Mutation of cotorsion pairs and its geometric realization arising from marked surfaces

(joint work with Jie Zhang and Bin Zhu)

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Plan of the talk:

Two parts:

1. **Classification of cotorsion pairs in the cluster category of a marked surface (which can be regarded as an application of the main result in Zhu's talk).**
2. **Mutation of cotorsion pairs and its geometric realization.**

1. Classification of (co)torsion pairs in the cluster category of a marked surface

- **Triangulated 2-Calabi-Yau category**

k: **an algebraically closed field.**

C: **a Hom-finite Krull-Schmidt triangulated 2-CY category over** k**.**

$[1]$: **the shift functor of** C**.**

1. Classification of (co)torsion pairs in the cluster category of a marked surface

- **Cotorsion pair and its core**

A pair $(\mathscr{X}, \mathscr{Y})$ of subcategories closed under isomorphisms, direct sums and summands is a cotorsion pair if

- **Hom**$(X, Y[1]) = 0$**,** $\forall X \in \mathscr{X}, Y \in \mathscr{Y}$**;**
- $C = \mathscr{X} * \mathscr{Y}[1]$**, i.e.** $\forall C \in C$**,** $\exists$ **triangle**

$$X \to C \to Y[1] \to X[1]$$

with $X \in \mathscr{X}, Y \in \mathscr{Y}$**.**

$I = \mathscr{X} \cap \mathscr{Y}$**: the core of cotorsion pair** $(\mathscr{X}, \mathscr{Y})$**.**

1. Classification of (co)torsion pairs in the cluster category of a marked surface

- **Marked surface (with unpunctured)**

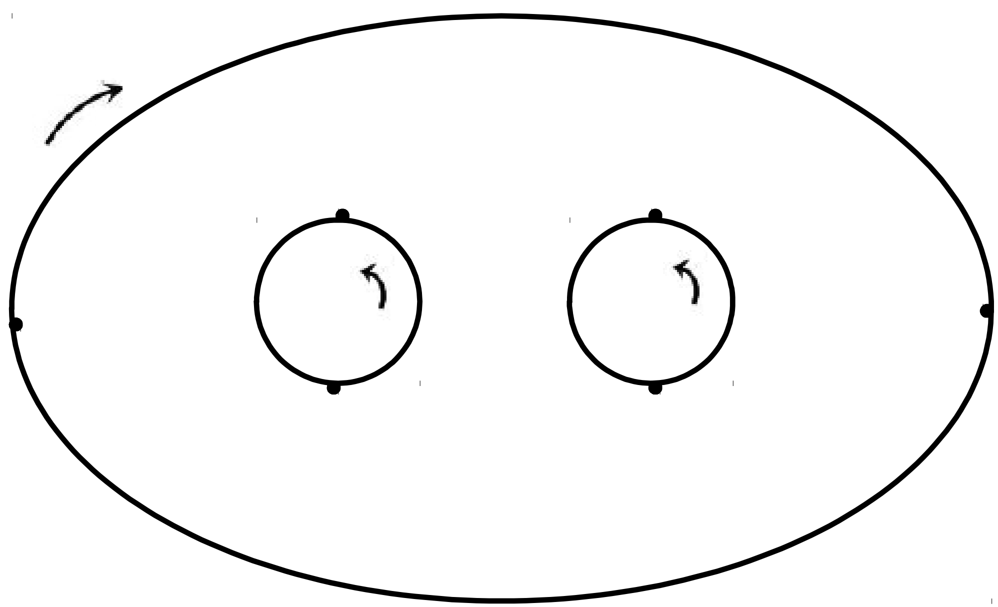
A marked surface (with unpunctured) is a pair (S, M):

- **S is a compact oriented Riemann surface with $\partial S \neq \emptyset$;**
- **M is a finite set of marked points lying on ∂S such that every boundary component contains at least one marked point.**

1. Classification of (co)torsion pairs in the cluster category of a marked surface

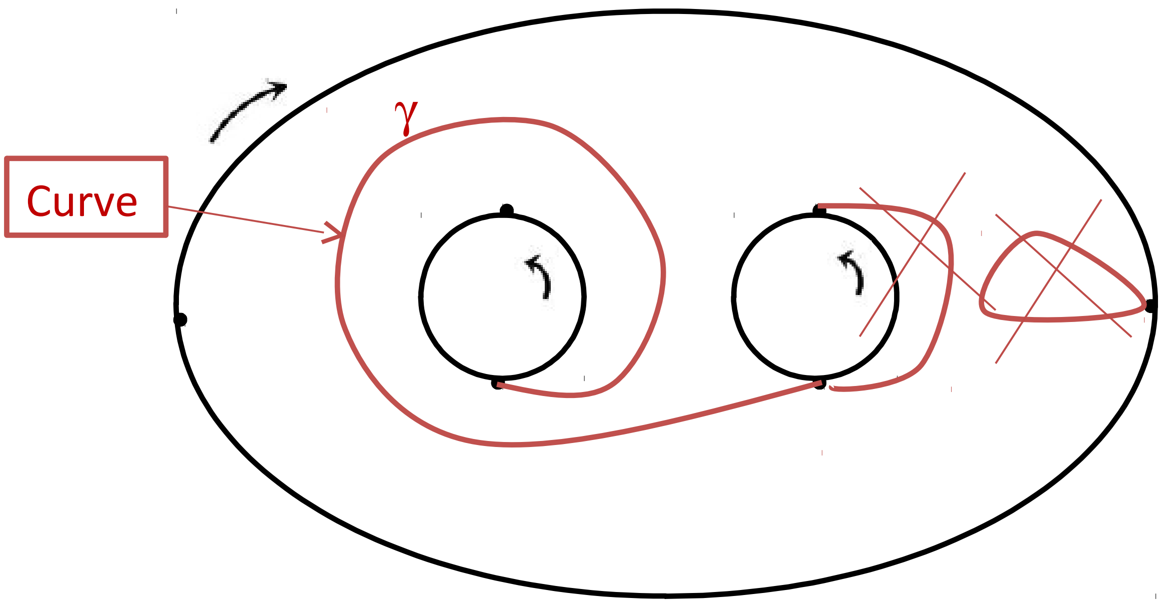
- **Example**

(S,M) is obtained from a sphere by removing 3 disks, with two marked points in each boundary component.



1. Classification of (co)torsion pairs in the cluster category of a marked surface

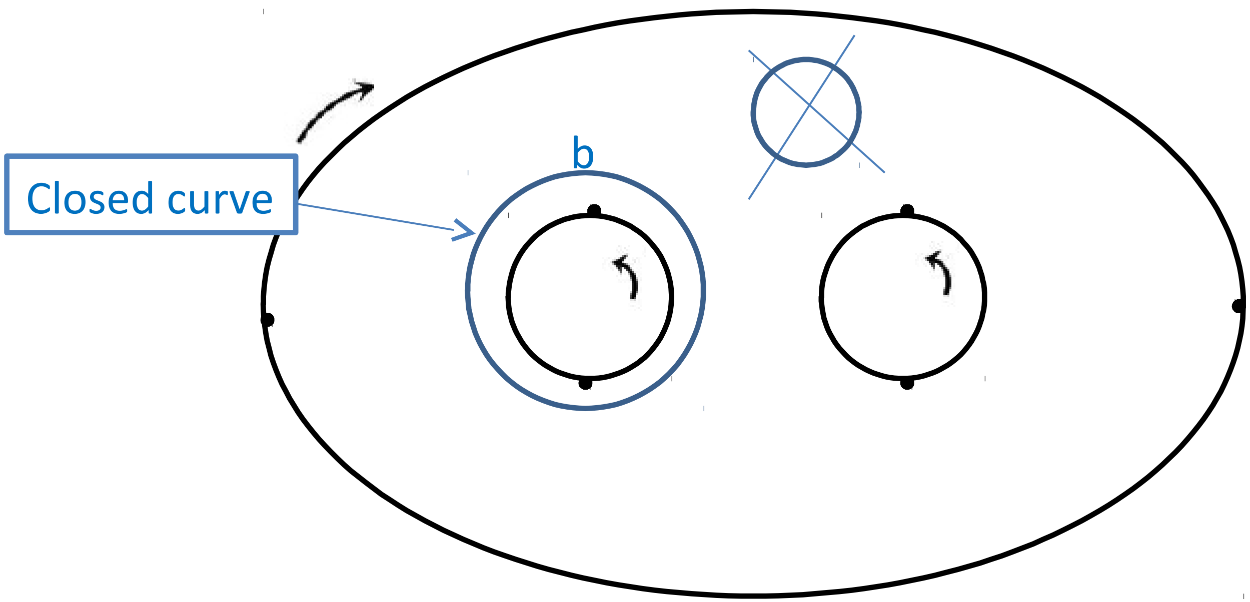
- **Curves in (S,M)**



1. Classification of (co)torsion pairs in the cluster category of a marked surface

- **Valued closed curves in (S,M)**

a pair (b, λ): b a closed curve, $0 \neq \lambda \in k.$



1. Classification of (co)torsion pairs in the cluster category of a marked surface

- **Cluster category C(S,M)**

Triangulation of (S, M)

↓ (Labardini-Fragoso, Assem-Brustle-Charbonneau-Jodoin-Plamondon)

Quiver with potential (Q, W) $(\dim_k J(Q, W) < \infty)$

↓ Amiot

Hom-finite Krull-Schmidt triangulated 2-CY category $C_{(S,M)}$ **with cluster tilting objects**

↓ Keller-Yang

$C_{(S,M)}$ **is independent of the choice of triangulation of** (S, M)

↓ Brustle-Zhang

Indecomposable objects in C(S,M) ←1-1→ **Curves and valued closed curves in (S,M)**

X_γ ← γ

1. Classification of (co)torsion pairs in the cluster category of a marked surface

- **Cluster category C(S,M)**

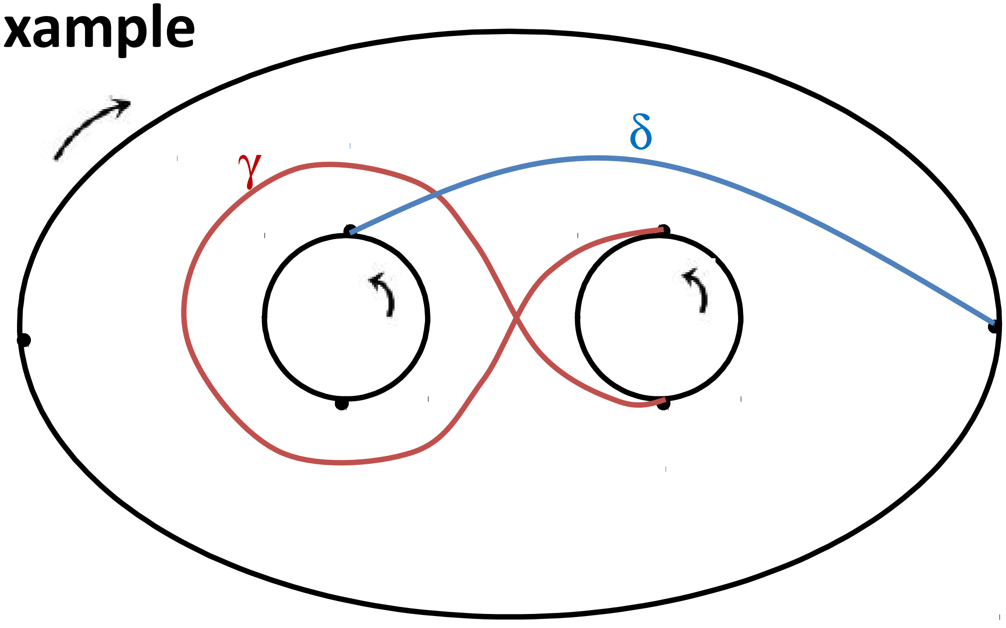
Subcategories in C(S,M) (closed under isomorphisms, direct sums and summands)	$\overset{1\text{-}1}{\longleftrightarrow}$	Collections of curves and valued closed curves in (S,M)
I	$\longleftrightarrow$	I

1. Classification of (co)torsion pairs in the cluster category of a marked surface

- **Intersections of curves**

Int(γ, δ): the minimal intersection number of two representatives of the homotopy classes of γ and δ.

Example

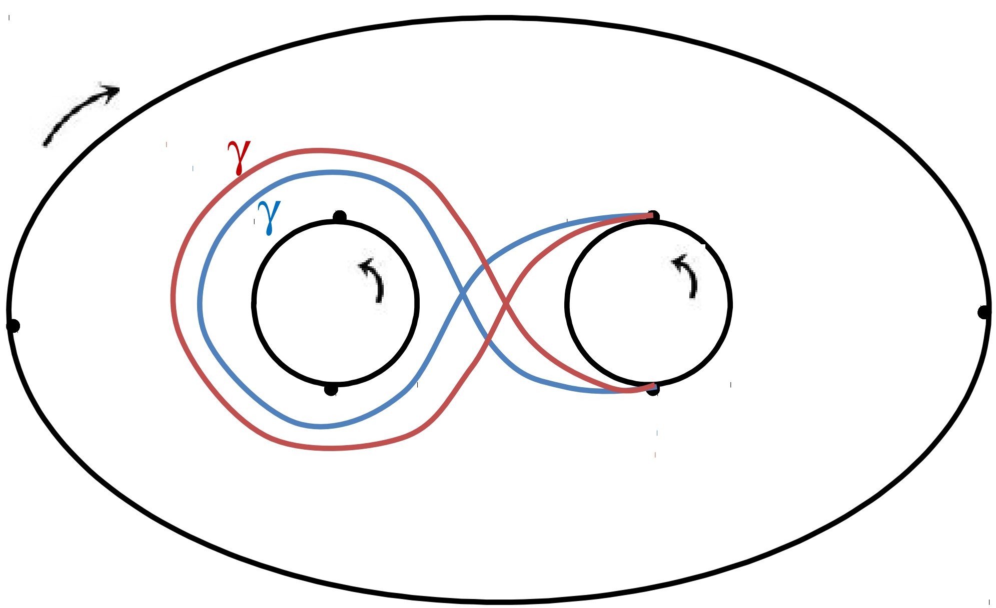


$\mathrm{Int}(\gamma,\delta) = 1$

1. Classification of (co)torsion pairs in the cluster category of a marked surface

- **Intersections of curves**

Another example



$\text{Int}(\gamma,\gamma) = ?$

$\text{Int}(\gamma,\gamma) = 2$

1. Classification of (co)torsion pairs in the cluster category of a marked surface

- **Intersections and dimensions of extensions**

Theorem A

Let γ and δ be two curves (which are not necessarily different) in (S, M). Then

$$\dim_k \mathbf{Ext}^1_{C_{(S,M)}}(X_\gamma, X_\delta) = \mathbf{Int}(\gamma, \delta).$$

1. Classification of (co)torsion pairs in the cluster category of a marked surface

- **Paintings of (S,M)**

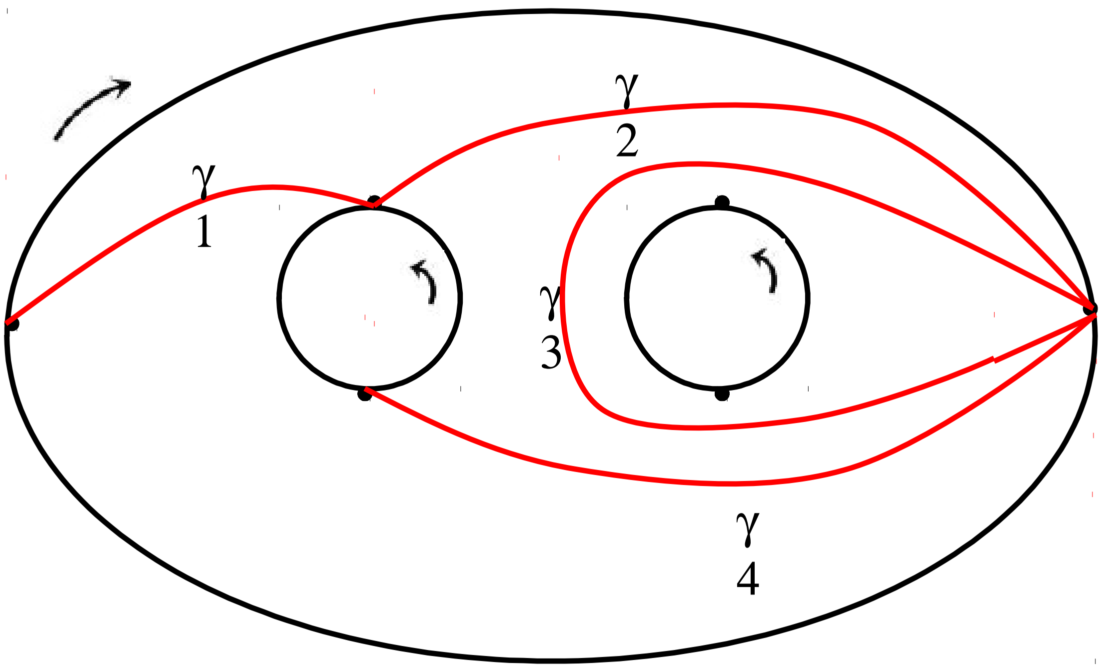
I: a collection of curves in (S, M) with
$\mathbf{Int}(\gamma, \delta) = 0, \ \forall \gamma, \delta \in I.$

An I−painting of (S, M) is obtained from (S, M) by filling black in some non-triangle components of (S, M) divided by curves in I and leaving other components white.

1. Classification of (co)torsion pairs in the cluster category of a marked surface

- **Example**

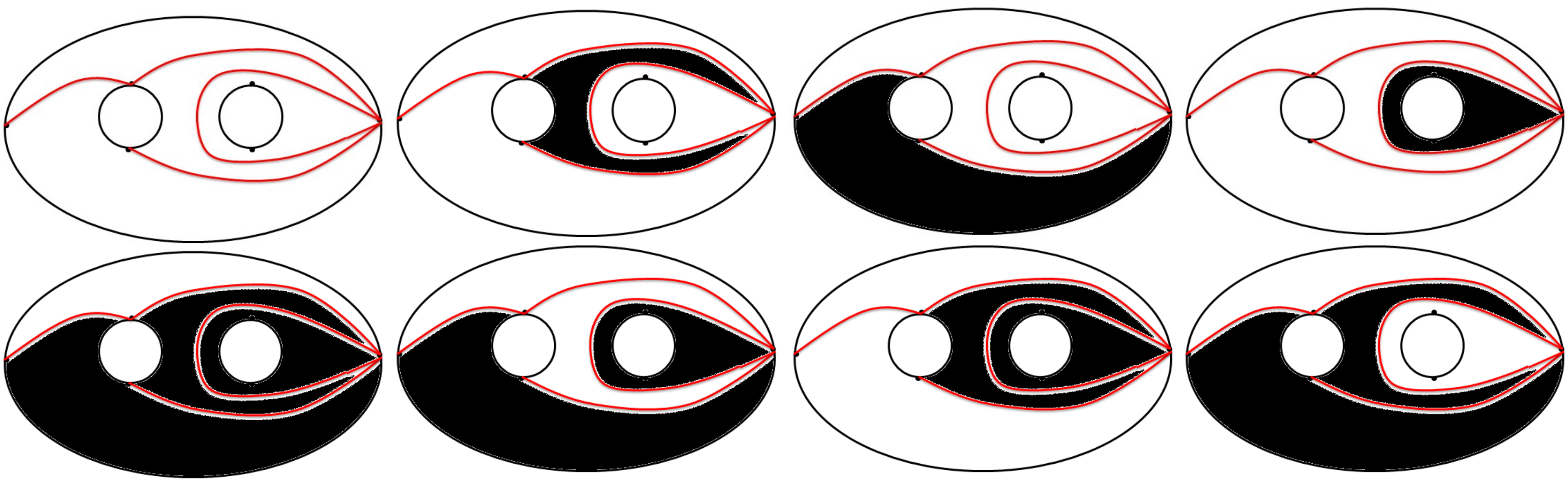
$(S, M) \qquad I = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$



1. Classification of (co)torsion pairs in the cluster category of a marked surface

- **Example**

There are 2^3 I−paitings.



1. Classification of (co)torsion pairs in the cluster category of a marked surface

- **Cotorison pairs and paintings**

Theorem B

There is a bijection between cotorsion pairs in $C_{(S,M)}$ with core I and I−paintings of (S, M).

I**−painting** G **cotorsion pair** $(\mathscr{X}_G, \mathscr{Y}_G)$

$\mathscr{X}_G =$ {**curves and value closed curves in black regions**} $\bigcup I$

$\mathscr{Y}_G =$ {**curves and value closed curves in white regions**} $\bigcup I$

Part I

2. Mutations of cotorsion pairs and their geometric realizations

- **Mutations of cotorsion pairs**

$(\mathscr{X}, \mathscr{Y})$: **a cotorsion pair.**

I: **core of** $(\mathscr{X}, \mathscr{Y})$.

$\mathcal{D} \subset I$: **a functorially finite subcategory of** C.

We define the D**−mutation of** $(\mathscr{X}, \mathscr{Y})$:

$\mu^{-1}((\mathscr{X}, \mathscr{Y}); \mathcal{D}) := ((\mathcal{D} * \mathscr{X}[1]) \cap {}^{\perp}\mathcal{D}[1], (\mathcal{D} * \mathscr{Y}[1]) \cap {}^{\perp}\mathcal{D}[1])$

where ${}^{\perp}\mathcal{D}[1] = \{C \in C \mid \mathbf{Hom}(C, D[1]) = 0, \forall D \in \mathcal{D}\}$.

2. Mutations of cotorsion pairs and their geometric realizations

- **Mutations of cotorsion pairs**

Theorem C

Let $(\mathscr{X}, \mathscr{Y})$ be a cotorsion pair in C with core I and $\mathscr{D}$ be a functorially finite subcategory of C with $\mathscr{D} \subset I$. Then the $\mathscr{D}$–mutation of $(\mathscr{X}, \mathscr{Y})$ is a cotorsion pair in C with core $(\mathscr{D} * I[1]) \cap {}^{\perp}\mathscr{D}[1]$.

2. Mutations of cotorsion pairs and their geometric realizations

- **Rotations of paintings**

G: I−painting of (S, M).

$\mathcal{D}$: subcollection of I.

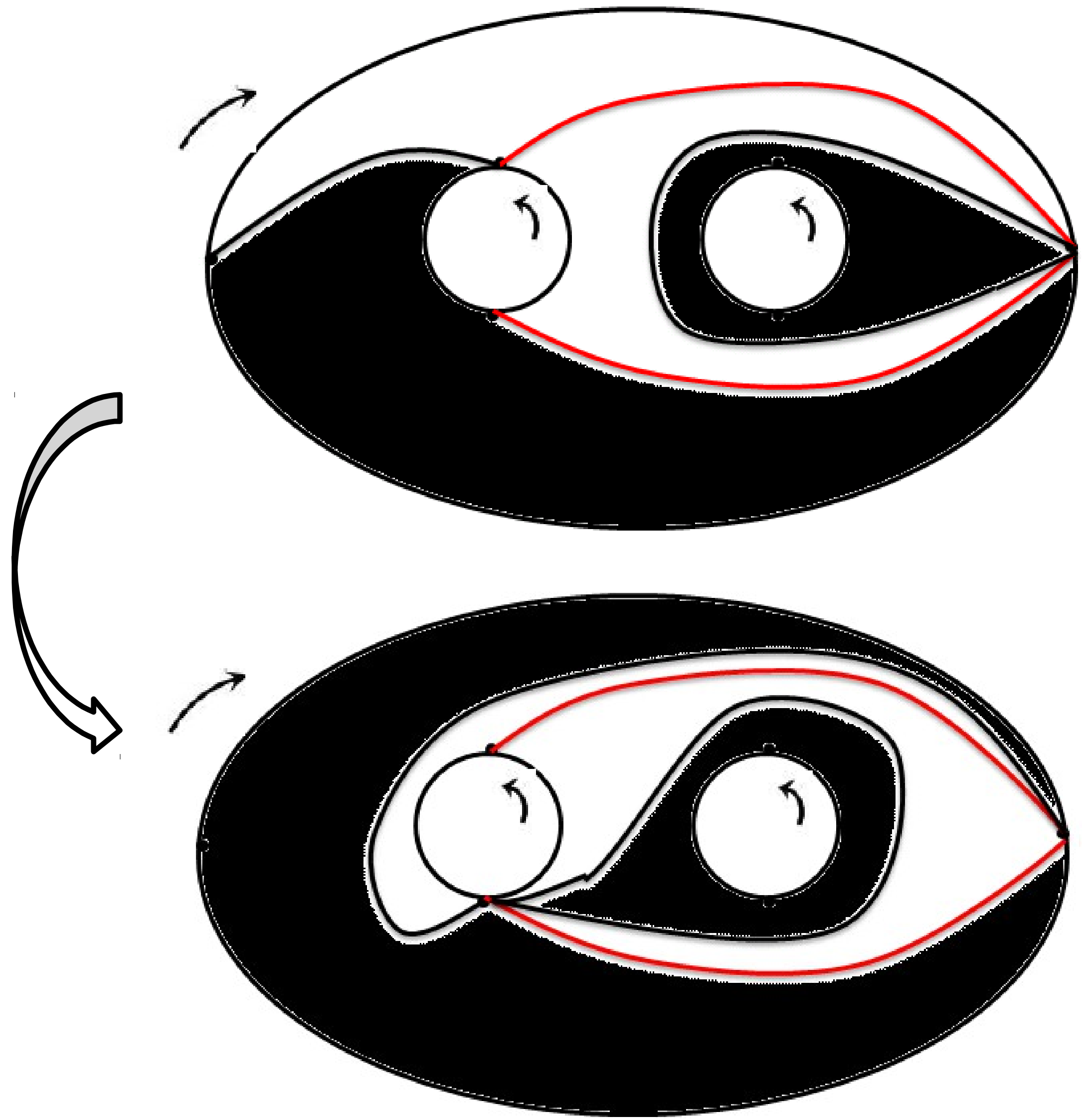
The $\mathcal{D}$−rotation of G is defined as rotating each region divided by I along the boundaries of S and curves in $\mathcal{D}$ in clockwise order to the next position.

2. Mutations of cotorsion pairs and their geometric realizations

- **Example**

$\mathcal{D}$ = {**red curves**}

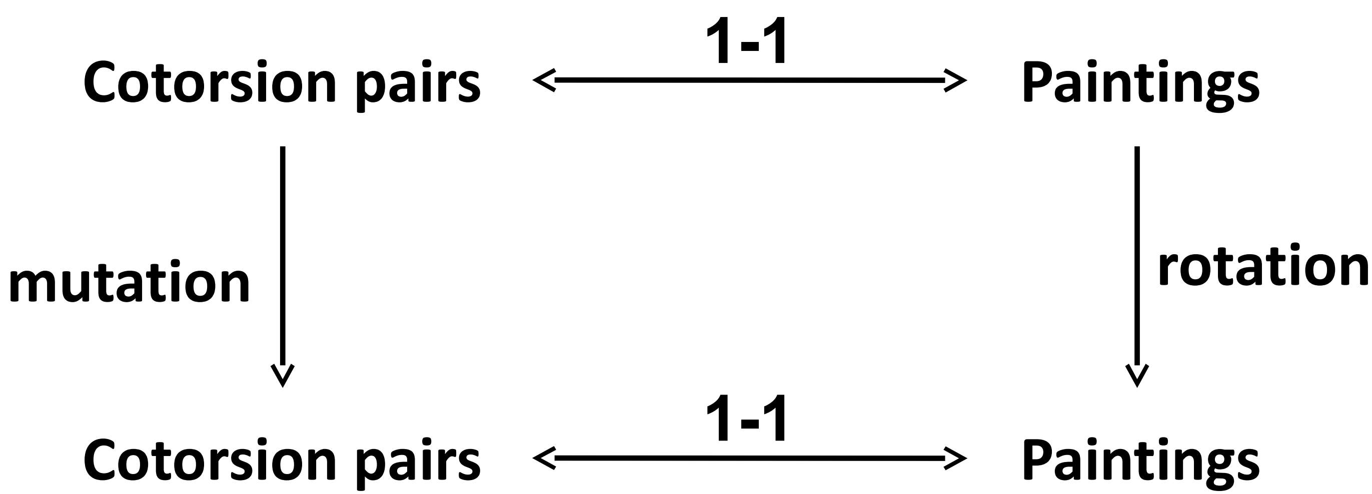
$\mathcal{D}$**−rotation**



2. Mutations of cotorsion pairs and their geometric realizations

- **Mutations and Rotations**

Theorem D



Part II

Thanks!