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A parameterization of the canonical basis of affine modified quantized enveloping algebras

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- Quantized enveloping algebra

  ⋆ $\mathbf{A} = (A, I, I^\vee, P, P^\vee)$: a Cartan datum associated with a generalized Cartan matrix A;

  ⋆ U: the quantized enveloping algebra associated to the above Cartan datum generated by E_i, F_i and K_μ, $i \in I$ and $\mu \in P^\vee$ over $\mathbb{Q}(v)$;

  ⋆ $\mathbf{f}$: the Lusztig's algebra associated to the above Cartan datum generated by θ_i, $i \in I$;

  ⋆ $^+ : \mathbf{f} \to U$ $(x \mapsto x^+)$: the algebra homomorphism with image U^+;

  ⋆ $^- : \mathbf{f} \to U$ $(x \mapsto x^-)$: the algebra homomorphism with image U^-;

  ⋆ B: the canonical basis of $\mathbf{f}$ introduced by Lusztig.

- Modified quantized enveloping algebra

  - $\star$ $\dot{U}$: the modified quantized enveloping algebra;
  - $\star$ $\dot{U} = \oplus_{\lambda \in P} \dot{U} 1_\lambda$, where $\dot{U} 1_\lambda = U / \sum_{\mu \in P^\vee} U(K_\mu - v^{\langle \lambda, \mu \rangle})$;
  - $\star$ $\{b^+ b'^- 1_\lambda | b, b' \in B\}$ is a basis of the $\mathbb{Q}(v)$-space $\dot{U} 1_\lambda$;
  - $\star$ $\dot{B} = \{b \diamondsuit_\lambda b' | b, b' \in B, \lambda \in P\}$: the canonical basis of $\dot{U}$;
  - $\star$ $b \diamondsuit_\lambda b' = b^+ b'^- 1_\lambda +$ lower terms.

- Aim

  ⋆ Give a parameterization of the canonical basis of affine modified quantized enveloping algebras using root categories.

- Quivers and path algebra

  ⋆ $Q = (I, H, s, t)$: a quiver;

    ⋆ I: the set of vertices;

    ⋆ H: the set of arrows;

    ⋆ $s, t : H \to I$: $s(i \to j) = i, t(i \to j) = j$;

- Ringel-Hall algebras

  ⋆ $\mathbb{F}_q$: a finite field with q elements;

  ⋆ $\Lambda = \mathbb{F}_q(Q)$: the path algebra of Q over $\mathbb{F}_q$;

  ⋆ L, M, N: three modules in mod-Λ;

  ⋆ g_{MN}^{L}: the number of Λ-submodules W of L such that $W \simeq N$ and $L/W \simeq M$ in mod-Λ;

- Ringel-Hall algebras

  ⋆ $\mathcal{H}_q^*(\Lambda)$: the Ringel-Hall algebra of Λ over $\mathbb{Q}(v_q)$ ($v_q = \sqrt{q}$);

    ⋆ Basis: $\{u_{[M]} | [M] \in \mathcal{P}\}$ ($\mathcal{P}$: the set of isomorphism classes of finite dimension Λ-modules);

    ⋆ Product:

$$u_{[M]} * u_{[N]} = v_q^{\langle \underline{\dim} M, \underline{\dim} N \rangle} \sum_{[L] \in \mathcal{P}} g_{MN}^L u_{[L]}.$$

  ⋆ $\mathcal{C}_q^*(\Lambda) := \langle u_{[S_i]} \rangle$: The composition algebra;

- Ringel-Hall algebras

  ⋆ $\mathcal{C}^*(Q)$: the generic composition algebra of quiver Q over $\mathbb{Q}(v)$;

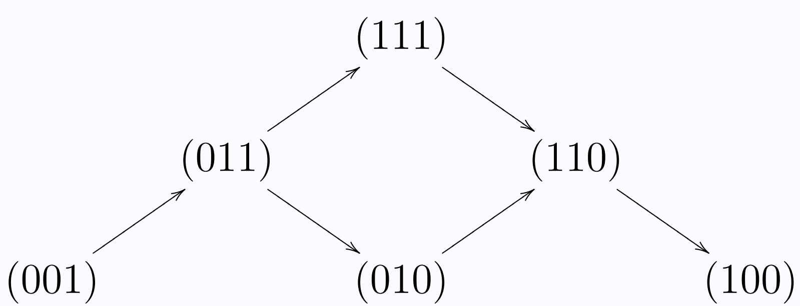
  ⋆ **Theorem 1 (Green-Ringel)** *Let Q be a connected quiver without oriented cycles. Then the correspondence $u_i \mapsto \theta_i$ induces an algebra isomorphism $\mathcal{C}^*(Q) \to$* $\mathbf{f}$.

- PBW-type basis of f: Finite type

  ⋆ $\mathcal{H}^*(Q) = \mathcal{C}^*(Q)$;

  ⋆ Example: Auslander − Reiten quiver of $Q = A_3$;

  *



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- PBW-type basis of $\mathbf{f}$: Finite type

  - ⋆ **Theorem 2 (Lusztig, Ringel)** *(1) The set* $\{\langle M\rangle | [M]\in\mathcal{P}\}$ *is a* $\mathcal{A}$*-basis of* $\mathcal{C}^*_{\mathcal{A}}(Q)$*;*

    (2) There exists a canonical basis

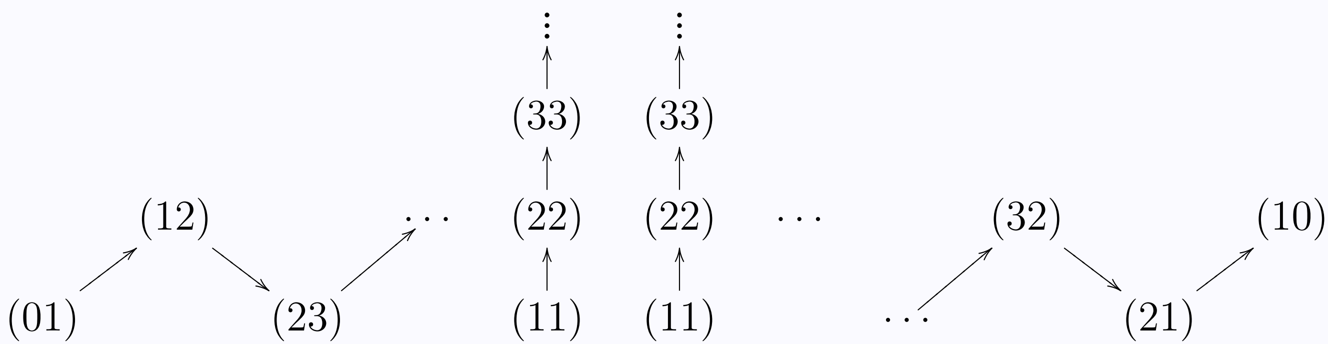
    $$\mathcal{E}_{[M]} = \langle M\rangle + \sum_{[M']\prec[M]} u_{[M'][M]}\langle M'\rangle,$$

    with $u_{[M'][M]}\in v^{-1}\mathbb{Q}[v^{-1}]$*;*

    (3) $\{\mathcal{E}_{[M]} | [M]\in\mathcal{P}\} = B.$

- PBW-type basis of **f**: Affine type

AR-quiver of $Q = \{1 \rightrightarrows 2\}$



- PBW-type basis of $\mathbf{f}$: Affine type

  ⋆ $Prep$ (resp. $Prei$): the iso-classes of indecomposable preprojective (resp. preinjective) modules;

  ⋆ $\mathcal{T}_i(i = 1, \dots, s)$: the non-homogeneous tubes with the period r_i;

  ⋆ $\mathbb{N}_f^{Prep} := \{\mathbf{a} : Prep \to \mathbb{N} \,|\, \mathbf{a} \text{ is support-finite}\}$;

  ⋆ $\mathbb{N}_f^{Prei} := \{\mathbf{b} : Prei \to \mathbb{N} \,|\, \mathbf{b} \text{ is support-finite}\}$;

  ⋆ Π_i^a: the set of aperiodic r_i-tuples of partitions;

  ⋆ $\mathcal{M} := \{\mathbf{c} = (\mathbf{a_c}, \mathbf{b_c}, \pi_\mathbf{c}, w_\mathbf{c}) \,|\, \mathbf{a_c} \in \mathbb{N}_f^{Prep}, \mathbf{b_c} \in \mathbb{N}_f^{Prei}, \pi_\mathbf{c} = (\pi_{1\mathbf{c}}, \pi_{2\mathbf{c}}, \dots, \pi_{s\mathbf{c}}) \in \Pi_1^a \times \Pi_2^a \times \cdots \times \Pi_s^a, w_\mathbf{c} = (w_1 \geq w_2 \geq \cdots \geq w_t) \text{ is a partition}\}$.

- PBW-type basis of $\mathbf{f}$: Affine type

⋆ For each $\mathbf{c} \in \mathcal{M}$ there is an element in $\mathcal{C}^*(Q)$:

$$e^{\mathbf{c}} := \langle M(\mathbf{a_c})\rangle * E_{\pi_{1\mathbf{c}}} * E_{\pi_{2\mathbf{c}}} * \cdots * E_{\pi_{s\mathbf{c}}} * S_{w_{\mathbf{c}}\delta} * \langle M(\mathbf{b_c})\rangle.$$

⋆ **Theorem 3 (Lin-Xiao-Zhang)** *(1) The set* $\{e^{\mathbf{c}} | \mathbf{c} \in \mathcal{M}\}$ *is a* $\mathbb{Q}(v)$*-basis of* $\mathcal{C}^*(Q)$*;*

(2) There exists a canonical basis

$$\mathcal{E}^{\mathbf{c}} = e^{\mathbf{c}} + \sum_{\mathbf{c}' \prec \mathbf{c}} u_{\mathbf{c}'\mathbf{c}} e^{\mathbf{c}'},$$

with $u_{\mathbf{c}'\mathbf{c}} \in v^{-1}\mathbb{Q}[v^{-1}]$*;*

(3) $\{\mathcal{E}^{\mathbf{c}} | \mathbf{c} \in \mathcal{M}\} = B$*.*

- Root category

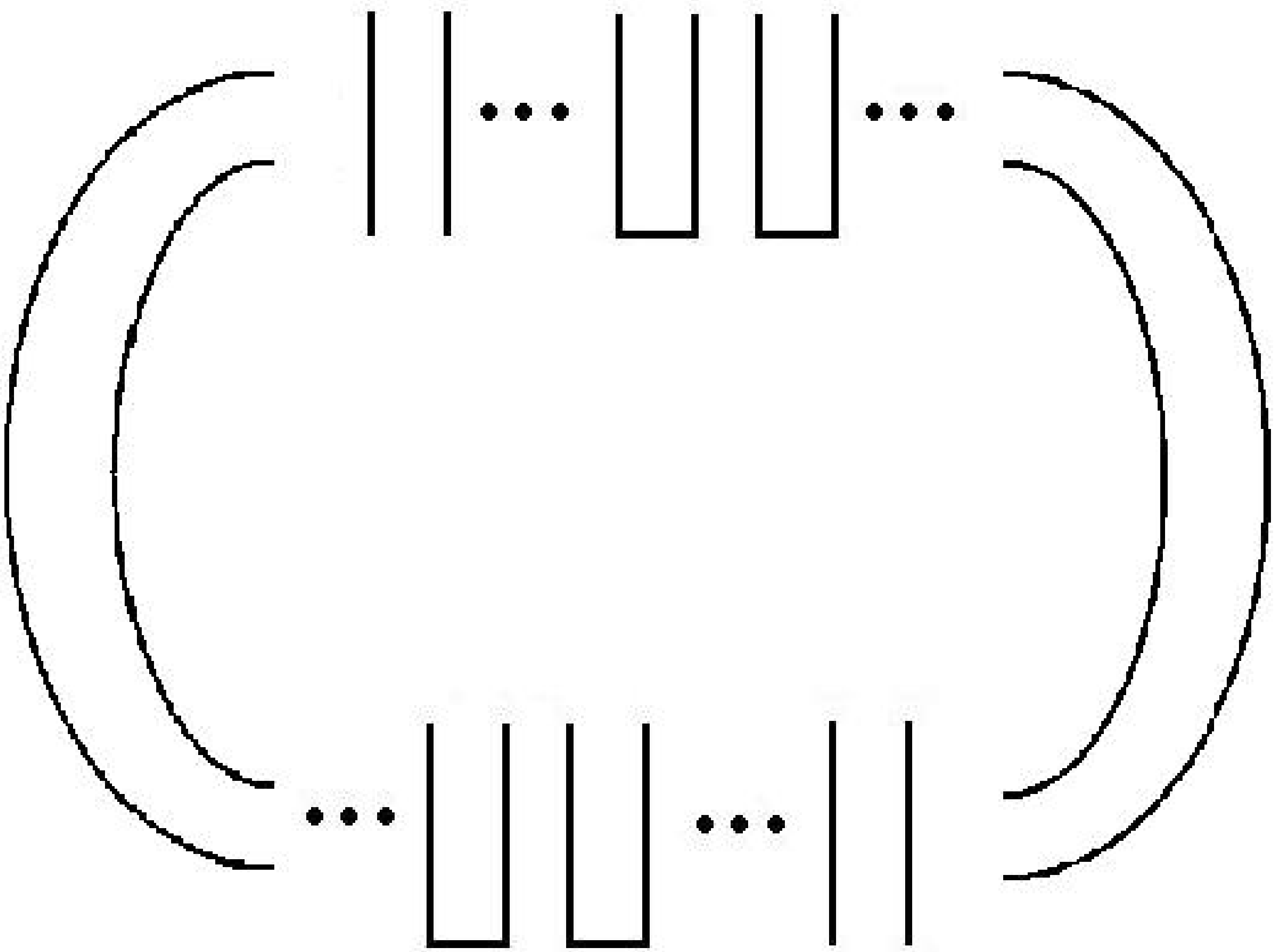
  - $\star$ Q: a connected tame quiver without oriented cycles;

  - $\star$ $A = kQ$: the path algebra of Q;

  - $\star$ $\mathcal{R}(Q) = D^b(A)/T^2$: the root category;

  - $\star$ mod-A can be embedded into $\mathcal{R}(Q)$ as a full subcategory and $\mathcal{R}(Q)$ is divided into two parts: mod-A and T(mod-A).

AR-quiver of the root category



- Root category

  ⋆ There is a bijection between the isomorphism classes of the indecomposable objects in root categories $\mathcal{R} = D^b(A)/T^2$ and the roots of the corresponding Lie algebra (Happle);

  ⋆ The root categories of finite dimensional hereditary algebras give a realization of all symmetrizable Kac-Moody algebra in a global way (Peng-Xiao).

- PBW-type basis of $\dot{U}1_\lambda$

  ⋆ $\mathcal{T}_i (i = 1, \dots, 2s)$: all the non-homogeneous tubes in $\mathcal{R}(Q)$;

  ⋆ $\mathcal{P}$: the indecomposable iso-classes in $\mathcal{R}(Q)$ excluding those in homogeneous tubes and non-homogeneous tubes;

  ⋆ $\tilde{\mathcal{M}}$: the set of quadruplets $\tilde{\mathbf{c}} = (\mathbf{d}_{\tilde{\mathbf{c}}}, \pi_{\tilde{\mathbf{c}}}, w_{\tilde{\mathbf{c}}}, w'_{\tilde{\mathbf{c}}})$,
  where $\mathbf{d}_{\tilde{\mathbf{c}}} \in \mathbb{N}_f^{\mathcal{P}}$,
  $\pi_{\tilde{\mathbf{c}}} = (\pi_{1\tilde{\mathbf{c}}}, \pi_{2\tilde{\mathbf{c}}}, \dots, \pi_{2s\tilde{\mathbf{c}}}) \in \Pi_1^a \times \Pi_2^a \times \cdots \times \Pi_{2s}^a$,
  $w_{\tilde{\mathbf{c}}} = (w_1 \geq w_2 \geq \cdots \geq w_t)$ and $w'_{\tilde{\mathbf{c}}} = (w'_1 \geq w'_2 \geq \cdots \geq w'_t)$ are partitions.

  ⋆ If Q' is another quiver such that $\mathcal{R}(Q) \simeq \mathcal{R}(Q')$, then they give the same $\tilde{\mathcal{M}}$.

- PBW-type basis of $\dot{U}1_\lambda$

  - ⋆ Consider the root category $\mathcal{R}$ corresponding to the quantized enveloping algebra U.

  - ⋆ Fix Q such that $\mathcal{R}(Q) \simeq \mathcal{R}$.

  - ⋆ There is a bijection $\psi_Q : \tilde{\mathcal{M}} \to \mathcal{M} \times \mathcal{M}$.

  - ⋆ $\tilde{e}^{\tilde{\mathbf{c}}} 1_\lambda := e^{\mathbf{c}_1 +} \cdot e^{\mathbf{c}_2 -} 1_\lambda$, where $(\mathbf{c}_1, \mathbf{c}_2) = \psi_Q(\tilde{\mathbf{c}}), \tilde{\mathbf{c}} \in \tilde{\mathcal{M}}$.

  - ⋆ $PBW_Q(\dot{U}1_\lambda) := \{\tilde{e}^{\tilde{\mathbf{c}}} 1_\lambda | \tilde{\mathbf{c}} \in \tilde{\mathcal{M}}\}$ is a PBW-type basis of $\dot{U}1_\lambda$.

- A bar invariant basis of $\dot{U}1_\lambda$

  ⋆ $\prec$: an order on $\tilde{\mathcal{M}} = \mathcal{M} \times \mathcal{M}$ by $\tilde{\mathbf{c}} \prec \tilde{\mathbf{c}}' \Leftrightarrow \mathbf{c}_1 \preceq \mathbf{c}_1'$ and $\mathbf{c}_2 \preceq \mathbf{c}_2'$ but $\tilde{\mathbf{c}} \neq \tilde{\mathbf{c}}'$.

  ⋆ A bar invariant basis $\mathcal{B}_Q(\dot{U}1_\lambda)$ of $\dot{U}1_\lambda$ from $PBW_Q(\dot{U}1_\lambda)$:

$$\tilde{\mathcal{E}}^{\tilde{\mathbf{c}}}1_\lambda = \tilde{e}^{\tilde{\mathbf{c}}}1_\lambda + \sum_{\tilde{\mathbf{c}}' \prec \tilde{\mathbf{c}}} \tilde{u}_{\tilde{\mathbf{c}}'\tilde{\mathbf{c}}} \tilde{e}^{\tilde{\mathbf{c}}'}1_\lambda,$$

  with $\tilde{u}_{\tilde{\mathbf{c}}'\tilde{\mathbf{c}}} \in v^{-1}\mathbb{Q}[v^{-1}]$.

  ⋆ **Theorem 4** $\mathcal{B}_Q(\dot{U}1_\lambda) = \{\tilde{\mathcal{E}}^{\tilde{\mathbf{c}}}1_\lambda | \tilde{\mathbf{c}} \in \tilde{\mathcal{M}}\} = \{b^+ b'^- 1_\lambda | b, b' \in B\}$.

- A parameterization of the canonical basis of $\dot{U}1_\lambda$

  ⋆ **Theorem 5** *There is a bijection*

$$\Psi_Q : \tilde{\mathcal{M}} \to \dot{B}_\lambda$$

given by

$$\tilde{\mathbf{c}} \mapsto \tilde{e}^{\tilde{\mathbf{c}}}1_\lambda \mapsto \tilde{\mathcal{E}}^{\tilde{\mathbf{c}}}1_\lambda = \mathcal{E}^{\mathbf{c}_1+}\mathcal{E}^{\mathbf{c}_2-}1_\lambda \mapsto \mathcal{E}^{\mathbf{c}_1}\diamondsuit_\lambda\mathcal{E}^{\mathbf{c}_2},$$

where $\dot{B}_\lambda$ *is the canonical basis of* $\dot{U}1_\lambda$ *and* $(\mathbf{c}_1, \mathbf{c}_2) = \psi_Q(\tilde{\mathbf{c}})$.

Thank You !