|\mathrm{Hom}(Y, X[1])_{L[1]}|}{|\mathrm{Aut}X|} \cdot \frac{\{Y, X[1]\}}{\{X, X\}} = \frac{|\mathrm{Hom}(L, Y)_{X[1]}|}{|\mathrm{Aut}L|} \cdot \frac{\{L, Y\}}{\{L, L\}}$$

and

$$\frac{|\mathrm{Hom}(Y[-1], X)_{L}|}{|\mathrm{Aut}Y|} \cdot \frac{\{Y[-1], X\}}{\{Y, Y\}} = \frac{|\mathrm{Hom}(X, L)_{Y}|}{|\mathrm{Aut}L|} \cdot \frac{\{X, L\}}{\{L, L\}}.$$

Example

Assume that $\mathcal{C} = \mathcal{D}^b(\mathcal{A})$ for a small Hom-finite abelian category $\mathcal{A}$ and X, Y and $L \in \mathcal{A}$. Then one can obtain

$$\mathrm{Hom}(Y, X[1])_{L[1]} = \mathrm{Ext}^1(Y, X)_L, \quad \{Y, X[1]\} = |\mathrm{Hom}_{\mathcal{A}}(Y, X)|^{-1},$$

$$g_{XY}^L = \frac{|\mathrm{Hom}(L, Y)_{X[1]}|}{|\mathrm{Aut} Y|} = \{L' \subseteq L \in L' \cong X, L/L' \cong Y\}$$

and

$$\{X, X\} = \{L, L\} = \{L, Y\} = 0.$$

Under the assumption, the theorem is reduced to the Riedtmann-Peng formula

$$\frac{|\mathrm{Ext}^1(Y, X)_L|}{|\mathrm{Hom}_{\mathcal{A}}(Y, X)|} = g_{XY}^L \cdot |\mathrm{Aut} X| \cdot |\mathrm{Aut} Y| \cdot |\mathrm{Aut} L|^{-1}.$$

Two versions of derived Hall algebras

Version-I (Toën, Xiao-Xu)
Version-II (Kontsevich-Soibelman)
Set $\mathcal{H}_{\mathcal{M}}(\mathcal{C}) = \bigoplus_{[X];X\in\mathcal{C}} \mathbb{Q}v_{[X]}$ with the multiplication defined by

$$\begin{aligned} v_{[X]} * v_{[Y]} &= \{Y, X[1]\} \cdot \sum_{[L]} |\mathrm{Hom}(Y, X[1])_{L[1]}| v_{[L]} \\ &= \{Y[-1], X\} \cdot \sum_{[L]} |\mathrm{Hom}(Y[-1], X)_{L}| v_{[L]} \end{aligned}$$

Fact: The derived Riedtmann-Peng formula gives the proof

Theorem

The map $\Phi : \mathcal{H}_{\mathcal{M}}(\mathcal{C}) \to \mathcal{H}(\mathcal{C})$ *by* $\Phi(v_{[X]}) = |\mathrm{Aut} X| \cdot \{X, X\} \cdot u_{[X]}$ *for any* $X \in \mathcal{C}$ *is an algebraic isomorphism between* $\mathcal{H}_{\mathcal{M}}(\mathcal{C})$ *and* $\mathcal{H}(\mathcal{C})$.

$$
\begin{array}{ccccccc}
Z & = & Z & & & & \\
\big\downarrow{\scriptstyle l'} & & \big\downarrow{\scriptstyle l} & & & & \\
L' & \xrightarrow{f'} & M & \xrightarrow{g'} & Y & \xrightarrow{h'} & L'[1] \\
\big\downarrow{\scriptstyle m'} & & \big\downarrow{\scriptstyle m} & & \| & & \big\downarrow{\scriptstyle m'[1]} \\
X & \xrightarrow{f} & L & \xrightarrow{g} & Y & \xrightarrow{h} & X[1] \\
\big\downarrow{\scriptstyle n'} & & \big\downarrow{\scriptstyle n} & & & & \\
Z[1] & = & Z[1] & & & &
\end{array}
\qquad (3)
$$

$\Longrightarrow$

$$L' \xrightarrow{\begin{pmatrix} f' & -m' \end{pmatrix}} M \oplus X \xrightarrow{\begin{pmatrix} m \\ f \end{pmatrix}} L \xrightarrow{\theta} L'[1]$$

The symmetry-I compares

$$L' \xrightarrow{\begin{pmatrix} f' & -m' \end{pmatrix}} M \oplus X \quad \text{and} \quad M \oplus X \xrightarrow{\begin{pmatrix} m \\ f \end{pmatrix}} L$$

The diagram induces **a new symmetry** comparing

$$L' \xrightarrow{f'} M \xrightarrow{m} L \quad \text{and} \quad L' \xrightarrow{m'} X \xrightarrow{f} L$$

The symmetry-II:

- Fix $\alpha \in \mathrm{Ext}^1(Y,X)_L$, then $\exists$ a map $f_*: \mathrm{Ext}^1(L,Z)_M \to \mathrm{Ext}^1(X,Z)_{L'}$ with the cardinality of fibre
$$|\mathrm{Hom}(Y,Z[1])| \cdot \{X \oplus Y, Z[1]\} \cdot \{L, Z[1]\}^{-1};$$
- Fix $\alpha' \in \mathrm{Hom}(X,Z[1])_{L'[1]}$, then $\exists$ a map $(m')_*: \mathrm{Ext}^1(Y,L')_M \to \mathrm{Ext}^1(Y,X)_L$ with the cardinality of fibre
$$|\mathrm{Hom}(Y,Z[1])| \cdot \{Y, X[1] \oplus Z[1]\} \cdot \{Y, L'[1]\}^{-1};$$
- $|f_*^{-1}(0)| \cdot \{Y, X[1]\} \cdot \{L, Z[1]\} = |(m')_*^{-1}(0)| \cdot \{X, Z[1]\} \cdot \{Y, L'[1]\}.$

The symmetry-I $\xrightarrow{\text{derived R.-P. formula}}$ **The symmetry-II** $\Longrightarrow$ The associativity of Kontsevich-Soibelman's Hall algebras over finite field.

Outline

Ind-constructible set

Throughout, fix the complex field $\mathbb{C}$. A ind-constructible set is a countable union of non-intersecting constructible sets.

Canonical example:

- A be a finite dimensional algebra over $\mathbb{C}$;
- Indecomposable projective: P_i, $i = 1, \cdots, l$;
- The projective dimension vector of $\oplus_{i=1}^{l} P_i^{a_i}$ is $(a_i)_{i=1}^{l}$.

Consider the affine variety $\mathcal{P}_{\mathbf{D}}$ dominated by the sequence of projective dimension vector $\mathbf{D} = (\mathbf{d}_k)_{k\in\mathbb{Z}}$ with finitely-many nonzero term and $\mathbf{d}_k = (a_i^{(k)})_{i=1}^{l}$. Then $\bigsqcup_{\mathbf{D}} \mathcal{P}_{\mathbf{D}}$ is a ind-constructible set.

Motivic functions

$\mathcal{X}$ a constructible stack. $Mot(\mathcal{X})$ the abelian group generated by isomorphism classes $[\pi : \mathcal{S} \to \mathcal{X}]$ of morphisms to $\mathcal{X}$ satisfying:

- $[(\mathcal{S}_1 \bigsqcup \mathcal{S}_2) \to \mathcal{X}] = [\mathcal{S}_1 \to \mathcal{X}] + [\mathcal{S}_2 \to \mathcal{X}]$
- $[\pi_1 : \mathcal{S}_1 \to \mathcal{X}] = [\pi_2 : \mathcal{S}_2 \to \mathcal{X}]$ if $\exists$ Zariski fibrations $f_i : \mathcal{S}_i \to \mathcal{S}$, $i = 1, 2$ and $h : \mathcal{S} \to \mathcal{X}$ with $\pi_i = h \circ f_i$.

$Mot(\mathcal{X})$ is naturally the $Mot(Spec(\mathbb{C}))$-module. Denote by $\mathbb{L}$ the identity element in $Mot(Spec(\mathbb{C}))$ and $Mot(\mathcal{X})[\mathbb{L}^{-1}]$ the localization of $Mot(\mathcal{X})$.

Motivic Hall algebras following Kontsevich-Soibelman

$\mathcal{C}$: a (left) homological-finite triangulated category.
Assumption: Objects in $\mathcal{C}$ form an ind-constructible set $\mathfrak{Obj}(\mathcal{C}) = \bigsqcup_{i\in I} \mathcal{X}_i$ for countable constructible stacks $\mathcal{X}_i$ with the action of an affine algebraic group G_i. The quotient stack of $\mathcal{X}_i$ by G_i is $[\mathcal{X}_i/G_i]$. Define

$$\mathcal{MH}(\mathcal{C}) = \bigoplus_{i\in I} Mot([\mathcal{X}_i/G_i])(\mathbb{L}^{-1})$$

with the multiplication

$$[\pi_1 : \mathcal{S}_1 \to \mathfrak{Obj}(\mathcal{C})] \cdot [\pi_2 : \mathcal{S}_2 \to \mathfrak{Obj}(\mathcal{C})] = [\pi : \mathcal{W}_n \to \mathfrak{Obj}(\mathcal{C})]\mathbb{L}^{-n}$$

where

$$\mathcal{W}_n = \{(s_1, s_2, \alpha) \mid s_i \in \mathcal{S}_i, \alpha \in \mathrm{Hom}_{\mathcal{C}}(\pi_2(s_2), \pi_1(s_1)[1]),$$

$$\dim_{\mathbb{C}}\{\pi_2(s_2), \pi_1(s_1)[1]\} = n.\}$$

Motivic Hall algebras following Kontsevich-Soibelman

The map π sends (s_1, s_2, α) to $Cone(\alpha)[-1]$.

Theorem

With the above multiplication, $\mathcal{MH}(\mathcal{C})$ becomes an associative algebra.

Inspired by [Kontsevich-Soibelman] and [Xiao-Xu], the proof is a motivic version of **The symmetry-II**.
By definition, the theorem is easily reduced to the case that $\mathcal{S}_i$ is just a point. Set $v_{[E]} = [\pi : pt \to \mathfrak{Obj}(\mathcal{C})]$ with $\pi(pt) = E$.

Motivic Hall algebras-Proof

Set $\{X,Y\} = \mathbb{L}^{\sum_{i>0}(-1)^i \dim_{\mathbb{C}} \mathrm{Hom}(X[i],Y)}$ and
$\dim_{\mathbb{C}}\{X,Y\} = \sum_{i>0}(-1)^i \dim_{\mathbb{C}} \mathrm{Hom}(X[i],Y)$.

$$\begin{aligned} v_{[X]} * v_{[Y]} &= \{Y, X[1]\}[\mathrm{Hom}(Y,X[1]) \to \mathfrak{Obj}(\mathcal{C})] \\ &=^{def} \{Y, X[1]\} \cdot \int_{\alpha \in \mathrm{Hom}(Y,X[1])_{[L[1]]}} v_{[L]}. \end{aligned}$$

Then, $v_{[Z]} * (v_{[X]} * v_{[Y]})$ is

$$\int_{\beta \in \mathrm{Hom}(L,Z[1])_{[M[1]]}} \int_{\alpha \in \mathrm{Hom}(Y,X[1])_{[L[1]]}} \{Y, X[1]\}\{L, Z[1]\} v_{[M]}.$$

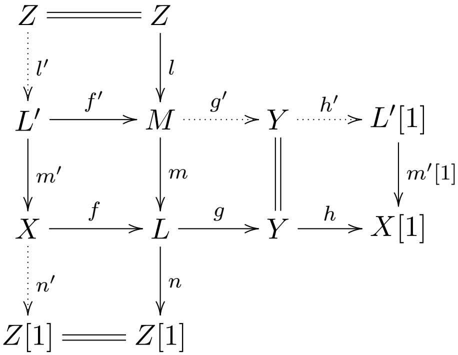
Motivic Hall algebras-Proof

In the same way, $(v_{[Z]} * v_{[X]}) * v_{[Y]}$ is

$$\int_{\beta' \in \mathrm{Hom}(Y, L'[1])_{[M[1]]}} \int_{\alpha' \in \mathrm{Hom}(X, Z[1])_{[L'[1]]}} \{X, Z[1]\}\{Y, L'[1]\} v_{[M]}.$$

Motivic Hall algebras-Proof

The relation between $v_{[Z]} * (v_{[X]} * v_{[Y]})$ and $(v_{[Z]} * v_{[X]}) * v_{[Y]}$ is illustrated by



Motivic Hall algebras-Proof

$v_{[Z]} * (v_{[X]} * v_{[Y]}) = (v_{[Z]} * v_{[X]}) * v_{[Y]} \Longleftrightarrow$ The motivic version of **The symmetry-II** as follows:

- Fix $\alpha \in \mathrm{Hom}(Y, X[1])_{L[1]}$, by the above diagram, there is a constructible bundle $\mathrm{Hom}(L, Z[1])_{M[1]} \to \mathrm{Hom}(X, Z[1])_{L'[1]}$ with fibre dimension

$$\mathrm{dimHom}(Y, Z[1]) + \dim\{X \oplus Y, Z[1]\} - \dim\{L, Z[1]\}$$

This follows the action of the functor $\mathrm{Hom}(-, Z[1])$ on the triangle

$$\alpha : X \to L \to Y \to X[1]$$

Combine the dimensions of coefficients, the sum of dimensions is

$$\dim\{Y, X[1]\} + \dim\{X, Z[1]\} + \dim\{Y, Z[1]\}.$$

Motivic Hall algebras-Proof

- Fix $\alpha' \in \mathrm{Hom}(X, Z[1])_{L'[1]}$, by the above diagram, there is a constructible bundle $\mathrm{Hom}(Y, L'[1])_{M[1]} \to \mathrm{Hom}(Y, X[1])_{L[1]}$ with fibre dimension

$$\mathrm{dimHom}(Y, Z[1]) + \dim\{Y, X[1] \oplus Z[1]\} - \dim\{Y, L'[1]\}$$

This follows the action of the functor $\mathrm{Hom}(Y, -)$ on the triangle

$$\alpha : Z \to L' \to X \to Z[1]$$

Combine the dimensions of coefficients, the sum of dimensions is also

$$\dim\{Y, X[1]\} + \dim\{X, Z[1]\} + \dim\{Y, Z[1]\}.$$