Bi-module Problems

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0. Introduction

In the paper "Matrix and Drozd's Theorem" (1990), Crawley-Boevey defined an algebraic structure called bi-module problem. Namely, let K be a k-category, M a K-K-bi-module, and $d: K \longrightarrow M$ a derivation with $d(UV) = d(U)V + Ud(V)$, then the triple (K, M, d) is called a bi-module problem. Usually we assume that k is a perfect field, and K is a Krull-Schmidt category with finitely many indecomposable objects.

In the present talk we will extend the notion of Bi-module problem to that over a minimal algebra R over an algebraically closed k, and give its representation category, which is as the same as that given by Crawley-Boevey for R being trivial.

Then we define the dual structure of Bi-module problem, the so-called bi-co-module problem, and its representation category.

1. $\Delta^{\otimes}$-modules

Let $\mathcal{T}$ be a vertex set whose elements are divided into two disjoint families : the subset $\mathcal{T}_0$ of trivial vertices and the subset $\mathcal{T}_1$ of non-trivial vertices. To each $X \in \mathcal{T}_1$, we associate an indeterminate x and a fixed non-zero polynomial ϕ_X in $k[x]$. Given any $X \in \mathcal{T}$, we define a k-algebra R_X with identity 1_X by $R_X = k1_X$ if X is trivial ; and otherwise, $R_X = k[x, \phi_{_X}(x)^{-1}]$ the localization of $k[x]$ at $\phi_{_X}(x)$, and x is called a parameter.

For the sake of convenience, for each $X \in \mathcal{T}_0$, we set $x = 1_{_X}$, $\phi_X(x) = 1_{_X}$ and $k[x] = k1_{_X}$. In this way, R_X is the localization of $k[x]$ at $\phi_X(x)$, for every $X \in \mathcal{T}$.

The k-algebra $R = \Pi_{X\in\mathcal{T}} R_X$ is said to be a minimal algebra over $\mathcal{T}$.

The following notions are proposed and formulated by S.Liu.

Let $\Delta = R \otimes_k R$. Then Δ is a k-algebra under the multiplication $(a \otimes_k b)(c \otimes_k d) = ac \otimes_k bd$ for any $a, b, c, d \in R$. Denote

$$\Delta^{\otimes p} = \underbrace{\Delta \otimes_R \Delta \otimes_R \cdots \otimes_R \Delta}_{p} \text{ for } p \geqslant 1, \quad \text{and} \quad \Delta^0 = R.$$

The following statements hold for an integer $p \geqslant 0$:

(i) $\Delta^{\otimes p} \cong \underbrace{R \otimes_k R \otimes_k \cdots \otimes_k R}_{p+1}$ is both a free left R-module and a free right R-module, as well as a free Δ-module.

(ii) $\Delta^{\otimes p}$ is a commutative R-semi-algebra with the multiplication defined by

$$(\delta_1 \otimes_R \cdots \otimes_R \delta_n)(\sigma_1 \otimes_R \cdots \otimes_R \sigma_p) = \delta_1\sigma_1 \otimes_R \cdots \otimes_R \delta_p\sigma_p,\ \delta_i, \sigma_i \in \Delta.$$

(iii) $\Delta^{\otimes p}$ is a Δ-bi-module with the scalar multiplications defined by

$$(r \otimes_k s) \otimes_\Delta (\delta_1 \otimes_R \cdots \otimes_R \delta_p) = (\delta_1 \otimes_R \cdots \otimes_R \delta_p) \otimes_\Delta (r \otimes_k s)$$
$$= r\delta_1 \otimes_R \cdots \otimes_R \delta_p s,$$

for $r, s \in R, \delta_i \in \Delta$.

(iv) $\Delta^{\otimes p} \otimes_R \Delta^{\otimes q}$ is a $\Delta^{\otimes 2}$-bi-module with the operation

$$(\delta \otimes_R \sigma) \otimes_{\Delta^{\otimes 2}} (\xi \otimes_R \eta) = (\xi \otimes_R \eta) \otimes_{\Delta^{\otimes 2}} (\delta \otimes_R \sigma)$$
$$= (\delta \otimes_\Delta \xi) \otimes_R (\sigma \otimes_\Delta \eta),$$

where $\delta, \sigma \in \Delta$, $\xi \in \Delta^{\otimes p}$, $\eta \in \Delta^{\otimes q}$.

Set

$$\Delta^{\otimes} = \oplus_{p=0}^{\infty} \Delta^{\otimes p}, \quad \text{and} \quad \Delta^{\otimes \geqslant 1} = \oplus_{p=1}^{\infty} \Delta^{\otimes p}.$$

It is easy to see that $\Delta^{\otimes}$ is an $\mathbb{N}$-graded k-algebra with the multiplication

$$(\sigma_1 \otimes_R \cdots \otimes_R \sigma_m) \otimes_R (\eta_1 \otimes_R \cdots \otimes_R \eta_n)$$
$$= \sigma_1 \otimes_R \cdots \otimes_R \sigma_m \otimes_R \eta_1 \otimes_R \cdots \otimes_R \eta_n, \ \forall\, \sigma_i, \eta_i \in \Delta.$$

And $\Delta^{\otimes}$ is a Δ-module.

Definition 1.1 (i) A Δ-module (left and right) $\mathcal{S}_1$ is called a quasi-free Δ-module finitely generated by $u_1, \cdots, u_m$, if the morphism

$$\Delta_{X_1,Y_1} \oplus \cdots \oplus \Delta_{X_m Y_m} \rightarrow \mathcal{S}_1$$

sending each $1_{X_i} \otimes_k 1_{Y_i}$ to u_i is an isomorphism. In this case, $\{u_1, \ldots, u_m\}$ is called a Δ-quasi-free basis of $\mathcal{S}_1$.

(ii) Set $\mathcal{S}_p = \Delta^{\otimes p} \otimes_\Delta \mathcal{S}_1, p \geqslant 1$. Then $\mathcal{S}_p$ is a quasi-free $\Delta^{\otimes p}$-module with a quasi-basis $\{1_{\Delta^{\otimes p}} \otimes_\Delta u_i, \ldots, 1_{\Delta^{\otimes p}} \otimes_\Delta u_m\}$, and for $\eta, \sigma \in \Delta^{\otimes p}$,

$$\eta(\sigma \otimes_\Delta u_i) = (\eta\sigma) \otimes_\Delta u_i = (\sigma \otimes_\Delta u_i)\eta.$$

(iii) Set $\mathcal{S} = \oplus_{p=1}^{\infty} \mathcal{S}_p$, then $\mathcal{S}$ is a $\Delta^{\otimes \geqslant 1}$-module under the operator

$$\xi v = \sum_{p=1}^{\infty} \xi_p v_p, \quad \xi = \xi_1 + \xi_2 + \cdots \in \Delta^{\otimes \geqslant 1}, \; v = v_1 + v_2 + \cdots \in \mathcal{S},$$

which is called a quasi-free $\Delta^{\otimes \geqslant 1}$-module finitely generated by index 1.

Let $\mathcal{D}$, $\mathcal{G}$ be graded quasi-free $\Delta^{\otimes \geqslant 1}$-modules finitely generated by index 1. A $\Delta^{\otimes \geqslant 1}$-homomorphism $d : \mathcal{D} \to \mathcal{G}$ given by $d_p : \mathcal{D}_p \to \mathcal{G}_p$ is said to be determined by index 1, provided for $p \geqslant 1, \eta \in \Delta^{\otimes p}, x \in \mathcal{D}_1$,

$$\begin{gathered} d_p = \mathrm{id} \otimes_\Delta d_1 : \Delta^{\otimes p} \otimes_\Delta \mathcal{D}_1 \to \Delta^{\otimes p} \otimes_\Delta \mathcal{G}_1, \\ \eta \otimes_\Delta x \mapsto \eta \otimes_\Delta d_1(x). \end{gathered} \tag{1-1}$$

A $\Delta^{\otimes \geqslant 1}$-homomorphism $f : \mathcal{D} \otimes_R \mathcal{G} \to \mathcal{G}$ given by $f_{pq} : \mathcal{D}_p \otimes_R \mathcal{G}_q \to \mathcal{G}_{p+q}$ is said to be determined by index $(1, 1)$ provided for $p, q \geqslant 1$, $\eta \in \Delta^{\otimes p}, \sigma \in \Delta^{\otimes q}, x \in \mathcal{D}_1, y \in \mathcal{G}_1$,

$$\begin{gathered} f_{pq} = \mathrm{id} \otimes_R \mathrm{id} \otimes_{\Delta^{\otimes 2}} f_{11} : \\ (\Delta^{\otimes p} \otimes_\Delta \mathcal{D}_1) \otimes_R (\Delta^{\otimes q} \otimes_\Delta \mathcal{G}_1) \to \Delta^{\otimes p+q} \otimes_\Delta \mathcal{G}_1 \\ (\eta \otimes_\Delta x) \otimes_R (\sigma \otimes_\Delta y) \mapsto (\eta \otimes_R \sigma) \otimes_{\Delta^{\otimes 2}} f_{11}(x \otimes_R y). \end{gathered} \tag{1-2}$$

Definition 1.2 Let R be a minimal algebra with vertex set $\mathcal{T}$. We define a quasi-free R-module $\mathcal{K}_0 = \sum_{X\in\mathcal{T}} R_X E_X \cong R$ with a quasi-basis $\{E_X\}_{X\in\mathcal{T}}$; and a quasi-free Δ-module $\mathcal{K}_1$ with a quasi-basis $\mathcal{V} = \{V_1, V_2, \cdots, V_m\}$. Let $\mathcal{K}_{\geqslant 1} = \oplus_{i=1}^{\infty}\mathcal{K}_i$ be a $\Delta^{\otimes_{\geqslant 1}}$-module finitely generated by index 1, and $\mathcal{K} = \oplus_{i=0}^{\infty}\mathcal{K}_i$ be a $\Delta^{\otimes}$-module.

Moreover, there are a $\Delta^{\otimes}$-homomorphism $m : \mathcal{K} \otimes_R \mathcal{K} \to \mathcal{K}$, where restricting m into $\mathcal{K}_{\geqslant 1} \otimes_R \mathcal{K}_{\geqslant 1}$ is a $\Delta^{\otimes_{\geqslant 1}}$ homomorphism determined by index $(1,1)$; and a Δ-homomorphism $e : R \cong \mathcal{K}_0 \hookrightarrow \mathcal{K}$, such that $(\mathcal{K}, m, e)$ becomes an $\mathbb{N}$-graded k-algebra with identity $E = \sum_{X\in\mathcal{T}} E_X$.

Definition 1.3 Let the minimal algebra R and the $\mathbb{N}$-graded algebra $\mathcal{K}$ be given in 1.2. We define an $\Delta^{\otimes}$-module $\mathcal{M} = \oplus_{i=0}^{\infty}\mathcal{M}_i$, where $\mathcal{M}_0 = 0$, $\mathcal{M}_{\geqslant 1} = \oplus_{i=1}^{\infty}\mathcal{M}_i$ is a $\Delta^{\otimes_{\geqslant 1}}$-module finitely generated by index 1, and $\mathcal{M}_1$ has a Δ-quasi-basis $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$. Moreover, there are $\Delta^{\otimes}$-homomorphisms $l : \mathcal{K} \otimes_R \mathcal{M} \to \mathcal{M}$ and $r : \mathcal{M} \otimes_R \mathcal{K} \to \mathcal{M}$, where restricting l and r into $\mathcal{K}_{\geqslant 1} \otimes_R \mathcal{M}_{\geqslant 1}$ and $\mathcal{M}_{\geqslant 1} \otimes_R \mathcal{K}_{\geqslant 1}$ are $\Delta^{\otimes_{\geqslant 1}}$ homomorphisms determined by index $(1, 1)$ respectively, such that $\mathcal{M}$ becomes an $\mathbb{N}$-graded $\mathcal{K}$-$\mathcal{K}$-bi-module.

2. $\Delta^{\otimes}$-co-modules

Let $\mathcal{S} = \oplus_{p=1}^{\infty} \mathcal{S}_p$ be a $\Delta^{\otimes \geqslant 1}$-module finitely generated by index 1 with $\mathcal{S}_1$ a quasi-free Δ-module. Then the $\Delta^{\otimes \geqslant 1}$-*dual module of* $\mathcal{S}$ is defined to be $\mathcal{S}^* = \oplus_{p=1}^{\infty} \mathcal{S}_p^*$, where $\mathcal{S}_p^* = \mathrm{Hom}_{\Delta^{\otimes p}}(\mathcal{S}_p, \Delta^{\otimes p})$ for $p = 1, 2, \cdots$.

We have the following Lemma.

Lemma 2.1 $\mathcal{S}^* \simeq \oplus_{p=1}^{\infty} \Delta^{\otimes p} \otimes_{\Delta} \mathcal{S}_1^*$.

Let $\mathcal{D}$, $\mathcal{G}$ be $\Delta^{\bigotimes_{\geqslant 1}}$-modules finitely generated by index 1 with $\mathcal{D}_1$ and $\mathcal{G}_1$ being quasi-free Δ-modules. And let $d : \mathcal{D} \longrightarrow \mathcal{G}$ be a $\Delta^{\bigotimes_{\geqslant 1}}$-homomorphism determined by index 1 given in Formula (1-1). Then the $\Delta^{\bigotimes \geqslant 1}$-*dual* d^* of d is defined by $d_p^* : \mathcal{G}_p^* \longrightarrow \mathcal{D}_p^*$ for $p = 1, 2, \cdots$.

Lemma 2.2 $d_p^* = \text{id} \otimes_\Delta d_1^*$. Consequently d^* is still determined by index 1.

Let $\mathcal{D}$, $\mathcal{G}$ be $\Delta^{\otimes \geqslant 1}$-modules finitely generated by index 1 with $\mathcal{D}_1$, $\mathcal{G}_1$ being quasi-free Δ-modules. And let $f : \mathcal{D} \otimes_R \mathcal{G} \to \mathcal{G}$ be a $\Delta^{\otimes \geqslant 1}$-homomorphism determined by index $(1, 1)$ given in Formula (1-2). Then the $\Delta^{\otimes \geqslant 1}$-dual f^* of f is defined for $n = 2, 3, \cdots$:

$$f_n^* = \sum_{p+q=n} f_{pq}^* : \mathcal{G}_n^* \to \sum_{p+q=n} \mathcal{D}_p^* \otimes_R \mathcal{G}_q^*$$

Lemma 2.3 $f_{pq}^* = \mathrm{id}_{\Delta^{\otimes p}} \otimes_R \mathrm{id}_{\Delta^{\otimes q}} \otimes_{\Delta^{\otimes 2}} f_{11}^*$. Thus f^* is still determined by index $(1, 1)$.

The following notions concern bi-co-module problems is suggested by Y.Han.

Definition 2.4 Let R be a minimal algebra with vertex set $\mathcal{T}$. We define a quasi-free R-module $\mathcal{C}_0 = \sum_{X\in\mathcal{T}} R_X e_X \simeq R$ with a quasi-basis $\{e_X\}_{X\in\mathcal{T}}$; and a quasi-free Δ-module $\mathcal{C}_1$ with a quasi-basis $\mathcal{V}^* = \{v_1, v_2, \cdots, v_m\}$. Let $\mathcal{C}_{\geqslant 1} = \oplus_{i=1}^{\infty}\mathcal{C}_i$ be a $\Delta^{\otimes_{\geqslant 1}}$-module finitely generated by index 1, and let $\mathcal{C} = \oplus_{i=0}^{\infty}\mathcal{C}_i$ be a $\Delta^{\otimes}$-module.

Moreover, there exists a $\Delta^{\otimes}$-homomorphism $\mu : \mathcal{C} \to \mathcal{C} \otimes_R \mathcal{C}$, where restricting μ into $\mathcal{C}_{\geqslant 1}$ is a $\Delta^{\otimes_{\geqslant 1}}$ homomorphism determined by index $(1,1)$; and a Δ morphism $\varepsilon : \mathcal{C} \to \mathcal{C}_0 \simeq R$ which sends

e_X to 1_X, and v_j to 0 for all $1 \leqslant j \leqslant m$, such that $(\mathcal{C}, \mu, \varepsilon)$ becomes an $\mathbb{N}$-graded k-co-algebra.

Definition 2.5 Let the minimal algebra R and the $\mathbb{N}$-graded co-algebra $\mathcal{C}$ be given in 2.4. We define a $\Delta^{\otimes}$-module $\mathcal{N} = \oplus_{i=0}^{\infty}\mathcal{N}_i$, where $\mathcal{N}_0 = 0$, $\mathcal{N}_{\geqslant 1} = \oplus_{i=1}^{\infty}\mathcal{N}_i$ is a $\Delta^{\otimes_{\geqslant 1}}$-module finitely generated in degree 1 with $\mathcal{N}_1$ being a quasi-free Δ-module, having a Δ-quasi-basis $\mathcal{A}^* = \{a_1, a_2, \ldots, a_n\}$.

Moreover there exist graded $\Delta^{\otimes}$-homomorphisms $\iota : \mathcal{N} \to \mathcal{C} \otimes_R \mathcal{N}$ and $\tau : \mathcal{N} \to \mathcal{N} \otimes_R \mathcal{C}$, such that $\iota : \mathcal{N}^{\geqslant 2} \to \mathcal{C}^{\geqslant 1} \otimes_R \mathcal{N}^{\geqslant 1}$ and $\tau : \mathcal{N}^{\geqslant 2} \to \mathcal{N}^{\geqslant 1} \otimes_R \mathcal{C}^{\geqslant 1}$ are determined by index $(1, 1)$ respectively, thus $\mathcal{N}$ becomes an $\mathbb{N}$-graded $\mathcal{C}$-$\mathcal{C}$-bi-co-module.

3. Bi-module problems and their representations

Definition 3.1 A quadruple $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, d)$ is called a bi-module problem provided

(i) R is a minimal algebra with vertex set $\mathcal{T}$.

(ii) $\mathcal{K} = \oplus_{i=0}^{\infty}\mathcal{K}_i$ is an $\mathbb{N}$-graded k-algebra given by Definition 1.2.

(iii) $\mathcal{M} = \oplus_{i=0}^{\infty}\mathcal{M}_i$ is an $\mathbb{N}$-graded $\mathcal{K}$-$\mathcal{K}$-bi-module given by Definition 1.3.

(iv) There is an $\mathbb{N}$-graded derivation $d : \mathcal{K} \to \mathcal{M}$ determined by index 1 given in Formula (1-1) with $d(UV) = Ud(V) + d(U)V$ for any $U, V \in \mathcal{K}$.

Now we define the representation category of $\mathfrak{A}$.

Definition 3.2 Let $J(\lambda) = J_d(\lambda)^{e_d} \oplus J_{d-1}(\lambda)^{e_{d-1}} \oplus \cdots \oplus J_1(\lambda)^{e_1}$, with e_i non-negative integers, be a Jordan form. Set $m_j = e_d + e_{d-1} + \cdots + e_j$. The following partitioned matrix W_λ is called a Weyr matrix of eigenvalue λ :

$$W(\lambda) = \begin{pmatrix} \lambda I_{m_1} & W_{12} & 0 & \cdots & 0 & 0 \\ & \lambda I_{m_2} & W_{23} & \cdots & 0 & 0 \\ & & \lambda I_{m_3} & \cdots & 0 & 0 \\ & & & \ddots & \vdots & \vdots \\ & & & & \lambda I_{m_{d-1}} & W_{d-1,d} \\ & & & & & \lambda I_{m_d} \end{pmatrix}_{d \times d},$$

with $W_{j,j+1} = \begin{pmatrix} I_{m_{j+1}} \\ 0 \end{pmatrix}_{m_j \times m_{j+1}}$. A direct sum $W = W(\lambda_1) \oplus W(\lambda_2) \oplus \cdots \oplus W(\lambda_s)$ of finitely many Weyr matrices with distinct eigenvalues λ_i is said to be a Weyr matrix.

Let $X \in \mathcal{T}_1$. A Weyr matrix W over k is called R_X-regular if its eigenvalues are R_X-regular, i.e. $\phi_X(\lambda) \neq 0$, for all the eigenvalues λ.

Denote by mod-R the category consisting of finitely dimensional R-modules. Since R is commutative, mod-R can be viewed as both left and right module category. Let

$$P \in \text{mod-}R, \quad \text{with} \quad P_X = k^{m_X},\ P(x) = W_X, \forall\, X \in \mathcal{T},$$

where $W_X = I_{m_X}$ for $X \in \mathcal{T}_0$; or W_X is a R_X-regular Weyr matrix for $X \in \mathcal{T}_1$. And $(m_X)_{X\in\mathcal{T}}$ is said to be the dimension vector of P. Suppose the quasi-basis element $A_i = 1_{X_i} A_i 1_{Y_i}$, let

$$\begin{array}{c} {}_P(\mathcal{M}_1)_P = P \otimes_R \mathcal{M}_1 \otimes_R P \quad \Longrightarrow \\ {}_P(\mathcal{M}_1)_P \simeq \bigoplus_{i=1}^{n} \mathrm{IM}_{m_{X_i} \times m_{Y_i}}(k). \end{array} \tag{3-1}$$

In fact $P \otimes_R \mathcal{M}_1 \otimes_R P \simeq P \otimes_R (\oplus_{i=1}^{n} \Delta_{X_i Y_i}) \otimes_R P \simeq \oplus_{i=1}^{n} P \otimes_R R(1_{X_i} \otimes_k 1_{Y_i}) R \otimes_R P \simeq \oplus_{i=1}^{n} P 1_{X_i} \otimes_k 1_{Y_i} P \simeq \oplus_{i=1}^{n} \mathrm{IM}_{m_{X_i} \times m_{Y_i}}(k)$. Then ${}_P(\mathcal{M}_1)_P$ possesses an R-R-bi-modules structure :

$$xB = W_X B, \quad By = BW_Y, \quad \forall\ B \in \mathrm{IM}_{m_X \times m_Y}(k). \tag{3-2}$$

Taken $\alpha = (\alpha_1, \cdots, \alpha_n) \in_P (\mathcal{M}_1)_P$ with $\alpha_i \in \mathrm{IM}_{m_{X_i} \times m_{Y_i}}(k)$.

Write

$$P_1 = \sum_{i=1}^{n} \alpha_i \otimes_\Delta A_i, \tag{3-3}$$

where α_i can be viewed as a matrix coefficient of A_i.

Definition 3.3 A representation (P, P_1) of a bi-module problem $\mathfrak{A}$ consists of a module $P \in$ mod-R and an element P_1 given by Formulae (3-1)-(3-3).

Suppose (Q, Q_1) is again a representation of dimension vector $\underline{n}$, $Q_X = W'_X$ for $X \in \mathcal{T}$, and $Q_1 = \sum_{i=1}^{n} \beta_i \otimes_\Delta A_i$. Set the quasi-basis element $V_j = 1_{X'_j} V_j 1_{Y'_j}$, let

$$ {}_P(\mathcal{K}_1)_Q = P \otimes_R \mathcal{K}_1 \otimes_R Q \simeq \bigoplus_{j=1}^m \mathrm{I}M_{m_{X'_j} \times n_{Y'_j}}(k), \qquad (3\text{-}4) $$

with the left and right R-module actions :

$$ xU = W_X U, \quad Uy = UW'_Y, \quad \forall\, U \in \mathrm{I}M_{m_X \times n_Y}(k). \qquad (3\text{-}5) $$

Let $f_0 = (f_X \mid X \in \mathcal{T}) : P \to Q$ be an R-module homomorphism, and $f_1 = (f_1^1, \cdots, f_1^m) \in {}_P(\mathcal{K}_1)_Q$ with $f_1^j \in \mathrm{I}M_{m_{X'_j} \times n_{Y'_j}}(k)$. Write

$$ S_0 = \sum_{X \in \mathcal{T}} f_X \otimes_R E_X, \quad S_1 = \sum_{j=1}^m f_1^j \otimes_\Delta V_j, \qquad (3\text{-}6) $$

where f_X, f_1^j can be viewed as matrix coefficients of E_X, V_j respectively.

Definition 3.4 A morphism of a bi-module problem $\mathfrak{A}$ is $S = S_0 + S_1 : (P, P_1) \longrightarrow (Q, Q_1)$, where S_0, S_1 are given by (3-4)-(3-6), such that $P_1 \circ S_1 - S_1 \circ Q_1 = d(S_1)$ with the operations "d" and "$\circ$" being calculated below.

For the calculation of d, we have

$$d(V_i) = \sum_l \zeta_{il} \otimes_\Delta A_l \implies d(f_1^i \otimes_\Delta V_i) = \sum_l \left(\zeta_{il} \otimes_\Delta f_1^i\right) \otimes_\Delta A_l$$

by duality. For the calculation of $\circ$, we have :

$$V_i \otimes_R A_j = \sum_l \eta_{ijl} \otimes_{\Delta^{\otimes 2}} (1_{\Delta^{\otimes 2}} \otimes_\Delta A_l) \implies$$
$$(f_1^i \otimes_\Delta V_i) \circ (\alpha_j \otimes_\Delta A_j) = \sum_l [\eta_{ijl} \otimes_{\Delta^{\otimes 2}} (f_1^i \otimes_R \alpha_j)] \otimes_\Delta A_l$$

by duality. $(\alpha_i \otimes_\Delta A_j) \circ (f_1^j \otimes_\Delta V_j)$ can be calculated similarly.

Corollary 3.5 Let $S : (P, P_1) \to (Q, Q_1)$, $S' : (Q, Q_1) \to (U, U_1)$ be two morphisms over $\mathfrak{A}$. Then $S \circ S' : (P, P_1) \to (U, U_1)$ is also a morphism over $\mathfrak{A}$ with $(S \circ S')_0 = S_0 S'_0$, $(S \circ S')_1 = S_0 S'_1 + S_1 S'_0 + S_1 \circ S'_1$.

We denote by $R(\mathfrak{A})$ the category of representations of the bi-module problem $\mathfrak{A}$.

4. Bi-co-module problems and their representations

Definition 4.1 A quadruple $\mathfrak{C} = (R, \mathcal{C}, \mathcal{N}, \partial)$ is called a bi-co-module problem provided

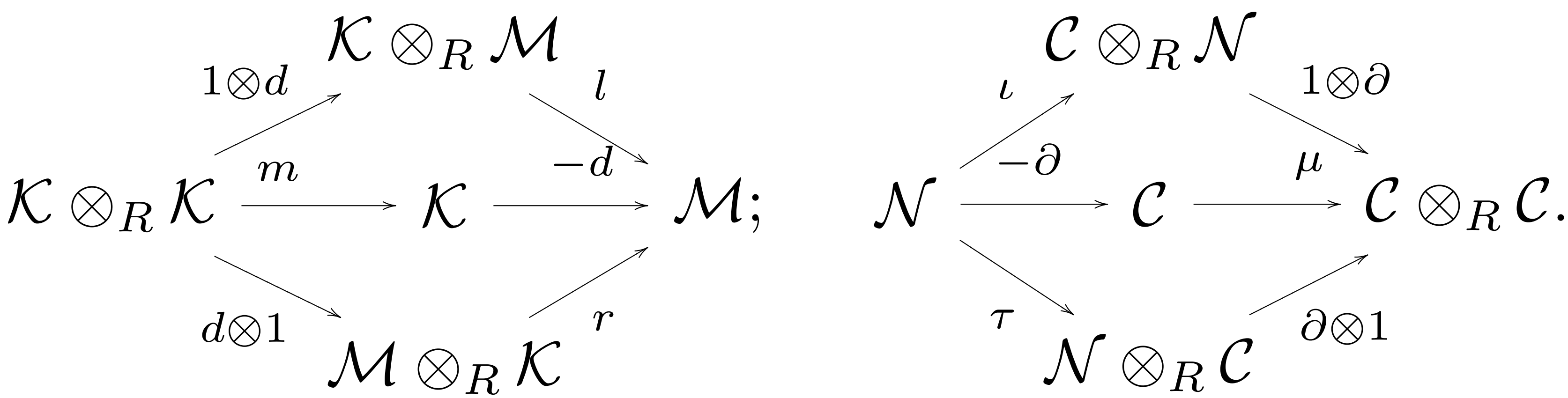
(i) R is a minimal algebra with a vertex set $\mathcal{T}$;

(ii) $\mathcal{C} = \oplus_{i=0}^{\infty} \mathcal{C}_i$ is an $\mathbb{N}$-graded co-algebra given by Definition 2.4.

(iii) $\mathcal{N} = \oplus_{i=0}^{\infty} \mathcal{N}_i$ is an $\mathbb{N}$-graded $\mathcal{C}$-$\mathcal{C}$-bi-co-module given by Definition 2.5.

(iv) There is an $\mathbb{N}$-graded co-derivation $\partial : \mathcal{N} \to \mathcal{C}$ determined by index 1.

The derivation law and co-derivation law are shown in the picture below :



Definition 4.2 A representation (P, α) of bi-co-module problem $\mathfrak{C}$ is a pair (P, α), where P is a finite-dimensional left R-module, and $\alpha \in \mathrm{Hom}_R(\mathcal{N}_1 \otimes_R P, P)$.

Let (P, α) and (Q, β) be two representations of $\mathfrak{C}$ of dimension $\underline{m}, \underline{n}$ respectively. A morphism $f : (P, \alpha) \to (Q, \beta)$ of a bi-co-module problem $\mathfrak{C}$ is an R-morphism $f \in \mathrm{Hom}_R(\mathcal{C}_{\leqslant 1} \otimes_R P, Q)$ satisfying the following commutative diagram :

$$
\begin{array}{ccccc}
\mathcal{N}_1 \otimes_R P & \xrightarrow{\iota_{11}\otimes 1} & \mathcal{C}_{\leqslant 1} \otimes_R \mathcal{N}_1 \otimes_R P & \xrightarrow{1\otimes\alpha} & \mathcal{C}_{\leqslant 1} \otimes_R P \\
 & \searrow^{\partial_1\otimes 1} & & & \\
\Big\downarrow{\scriptstyle -\tau_{11}\otimes 1} & & \mathcal{C}_{\leqslant 1} \otimes_R P & & \Big\downarrow{\scriptstyle f} \\
 & & & \searrow^{f} & \\
\mathcal{N}_1 \otimes_R \mathcal{C}_{\leqslant 1} \otimes_R P & \xrightarrow[1\otimes f]{} & \mathcal{N}_1 \otimes_R Q & \xrightarrow[\beta]{} & Q
\end{array}
$$

If $g : Q \to U$ is another morphism, then the composition $h = fg$ is defined to be the composition of the following maps :

$$
\mathcal{C}_{\leqslant 1} \otimes_R P \xrightarrow{\mu\otimes 1} \mathcal{C}_{\leqslant 1} \otimes_R \mathcal{C}_{\leqslant 1} \otimes_R P \xrightarrow{1\otimes f} \mathcal{C}_{\leqslant 1} \otimes_R Q \xrightarrow{g} U.
$$

We denote by $R(\mathfrak{C})$ the category of representations of the bi-co-module problem $\mathfrak{C}$.

Theorem 4.4 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, d)$ be a bi-module problem. Let $\mathcal{C}$ and $\mathcal{N}$ be the $\Delta^{\otimes}$-dual of $\mathcal{K}$ and $\mathcal{M}$ respectively, and ι, τ, ∂ be the $\Delta^{\otimes}$-dual of the left module action l, the right module action r, the derivation d respectively. Then $\mathfrak{C} = (R, \mathcal{C}, \mathcal{N}, \partial)$ is a bi-co-module problem, which is called the $\Delta^{\otimes}$-dual of $\mathfrak{A}$.

Furthermore, representation categories $R(\mathfrak{A})$ and $R(\mathfrak{C})$ are equivalent.

Moreover, we can construct a layered bocs $B = (\Gamma; \Omega)$ from a bi-co-module problem $\mathfrak{C}$ corresponding to a bi-module problem $\mathfrak{A}$ such that the representation categories $R(\mathfrak{A}); R(\mathfrak{C})$ and $R(\mathfrak{B})$ are equivalent.

We can also introduce the reduction operations for bi-module problems and bi-co-module problems which correspond to the reductions for bocses. These reduction techniques can not only provide some inductive step in the proof of foundation Tame Wild Theorem, but also play a key role to find some "good" or "bad" configurations in representation categories.

In addition, a class of special bi-module problems in terms of matrices, the so-called matrix bi-module problem, is very important and interesting. Perhaps, the notions of bi-module problem, bi-co-module problem and bocs originated with it.

Unfortunately, for the limit of time I am not able to show the above subjects.

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Thanks