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Torsionless modules over cluster-tilted algebras of type A_n

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Contents

1 cluster categories and cluster-tilted algebras

- k— a field

- H— a finite dimensional hereditary k-algebra

- $\mathcal{D} = \mathcal{D}^b(H)$— bounded derived category of finitely generated H-modules

- [1]— shift functor

- τ— Auslander-Reiten translation in $\mathcal{D}$

Buan, Marsh, Reineke, Reiten, Todorov [4] defined a new category:

$\mathcal{C} = \mathcal{D}^b(H)/F$ with the functor $F = \tau^{-1}[1]$

Objects: the F-orbits of objects in $\mathcal{D}$

Morphisms: $\mathrm{Hom}_{\mathcal{C}}(\widetilde{X}, \widetilde{Y}) = \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(F^i X, Y)$. Here X, Y are the objects in $\mathcal{D}$, and $\widetilde{X}, \widetilde{Y}$ are the corresponding objects in $\mathcal{C}$.

This category $\mathcal{C}$ is called a **cluster category**.

Recall: For a hereditary algebra H, an H-module T is said to be a **tilting module** if
(a) $\mathrm{Ext}^1_H(T,T) = 0$, that is T is **exceptional**, and there exists an exact sequence $0 \to H \to T_0 \to T_1 \to 0$ with $T_0, T_1 \in \mathrm{add}T$ (see [7]).
some useful equivalent characterizations [1]:
(b) T is exceptional and has n non-isomorphic indecomposable direct summands, where n is the number of simple H-modules.
(c) T is exceptional and has a maximal number of non-isomorphic indecomposable direct summands.

A tilting module is said to be **basic** if all of its direct summands are non-isomorphic.

For the case of $\mathcal{C} = \mathcal{D}^b(H)/F$, we say that T in $\mathcal{C}$ is a **cluster tilting object** if $\mathrm{Ext}^1_{\mathcal{C}}(T,T) = 0$ and T has a maximal number of non-isomorphic direct summands.

Lemma 1.1 [4] Let T be an H-module. Then T is exceptional if and only if T is an exceptional object in $\mathcal{C}$.

Theorem 1.2 [4] (a) Let T be a basic tilting object in $\mathcal{C} = \mathcal{D}^b(H)/F$, where H is a hereditary algebra with n simple modules.
(i) T is induced by a basic tilting module over a hereditary algebra $H^{'}$, derived equivalent to H.
(ii) T has n indecomposable direct summands.
(b) Any basic tilting module over a hereditary algebra H induces a basic tilting object for $\mathcal{C} = \mathcal{D}^b(H)/F$.

Let $\mathcal{C}$ be a cluster category and T is a cluster tilting object in $\mathcal{C}$, we call $\Gamma = \mathrm{End}_{\mathcal{C}}(T)^{op}$ the **cluster-tilted algebra**.

Here we consider H the path algebra of a Dynkin quiver Q. If the underlying graph of Q is the Dynkin graph Δ, then we say that Γ is **cluster-tilted algebra of type** Δ.

Theorem 1.3 [2] Let T be a tilting object in $\mathcal{C}$, and

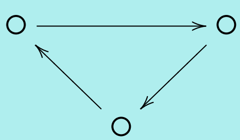
$$G = \mathrm{Hom}_{\mathcal{C}}(T, -) : \mathcal{C} \longrightarrow \mathrm{mod}\Gamma.$$

Then there is an induced functor $\overline{G} : \mathcal{C}/\mathrm{add}(\tau T) \longrightarrow \mathrm{mod}\Gamma$, which is an equivalence.

2 Cluster-tilted algebras of type A_n

Describe all the cluster-tilted algebras of type A_n in the following procedure:

Start with two quivers:


and

.

• All these vertices are free.

• New quivers can be obtained by identifying two free vertices.

• When we identify the two free ones, the new vertex we get is no longer free.

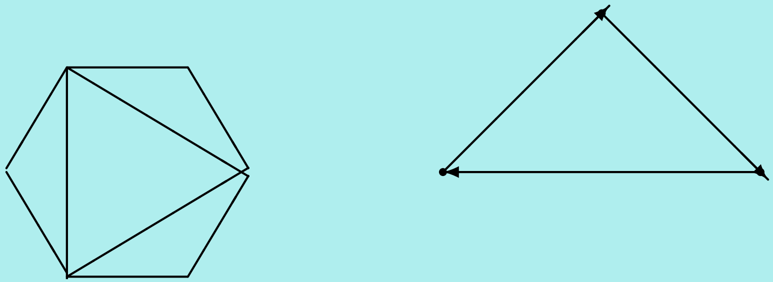
• Denote $\mathcal{A}_n$ by all the quivers of n vertices obtaining from this way.

- n— a positive integer

- $\mathcal{P}_{n+3}$— a regular polygon with $n+3$ vertices

Let T be a triangulation of $\mathcal{P}_{n+3}$, define a quiver Q_T:

Example:



• $\mathcal{T}_n$— the set of all triangulations of $\mathcal{P}_{n+3}$

Define a function $\varphi : \mathcal{T}_n \longrightarrow \mathcal{A}_n$

$$\varphi(T) = Q_T \ .$$

For a triangulation T of $\mathcal{P}_{n+3}$, denote by $T(i)$ the triangulation obtained from T by rotating T i steps in the clockwise direction. Define an equivalence relation $\sim$ on $\mathcal{T}_n$: let $T \sim T(i)$ for all i. Then we get a new function $\overline{\varphi} : (\mathcal{T}_n / \sim) \longrightarrow \mathcal{A}_n$ induced by φ.

Proposition 2.1 The function $\overline{\varphi} : (\mathcal{T}_n / \sim) \longrightarrow \mathcal{A}_n$ is a bijection.

Remark Torkildsen [9] proved there is a bijection between $(\mathcal{T}_n / \sim)$ and the cluster-tilted algebras of type A_n.

3 The torsionless modules

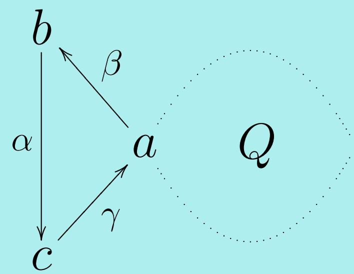
In the following we want to describe $\mathrm{Sub}(A)$ and $\underline{\mathrm{Sub}}(A)$ of a cluster-tilted algebra A of type A_n.

• $\mathrm{Sub}(A) = \{_AX | X$ is torsionless, i.e., X is a submodule of a projective module$\}$

• $\underline{\mathrm{Sub}}(A)$=$\mathrm{Sub}(A)/\mathrm{Proj}(A)$ where $\mathrm{Proj}(A)$ denotes all the projective A-modules.

A particular example is that A is a self-injective cluster-tilted algebra. Then $\mathrm{Sub}(A)$= $\mathrm{mod}(A)$.

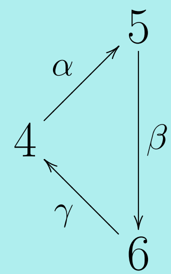
Lemma 3.1 Let A be a finite dimensional k-algebra. B is the algebra of the following quiver Q':



where Q is the quiver of the algebra A. Here the arrow β and γ are not involved in any relations except $\alpha\beta = 0$, $\gamma\alpha = 0$, $\beta\gamma = 0$. Then there is a bijection between indSub(A) and $\text{indSub}(B)/\{P(b), P(c), S(b), S(c), \text{rad}P(c)\}$ with $P(b), P(c)$ are the indecomposable projective B-modules and $S(b), S(c)$ the simple B-modules corresponding to the vertex b and c. Hence there is a bijection between $\text{ind}\underline{\text{Sub}}(A)$ and $\text{ind}\underline{\text{Sub}}(B)/\{S(b), S(c), \text{rad}P(c)\}$.

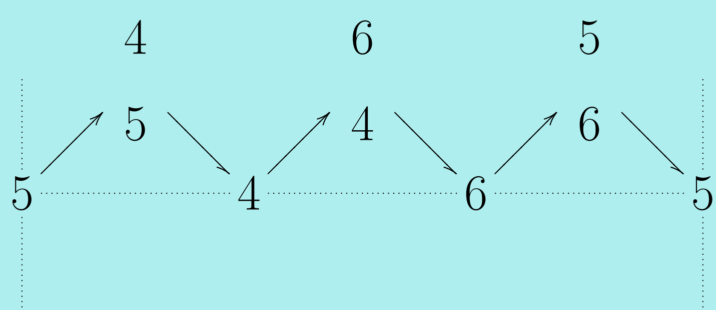
Examples:

(1) A is the path algebra of the quiver

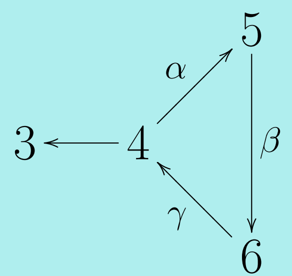


with the relations $\beta\alpha = 0, \ \gamma\beta = 0, \ \alpha\gamma = 0.$

Note that it is a self-injective algebra. Hence we have $\mathrm{Sub}(A) = \mathrm{mod}(A)$.

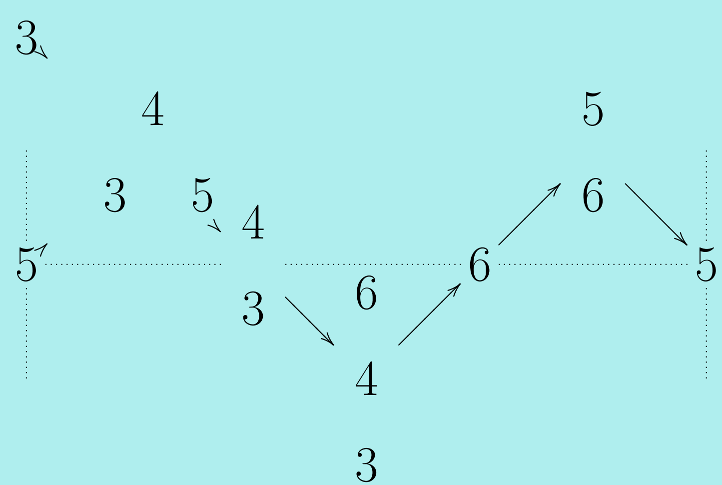


(2) A is the algebra of the quiver

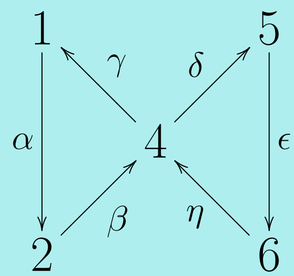


with the relations $\beta\alpha = 0, \ \gamma\beta = 0, \ \alpha\gamma = 0.$

We can get the indecomposable torsionless modules are:



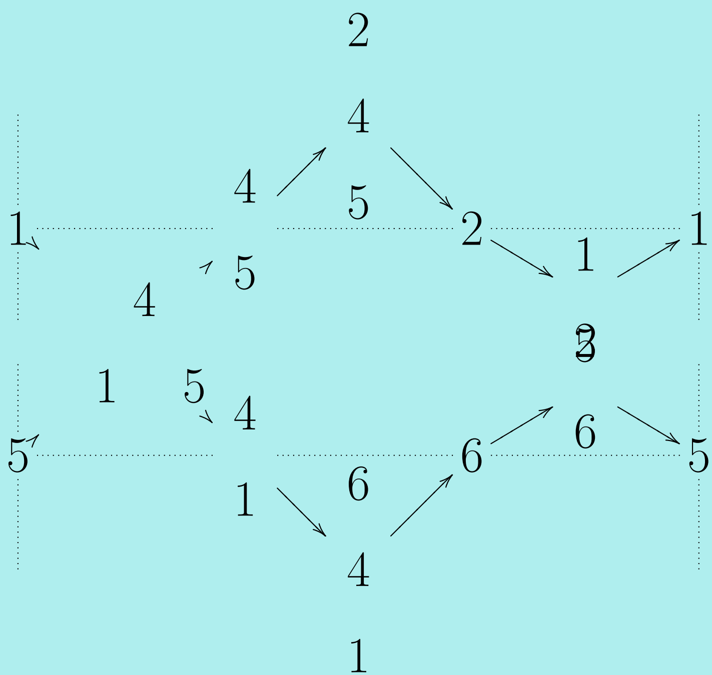
(3) A is the algebra of the quiver



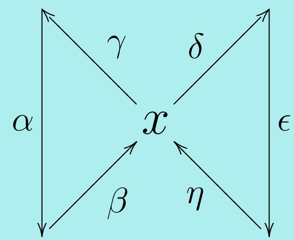
with the relations $\beta\alpha = 0,\ \gamma\beta = 0,\ \alpha\gamma = 0,\ \epsilon\delta = 0,\ \eta\epsilon = 0,\ \delta\eta = 0.$

Home Page
Title Page
Page 15 of 24
Go Back
Full Screen
Close
Quit

All the indecomposable torsionless modules are:



We call a vertex x a connecting vertex if two arrows end in x and two arrows start in x, i.e.,



with the relation $\gamma\beta = 0$ and $\delta\eta = 0$.

We say a vertex x is normal if it is one of the following cases:

$$x \longleftarrow$$

$$x \longrightarrow$$

$$\longrightarrow x \longrightarrow$$

$$\longrightarrow x \longleftarrow$$

$$\longleftarrow x \longrightarrow$$

Proposition 3.2 If A is a cluster-tilted algebra of type A_n, then $l(A) = 6c - d + e$, where c is the number of the 3-cycles in the quiver of A, d is the number of connecting vertices and e is the number of normal vertices.

Proposition 3.3 If M is a torsionless module, then M is local.

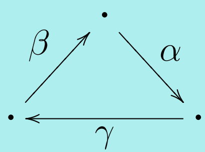
Proposition 3.4 If M is a torsionless but not projective module, then M is serial.

Theorem 3.5 If M is serial, then M is torsionless if and only if

(1) $\mathrm{Hom}(T, M) = 0$ for all non torsionless simple modules $T = S(x)$ and for all $T = S(\alpha)$, where α is an arrow not in any 3-cycle and $S(\alpha)$ not torsionless but $S(t(\alpha))$ torsionless;

(2) $M \neq L(\alpha)$ for α not in any 3-cycle.

Next we consider a special case: the cluster-tilted algebras of type A_n generated by



with relations $\alpha\beta = 0, \beta\gamma = 0, \gamma\alpha = 0$.

Recall the dominant dimension of an algebra A:

dom.dim $A \geqslant n :\Leftrightarrow$ if $0 \to A \to I_0 \to I_1 \to \cdots$ is a minimal injective resolution, then $I_0, I_1, \cdots, I_{n-1}$ are projective.

Proposition 3.6 If A is a cluster-tilted algebra as above. Then

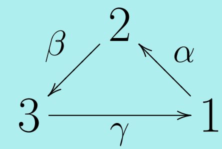
$$\text{dom.dim A} \geqslant 1.$$

4 The Auslander-Reiten structure

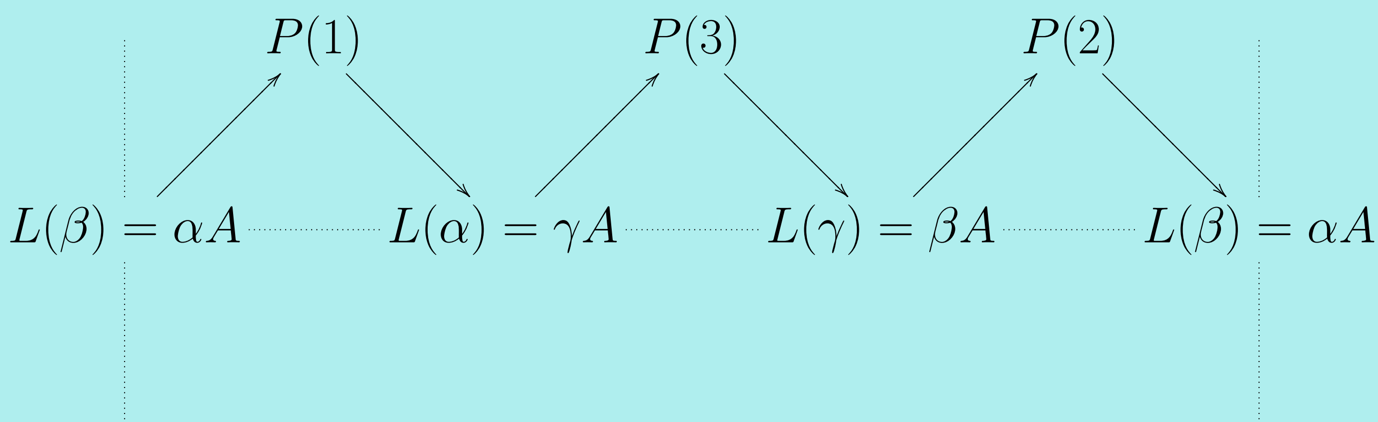
First we introduce the notion of $L(\alpha)$ and $Q(\beta)$. Actually this is the notion in the paper by Butler and Ringel [5]. For each arrow $\alpha : x \to y$, we have the exact sequence $0 \to Q(\alpha) \to M(\alpha) \to L(\alpha) \to 0$ where $0 \to Q(\alpha) \to I(y) \stackrel{I(\alpha)}{\to} I(x)$ and $P(y) \stackrel{P(\alpha)}{\to} P(x) \to L(\alpha) \to 0$.

Theorem 4.1 All the indecomposable torsionless modules of a cluster-tilted algebra A are the indecomposable projective modules and all the $L(\alpha)$'s where α is a cyclic arrow.

Theorem 4.2 A is a cluster-tilted algebra of type A_n, and the corresponding quiver is Q. If $1 \longrightarrow 2$ is a full subquiver of Q, then the Auslander-Reiten structure of all the indecomposable torsionless modules related to this part is $P(2) \to P(1)$. If



with the relations $\beta\alpha = 0, \gamma\beta = 0, \alpha\gamma = 0$ is a full subquiver of Q, then the Auslander-Reiten structure of all the indecomposable torsionless modules related to this part is



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Thank you!