MORITA ALGEBRAS AND THE DOUBLE CENTRALIZER PROPERTY OF A BIMODULE

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This is a part of joint work with Otto Kerner ([2], [3]).

All algebras and modules are finite dimensional over a field K,
A^{op} is the opposite algebra of an algebra A,
$D = \mathrm{Hom}_K(-, K)$.

Let A be an algebra and M a right A-module. For an automorphism $\sigma \in \mathrm{Aut}(A)$, M_σ denotes the right A-module such that $M_\sigma = M$ as K-spaces and $m \cdot a = m\sigma(a)$ for all $m \in M$, $a \in A$. Similarly, ${}_\sigma N$ is defined for a left A-module N.

• An algebra A is a *Frobenius algebra* if and only if $D(A) \cong A$ as left A-modules, or equivalently, as right A-modules.
• An algebra A is *symmetric* if and only if $D(A) \cong A$ as A-bimodules.
• An automorphism σ of A is called a *Nakayama automorphism* if and only if $D(A)_\sigma \cong A$ as A-bimodules, which is uniquely determined up to inner automorphism, and denoted by ν_A. Thus,
– a Frobenius algebra A is symmetric if and only if ν_A is an inner automorphism.
• For a right A-module M and $a \in A$, the right multiplication map $r_a : M \to M$ is defined by $r_a(m) = ma$ for all $m \in M$, and left multiplication map $l_a : N \to N$ is defined for a left A-module N by $l_a(n) = an$ for all $n \in N$.

The aim of this talk is to give a new characterization of the Morita algebra by using the bimodule introduced by M. Fang and S. König.

1. Morita theorem

In 1958, K. Morita [4] studied generalizations of quasi-Frobenius rings, and characterized the endomorphism rings of finitely generated faithful modules over a quasi-Frobenius ring (= self-injective, left and right Artinian ring). In case A is an algebra, the characterization is stated as follows.

Theorem 1.1 (K. Morita). *For an algebra A, the following conditions are equivalent:*
(1) $A \cong \mathrm{End}_B(M)$ *for a faithful right module M over a self-injective algebra B.*
(2) $A \cong \mathrm{End}_B(N)^{op}$ *for a faithful left module N over a self-injective algebra B.*
(3) *There is an idempotent e of A such that ${}_A Ae$ and eA_A are faithful injective A-modules and the mapping $l : A \to \mathrm{End}_{eAe}(Ae) : a \mapsto l_a$, is bijective.*
(4) *There is an idempotent e of A such that ${}_A Ae$ and eA_A are faithful injective A-modules and the mapping $r : A \to \mathrm{End}_{eAe}(eA)^{op} : a \mapsto r_a$, is bijective.*

Definition. An algebra A is called a *Morita algebra* (over a self-injective algebra B) if it satisfies the equivalent conditions in Theorem 1.1

Recently M. Fang and S. König [1] studied Morita algebras and proved

Theorem 1.2 (M. Fang and S. König)**.** *An algebra A is a Morita algebra over a symmetric algebra if and only if* $\mathrm{Hom}_A({}_AD(A), {}_AA) \cong A$ *as A-bimodules.*

2. Main theorems

Lemma 2.3. *For an algebra A,* $\mathrm{Hom}_A({}_AD(A), {}_AA)$ *and* $\mathrm{Hom}_A(D(A)_A, A_A)$ *are isomorphic as A-bimodules.*

In view of Lemma 2.3 we identify $\mathrm{Hom}_A({}_AD(A), {}_AA)$ and $\mathrm{Hom}_A(D(A)_A, A_A)$, and simply denote them by V. The main theorem is stated as follows.

Theorem 2.4. *For an algebra A, the following conditions are equivalent:*
(1) *A is a Morita algebra over a self-injective algebra.*
(2) *The canonical mapping $l : A \to \mathrm{End}(V_A), a \mapsto l_a$, is bijective.*
(3) *The canonical mapping $r : A \to \mathrm{End}({}_AV)^{op}, a \mapsto r_a$, is bijective.*
(4) *The A-bimodule V has the double centralizer property.*

The following lemma suggests that faithfulness of V ensures the existence of an idempotent e in the Morita theorem. Notice that the A-bimodule V satisfying (2) or (3) of Theorem 2.4 is faithful as a left or right A-module, respectively.

Lemma 2.5. *For an algebra A, the following conditions are equivalent:*
(1) *${}_AV$ is faithful.*
(2) *V_A is faithful.*
(3) *There is an idempotent e of A such that ${}_AAe$ and eA_A are injective and faithful.*
(4) *There is an idempotent e of A such that ${}_AAe$ is faithful and $D(Ae)_A \cong eA_A$.*

As an application of Theorem 2.4 we have the following theorem generalizing the Fan-König's Theorem 1.2.

Theorem 2.6. *For an algebra A, the following conditions are equivalent:*
(1) $V_A \cong A_A$.
(2) ${}_AV \cong {}_AA$.
(3) *$A \cong \mathrm{End}(M_B)$, where M is a faithful right module over a Frobenius algebra B such that $M \cong M_{\nu_B}$ as right B-modules.*
(4) *$A \cong \mathrm{End}({}_BN)^{op}$, where N is a faithful left module over a Frobenius algebra B such that $N \cong {}_{\nu_B}N$ as left B-modules.*

Obviously the algebra A satisfying the equivalent conditions of Theorem 2.6 is a Morita algebra over B.

References

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