$t-$structures in $2-$Calabi-Yau triangulated categories with cluster tilting objects

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Aims:

- To give a classification of cotorsion pairs in a $2-$CY triangulated category with a cluster tilting object.

Outline

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- $d-$**cluster tilting subcategories**

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(ii). For $X \in \mathcal{C}$,
$X \in \mathcal{T}$ **if and only if** $Ext^i_{\mathcal{C}}(X, \mathcal{T}) = 0, \forall 1 \leq i \leq d-1;$

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$\mathcal{C}$ **is called** $d-$**Calabi-Yau if there are functorial isomorphisms**
$Hom_{\mathcal{C}}(X,Y) \simeq DHom_{\mathcal{C}}(Y,X[d])$ **for all** $X,Y \in \mathcal{C}$. **Where**
$D = Hom_k(-,k)$

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2. For any $M \in \mathcal{C}$, there is a triangle $X \to M \to Y[1] \to X[1]$, with $X \in \mathcal{X}, Y \in \mathcal{Y}$. This means that $\mathcal{C} = \mathcal{X} * \mathcal{Y}[1]$

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For a cotorsion pairs $(\mathcal{X}, \mathcal{Y})$,
$\mathcal{X} = {}^{\perp}(\mathcal{Y}[1]) := \{M \in \mathcal{C} | Hom(M, \mathcal{Y}[1]) = 0\}$.

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A pair $(\mathcal{X}, \mathcal{Y})$ of subcategories in $\mathcal{C}$ is called a t–structure if $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair and $\mathcal{X}[1] \subseteq \mathcal{X}$, $\mathcal{Y}[-1] \subseteq \mathcal{Y}$.

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A pair $(\mathcal{X}, \mathcal{Y})$ of subcategories in $\mathcal{C}$ is called a co-t–structure if $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair and $\mathcal{X}[-1] \subseteq \mathcal{X}$, $\mathcal{Y}[1] \subseteq \mathcal{Y}$.

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1. $(\mathcal{X}, \mathcal{Y})$ is a t–structure iff $I = 0$;
2. $\mathcal{X}$ is a cluster tilting subcategory iff $I = \mathcal{X} = \mathcal{Y}$

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4. Mutation of cotorsion pairs in triangulated categories $\mathcal{C}$ is defined and the geometric interpretation is given by Zhang-Zhou-Zhu when $\mathcal{C}$ is $\mathcal{C}_{A_n}$, $\mathcal{C}_{A_\infty}$, $\mathcal{C}_{(S,M)}$ (Yu Zhou's talk).

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Denote by $\mathcal{C} = \oplus_{i \in A} \mathcal{C}_i$.

- $\mathcal{C}$ is called indecomposable (or connected) if $\mathcal{C}$ can not be decomposed as a direct sum of two non-zero triangulated subcategories.

- **Definition.** Let $\mathcal{T}$ be a d−cluster tilting subcategory in a triangulated category $\mathcal{C}$, $\mathcal{T}_i, i \in A$ subcategories of $\mathcal{T}$. Then $\mathcal{T}$ is called a direct sum of $\mathcal{T}_i$ provided that:

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Denote by $\mathcal{T} = \oplus_{i \in A} \mathcal{T}_i$.

- **Remark:** When $d = 2$, the condition 3 above is empty.

Decomposition theorem

- **Theorem: Let $\mathcal{C}$ be a $d-$CY triangulated category and $\mathcal{T}$ be a $d-$cluster tilting subcatgeory. Then $\mathcal{C} = \oplus_{i \in A} \mathcal{C}_i$, where $\mathcal{C}_i, i \in A$ are triangulated subcategories of $\mathcal{C}$, if and only if $\mathcal{T} = \oplus_{i \in A} \mathcal{T}_i$.**

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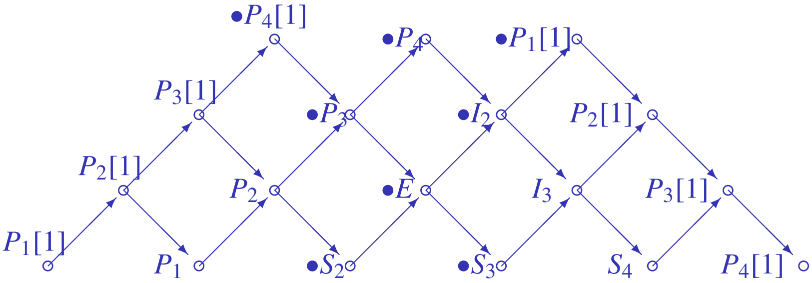
Moreover in this case, $\mathcal{C}_i = \mathcal{T}_i * \mathcal{T}[1] * \cdots \mathcal{T}_i[d-1]$, and $\mathcal{T}_i$ is d–cluster tilting in $\mathcal{C}_i$.

- **Example: Let $Q : 4 \to 3 \to 2 \to 1$, $C = C_Q$, the cluster catgeory of Q:**



We take $\mathcal{X} = add(E)$**,**
$^{\perp}(\mathcal{X}[1]) = add(\{E, P_3, P_4[1], P_4, I_2, P_1[1], S_2, S_3\}$**.**

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By Iyama-Yoshino, the subquotient category $^{\perp}(\mathcal{X}[1])/\mathcal{X} = \{P_3, P_4[1], P_4, I_2, P_1[1], S_2, S_3\}$ **is triangulated, and** $2-$**CY**

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This subqeotient category admits cluster tilting objects, for example the object $T = P_4[1] \oplus P_3 \oplus E \oplus S_3$**. We have that in this subquotient category,** $addT = add(S_3) \oplus add(P_3 \oplus P_4[1])$**.**

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Then this subquotient category $^{\perp}(\mathcal{X}[1])/\mathcal{X} = add(\{S_2, S_3\}) \oplus add(\{P_3, P_4[1], P_4, I_2, P_1[1]\})$. The first direct summand is the cluster category of the quiver A_1, the second direct summand is equivalent to C_{A_2}.

- **Corollary: Let C be a $2-$CY triangulated category, T and T' be cluster tilting objects in C. Then the quiver Q_T of $\mathrm{End}_C T$ is connected if and only if $Q_{T'}$ is connected.**

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- **Example:** $\mathcal{C} = D^b(kQ)$**, where** $Q : 3 \to 2 \to 1$**.**

$\mathcal{T} = add(\tau^{-n}kQ[n]; n \in \mathbf{Z})$ **is a cluster tilting subcategory in** $\mathcal{C}$**. Let** $\mathcal{T}_i = add(\tau^{-i}KQ[i]), i \in \mathbf{Z}$**. Then** $\mathcal{T} = \oplus_{i\in \mathbf{Z}}\mathcal{T}_i$**. But** $\mathcal{C}$ **is indecomposable.**

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For cluster category of type A_n, this result was proved by Holm-Jörgensen-Rubey;

For (generalized) cluster category $\mathcal{C}_{(S,M)}$, associated to a marked surface without punctures, this result was proved by Zhang-Zhou-Zhu.

- **Remark: The theorem is not true for $2-$CY category with cluster tilting subcategory containing infinitely many indecomposables.**
For example, the cluster category of type A_∞, introduced by Holm-Jörgensen contains non-trivial $t-$structutes by Ng's work.

Classification of cotorsion pairs

- **Theorem: Let $\mathcal{C}$ be a $2-$CY category with a cluster tilting object T, I a rigid object, and ${}^{\perp}(I[1])/I = \oplus_{j\in J} I_j$ be the complete decomposition of the triangulated category ${}^{\perp}(I[1])/I$.**

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Then all cotorsion pairs with core I are obtained as preimages under the functor $\pi : {}^{\perp}(I[1]) \to {}^{\perp}(I[1])/I$ of the pairs $(\oplus_{j\in L} I_j, \oplus_{j\in J-L} I_j)$, where L is a subset of J, and $J-L$ is the complement of L in J.

Moreover there are 2^n cortorsion pairs with core I in $\mathcal{C}$, where $n = |J|$.

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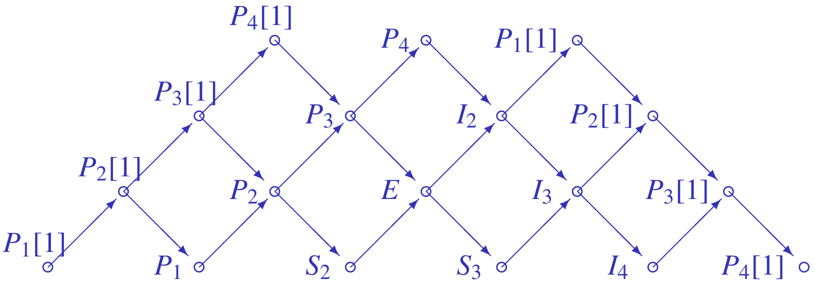
1. If $(\mathcal{X}, \mathcal{Y})$ **is a cotorsion pair, then** $(\mathcal{Y}, \mathcal{X})$ **is also a cotorsion pair.**

2. The co-$t-$**structures in** $\mathcal{C}$ **are only trivial, i.e.** $(\mathcal{C}, 0)$, $(0, \mathcal{C})$.

- **Example:** $Q : 4 \to 3 \to 2 \to 1$, $\mathcal{C} = \mathcal{C}_Q$ **the cluster category of** Q **as before.**



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This subquotient category has the complete decomposition:
$^{\perp}(I[1])/I = add(\{P_1, S_2\}) \oplus add(\{P_3, P_4 S_4, P_3[1], P_4[1]\})$.

There are 4 cortorsion pairs with core I:

$(\{P_2\},\{P_1,P_2,P_3,P_4,S_2,S_4,P_3[1],P_4[1]\})$;
$(\{P_2,P_1,S_2\},\{P_2,P_3,P_4,S_2,S_4,P_3[1],P_4[1]\})$;
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Thank You!