Monomorphism Category Associated to Symmetric Group and Parity in Finite Groups

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PARITY

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What is parity?

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- Where there is a semigroup or a ring, there might be a parity over it. Since parity is essentially given by one or two operations of the 2-element field $GF(2)$:

$+$	0	1
0	0	1
1	1	0

$\times$	0	1
0	0	0
1	0	1

PARITY OVER $\mathbb{Z}$

- The first example: $\mathbb{Z}$.

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- The two homomorphisms ι and π determine each other, either makes the parity over $\mathbb{Z}$. Indeed, the injection ι assigns all even elements (hence also assigns all "odd" elements), while the canonical surjection π determines the operation table(s) of the parity.

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PARITY OVER OTHER SYSTEMS

- Another example is the so-called even and odd functions defined over, say, $\mathbb{R}$. The parity over the set F of all even and odd functions is a monoid under "multiplication". But this parity only makes sense when we misunderstand this multiplication as addition "$+$": since

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- The parity over symmetric group S_n on n letters is entirely at the same situation as above and is determined completely by the following short exact sequence of groups

$$1 \longrightarrow A_n \xrightarrow{\iota} S_n \xrightarrow{\pi} S_2 \longrightarrow 1$$

PARITY OVER ALGEBRAIC SYSTEMS

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  Parity (over an algebraic system) is a short exact sequence with end term $\mathbb{Z}_2$ (as group or ring).

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- Equivalently:
  Parity is a monomorphism (=injective homomorphism here) of index 2.

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PARITY OVER ALGEBRAIC SYSTEMS

- BUT: Parity is clearly well-defined over $3\mathbb{Z}, 5\mathbb{Z}, \cdots$, so the natural index-3 monomorphism $3\mathbb{Z} \longrightarrow \mathbb{Z}$ also deserves a name of "even" or "odd". So, the reasonable question should be:

  Can we call a monomorphism even or odd?

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$$A \xrightarrow{\phi} B$$

  with ϕ a monomorphism. Such system is called to be a **monomorphism category**.

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- Every object in mathematics is some kind of monomorphism.

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Background: Counting the subobjects

- Question 1 (G.Birkhoff 1934). How many pairs of subgroups of a finite abelian group? i.e. let G be a finite abelian group of order n, $S_2(n)$- the set of pairs (A, B), where $A \leq B \leq G$("$\leq$" means "subgroup"), then

$$|S_2(n)| := \sharp\{\text{indecomposable isoclasses of } S_2(n)\} = ?$$

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- Since group homomorphism preserves p-subgroup (p- a prime), Question 1 can be reduced to abelian p-groups, thus may assume $|G| = p^n$ and

$$G = \sum_{i=1}^{t} \oplus \mathbb{Z}/p^{n_i}, n_1 + n_2 + \cdots + n_t = n$$

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- 3. 1999, Richman-Walker, $|S_2(5)| = 50$.

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- 4. But $|S_2(6)|$ depends on p and G, say, for $G = \mathbb{Z}/p^4 \oplus \mathbb{Z}/p^2$, $|S_2(6)| = p + 7$.

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Background: Counting the subobjects(III)

- 1989, 2000, D.M.Arnold[Ar]
  Filtered chain category $S_n(R)$ over commutative Artin uniserial ring R
  - objects in $S_n(R)$: $C = (C_1 \leq C_2 \leq \cdots \leq C_n)$ with all C_i's modules over R.
  - morphisms in $S_n(R)$: $f : C \to C', f(C_i) \leq C_i', 1 \leq i \leq n$.

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- 2002, D.Simon[S], $R = k[x]/x^m$.
  1. $S_n(R)$ representation-finite $\iff$ $m \leq 2$ or $n = 1$ or $m = 3 \wedge (3 \leq n \leq 4)$ or $m = 4, 5 \wedge n = 2$;
  2. $S_n(R)$ representation-tame $\iff$ $m = 6 \wedge n = 2$ or $m = 4 \wedge (3 \leq n \leq 4)$ or $m = 3 \wedge n = 5$;
  3. $S_n(R)$ representation-wild $\iff$ $m \geq 7 \wedge n \geq 2$ or $m \geq 5 \wedge n \geq 3$ or $m \geq 4 \wedge n \geq 5$ or $m \geq 3 \wedge n \geq 6$.

Morphism Category of A Category

Let A be a category (say: A=category of all finite groups)

- $\mathrm{Mor}(A)$-morphism category of A
  - Object of $\mathrm{Mor}(A)$ is $(X,\phi) = X_1 \xrightarrow{\phi} X_2$ with A-morphism $\phi : X_1 \to X_2$;
  - A morphism $f : (X,\phi) \to (Y,\psi)$ of $\mathrm{Mor}(A)$ is $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ with A-morphism $f_i : X_i \to Y_i$ such that $\psi f_1 = f_2 \phi$

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\phi} & X_2 \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\
Y_1 & \xrightarrow{\psi} & Y_2
\end{array}
$$

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Monomorphism Category (= subobjects category) $S(A)$

- Let $S(A)$ (**the monomorphism category**)- the full subcategory of $\mathrm{Mor}(A)$ consisting of all the objects (X, ϕ) where $\phi : X_1 \hookrightarrow X_2$ is a monomorphism.
- Can define dually $F(A)$ (**the epimorphism category**=(factor-objects category))-we do not use it in this talk.

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Subgroups of Symmetric Groups

- $\mathcal{S}$ = the monomorphism category of all subgroups of symmetric groups:

(1) Object= $(G, \phi, n) = G \xrightarrow{\phi} S_n$ with ϕ an injective group homomorphism.
(2) Morphism=commutative square of group homomorphisms.

AIM: Find "parity" in this monomorphism category and apply.

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Semigroup Structure over All Symmetric Groups

- Let $\mathcal{G}$ be the category of all symmetric groups (with group homomorphism as morphism). Let S_m be the symmetric group on letters $X^m = \{x_1^m, \cdots, x_m^m\}$. For $\sigma = (x_1 x_2 \cdots x_r) \in S_m, \tau = (y_1 y_2 \cdots y_t) \in S_n$. Define an operation $*$ of σ and τ by

$$\sigma * \tau = \sigma\tau$$

which is an element of the symmetric group S_{m+n} on $m + n$ letters $X^m \cup X^n = \{x_1^m, \cdots, x_m^m, x_1^n, \cdots, x_n^n\}$. The above operation can be extended over all permutations and thus defines a commutative semigroup structure over $\mathcal{G}$:

$$S_m * S_n = S_{m+n} = S_n * S_m$$

with no identity element.

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Semigroup Structure over Monomorphism Category

- Let $(X, \phi, m), (Y, \psi, n) \in \mathcal{S}$. Define
$$(X, \phi, m) * (Y, \psi, n) = (X \times Y, \phi * \psi, mn)$$
where $\phi * \psi(x, y) = \phi(x) * \psi(y)$.
- This equips $\mathcal{S}$ with a commutative semigroup structure.

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Equivalence over Monomorphism Category

- Let $\mathcal{I}$ be the full subcategory consisting of such objects (X, ϕ, n) of $\mathcal{S}$ such that $\phi(X) \subseteq A_n$, the alternating group on n letters.
- Define a relation $\sim$ over $\mathcal{S}$ by $\mathcal{I}$:

$$(G, \phi, m) \sim (H, \psi, n) \iff (G, \phi, m) * A = (H, \psi, n) * B$$

for some $A, B \in \mathcal{I}$.
- Fact: $\sim$ is an equivalence relation.

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Quotient Category of Monomorphism Category

- Theorem
  - The quotient category $\mathcal{Q} := \mathcal{S}/\sim$ is equivalent to a category with two objects

$$\mathcal{Q} \simeq \{(S_1, Id, 1), (S_2, Id, 2)\}.$$

  - $(\mathcal{Q}, *)$ is a monoid (=semigroup with identity) under the induced operation * with operation table

*	a	b
a	a	a
b	a	b

where $a := (S_1, Id, 1)$ and $b := (S_2, Id, 2)$, so a serves as the zero element and b the identity element.

Proof of the Theorem

- Let $(G, \phi, n) \in \mathcal{S}$. $A(G) = \phi^{-1}(\phi(G) \cap A_n) \lhd G$. Claim $(G, \phi, n) \simeq (S_1, Id, 1)$ or $(G, \phi, n) \simeq (S_2, Id, 2)$. If $A(G) = G$, then it is clear $(G, \phi, n) \simeq (S_1, Id, 1)$. Else, the following commutative diagrams of natural homomorphisms

$$\begin{array}{ccc} G & \xrightarrow{\quad\phi\quad} & S_n \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ G/A(G) & \xrightarrow{Id} & S_n/A_n \end{array}$$

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Proof of the Theorem (continued)

- $$\begin{array}{ccc} A(G) & \xrightarrow{\iota} & A_n \\ {\scriptstyle i_G}\downarrow & & \downarrow{\scriptstyle i} \\ G & \xrightarrow{\phi} & S_n \\ {\scriptstyle \pi_G}\downarrow & & \downarrow{\scriptstyle \pi} \\ G/A(G) & \xrightarrow{Id} & S_n/A_n \end{array}$$

  give the desired isomorphism between (G, ϕ, n) and $(S_2, Id, 2)$.
- Corollary. There is a parity over $\mathcal{S}$, where all (G, ϕ, n) with $\phi(G) \subseteq A_n$ are even elements, the other odd ones.

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Applications: Parity over Finite Groups

- Application 1. Let $|G|$ be odd. Then all injection $G \longrightarrow S_n$ is even. In particular, G can be embedded into $A_{|G|}$ (saving 50% space, comparing with Cayley's Embedding Theorem ☺)
- Application 2. Let G be commutative and $|G|$ be even. Then there is at least one injection $G \longrightarrow S_n$ is not even. That is, there is a parity over G.
- Remark. Application 1 and 2 can be deduced from Lagrange's Theorem.

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Parity over Noncommutative Groups of Even Orders

- Application 3. Let G be noncommutative and $|G|$ be even. Then the parity is independent of $|G|$.
- Example 1. All injections $A_n \hookrightarrow S_n$ are even so there is no parity over any alternating group.(When n is large, this can be deduced from the simplicity of A_n.)
- Example 2. Let $D_n =< (12\cdots n), \prod_{2\leq j<n+2-j}(j \quad n+2-j) >$ be the dihedral group ($|D_n| = 2n$). Then there is a parity over $D_n \iff n \neq 3(mod 4)$.

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