Singular equivalences of Morita type

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Origin

The singular category was defined by

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- Buchweitz (1987) (unpublished) as stable derived category
- Orlov (2009) rediscovered independently in the context of mirror symmetry and mathematical physics

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- let $D^b(A)$ be the derived category of bounded complexes of A-modules.
- Let $D^b(A - proj)$ be the full subcategory generated by bounded complexes of projective A-modules.

Then let

$$D_{sg}(A) := D^b(A)/D^b(A - proj)$$

be the Verdier quotient, (which is automatically a triangulated category).

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- and a morphism from X to Y is an "equivalence class" of triples
$$X \stackrel{\nu}{\leftarrow} Z \stackrel{\alpha}{\rightarrow} Y$$
where **the cone of ν is supposed to be in** $D^b(A - proj)$.

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Two triples

$$X \overset{\nu_1}{\leftarrow} Z_1 \overset{\alpha_1}{\rightarrow} Y$$

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in order to get rid of the artificially included Z.

More properly use limits.

modules with finite projective dimension

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modules with finite projective dimension

Modules M with finite projective dimension are isomorphic to 0. Indeed, M is isomorphic to its projective resolution, and then the 0 endomorphism has cone in $D^b(A-proj)$, hence is invertible.

Conversely an object of $D_{sg}(A)$ which becomes 0 in $D_{sg}(A)$ has to be isomorphic to one in $D^b(A-proj)$. So precisely modules with finite projective dimension become 0 in $D_{sg}(A)$.

$D^b(A) \to D_{sg}(A)$

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given by the identity on the objects and mapping a morphism $\alpha \in Hom_{D^b(A)}(X, Y)$ to

$$X \stackrel{id}{\leftarrow} X \stackrel{\alpha}{\to} Y$$

Link to the stable category

The inclusion

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so that the diagram

$$\begin{array}{ccc} A - mod & \stackrel{N}{\to} & D^b(A) \\ \downarrow G & & \downarrow F \\ A - \underline{mod} & \stackrel{M}{\to} & D_{sg}(A) \end{array}$$

is commutative where F and G are the natural quotient functors.

Link to the stable category

Moreover

$$M \circ \Omega \simeq [-1] \circ M$$

where Ω is the syzygy functor on $A - \underline{mod}$ and $[-1]$ is shift to the right on $D_{sg}(A)$.

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- If A is **selfinjective,** then Rickard and independently Keller and Vossieck proved in 1987 that

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An example due to Xiao-Wu Chen

Recall that

$$M(S) \simeq 0 \Leftrightarrow projdim(S) < \infty.$$

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Let $A := K[X, Y]/(X^2, Y^2, XY)$ and let S be the unique simple.

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for all $n \in \mathbb{N}$.

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This proves that $dim(End_{D_{sg}(A)}(S)) = \infty$ and Krull-Schmidt fails.
(Example is due to Xiao-Wu Chen who studied $D_{sg}(A)$ with $rad^2(A) = 0$ in great detail)

Motivation from stable equivalences

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- This induces a functor $M \otimes_B - : B - \underline{mod} \to A - \underline{mod}$ iff ${}_AM$ is projective.

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- This induces a functor $M \otimes_B - : B - \underline{mod} \to A - \underline{mod}$ iff ${}_AM$ is projective.
- It is exact iff M_B is projective.

Motivation from stable equivalences

Definition

(Broué 1994) Suppose A and B are K-algebras. A couple of bimodules $({}_AM_B,\ {}_BN_A)$ induces a stable equivalence of Morita type if

- M and N are f.g. projective as A-module and as B-module
- $M \otimes_B N \simeq A \oplus P$ as $A-A$-bimodules, for a projective bimodule P.
- $N \otimes_A M \simeq B \oplus Q$ as $B-B$-bimodules, for a projective bimodule Q.

singular equivalences of Morita type

Xiao-Wu Chen and Longang Sun gave (2012) an analogous definition for singular categories.

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Definition

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- $M \otimes_B N \simeq A \oplus P$ as $A - A$-bimodules, for P having a finite projective resolution as $A - A$-bimodule.
- $N \otimes_A M \simeq B \oplus Q$ as $B - B$-bimodules, for Q having a finite projective resolution as $B - B$-bimodule.

singular equivalences of Morita type: immediate consequences

This implies that P and Q are projective when restricted to one side.

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Hence for every A-module X we get $P \otimes_A X$ is in $D^b(A - proj)$, whence 0 in $D_{sg}(A)$. Likewise for B and Q.

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Hence for every A-module X we get $P \otimes_A X$ is in $D^b(A - proj)$, whence 0 in $D_{sg}(A)$. Likewise for B and Q.

Therefore if (M, N) induce a singular equivalence of Morita type, then $M \otimes_B - : D_{sg}(B) \longrightarrow D_{sg}(A)$ is an equivalence with inverse $N \otimes_A -$.

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Hochschild homology and stable equivalence

Chang-Chang Xi and Yuming Liu showed for Hochschild homology $HH_n(A)$ of degree n:

Theorem

(Chang-Chang Xi and Yuming Liu 2005) Let A and B be f.d. K-algebras and suppose that (M, N) induce a stable equivalence of Morita type. Then $HH_n(A) \simeq HH_n(B)$ for all $n \geq 1$.

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A new proof was given by Yuming Liu, Guodong Zhou and A.Z. (2012) using Bouc's trace map.

Hochschild homology and stable equivalence, remark on the proof

Given a bimodule ${}_AM_B$, Serge Bouc generalised (preprint 1997, unpublished) the usual Hattori-Stallings trace

$$tr_M : HH_0(B) = B/[B, B] \to A/[A, A] = HH_0(A)$$

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to a trace map

$$tr_M : HH_n(B) \to HH_n(A)$$

for all n. (quite technical definition, using cycles)

Hochschild homology and singular equivalence

For singular equivalences of Morita type we get

Theorem

(Guodong Zhou and A.Z. 2012) Let A and B be finite dimensional K-algebras and suppose that the pair of bimodules (M, N) induce a singular equivalence of Morita type.
Then Bouc's trace map tr_M induces an isomorphism $HH_n(A) \simeq HH_n(B)$ for all $n \geq 1$.

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If A and B are general noetherian rings then we still obtain that Hochschild homology coincides in high enough degrees.

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If A and B are general noetherian rings then we still obtain that Hochschild homology coincides in high enough degrees.
The proof in Liu, Zhou, Z. could be adapted to the singular situation.

Adjoint pairs and singular equivalence

Is it possible to deduce informations on the functors $M \otimes_B - : A - mod \longrightarrow B - mod$ knowing that (M, N) induces a singular equivalence of Morita type?

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We can modify and adapt their arguments to get

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Theorem

Let K be a field and let A and B be finite dimensional indecomposable K-algebras.

- *Suppose A and B are not of finite projective dimension as bimodules and*
- *suppose that $A/rad(A)$ and $B/rad(B)$ are separable over K.*

Suppose $({}_AM_B, {}_BN_A)$ induce a singular equivalence of Morita type and that M is indecomposable as bimodule.
If $Hom_A({}_AM_B, {}_AA_A)$ is projective as B-module, then

$${}_BN_A \simeq Hom_A({}_AM_B, {}_AA_A)$$

as $B-A$-bimodules, and $(M \otimes_B -, N \otimes_A -)$ is a pair of adjoint functors between the module categories $\mathrm{mod}(A)$ and $\mathrm{mod}(B)$.