CLASSIFICATION THEOREMS FOR CENTRAL SIMPLE ALGEBRAS WITH INVOLUTION

WITH AN APPENDIX BY R. PARIMALA

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ABSTRACT. The involutions in this paper are algebra anti-automorphisms of period two. Involutions on endomorphism algebras of finite-dimensional vector spaces are adjoint to symmetric or skew-symmetric bilinear forms, or to hermitian forms. Analogues of the classical invariants of quadratic forms (discriminant, Clifford algebra, signature) have been defined for arbitrary central simple algebras with involution. In this paper it is shown that over certain fields these invariants are sufficient to classify involutions up to conjugation. For algebras of low degree a classification is obtained over an arbitrary field.

1. INTRODUCTION

Let F be a field of characteristic different from two and let IF be the fundamental ideal of all even-dimensional forms in the Witt ring WF of quadratic forms over F. Let also $I^3F = (IF)^3$. In 1974 Elman and Lam [7] obtained classification theorems for quadratic forms. In particular they showed that if $I^3F = 0$ then quadratic forms over F are classified up to isometry by dimension, discriminant, and Clifford invariant. More generally they showed that if I^3F is torsion-free then quadratic forms are classified up to isometry by dimension, discriminant, Clifford invariant, and total signature, (i.e. the signature at each ordering of F). More recently Bayer-Fluckiger and Parimala [2], [3] obtained classification results for hermitian forms over central simple algebras with involution. They showed that if I^3F is torsion-free then rank, discriminant, Clifford and Rost invariants, and total signature are the invariants needed for isometry classification. (In the case of unitary involutions a stronger hypothesis on F is needed, namely that F has virtual cohomological dimension at most two). These results were vital ingredients in their proof of Serre's Conjecture II for classical groups [2] and of the Hasse Principle over fields of virtual cohomological dimension 2 [3].

In this paper the corresponding classification problem is considered for isomorphism of pairs (A, σ) where A is a central simple F-algebra and σ is an involution on A. Note that (A, σ) is isomorphic to $(\mathrm{End}_D V, \mathrm{adj}(h))$ where V is a finite-dimensional vector space over a division ring D with an involution τ of the same type as σ, h is a hermitian form over (D, τ) defined on the space V, and $\mathrm{adj}(h)$ denotes the adjoint involution of h. Note that h is only determined up to a scalar multiple since h and λh have exactly the same adjoint involution for any λ in $F^\times$. The quadratic form type invariants defined for (A, σ) have to depend on h only up to a scalar multiple. For example the signature of (A, σ) is only determined up to sign, and the Clifford algebra of (A, σ) is the analogue of the even Clifford algebra of a quadratic form. The full Clifford algebra of a quadratic form changes when the form is multiplied by a scalar. Thus the classification of algebras with involution is coarser than the corresponding classification of forms but the available invariants are weaker. For any unit a in an algebra A, we denote by $\mathrm{Int}(a)$ the inner automorphism of A

defined by $\mathrm{Int}(a)(x) = axa^{-1}$. Involutions σ, σ' on A are called *conjugate* if there exists $a \in A^\times$ such that

$$\sigma' = \mathrm{Int}(a) \circ \sigma \circ \mathrm{Int}(a)^{-1}.$$

Two algebras with involution (A, σ) and (A', σ') are said to be *isomorphic* if there exists an algebra isomorphism $\gamma\colon\ A \longrightarrow A$ such that $\gamma \circ \sigma = \sigma' \circ \gamma$. If A is central simple over a field F, two involutions σ and σ' on A are conjugate if and only if (A, σ) and (A, σ') are F-isomorphic since, by the Skolem-Noether theorem, all F-automorphisms of A are inner. (However one may also enquire whether (A, σ) and (A, σ') are isomorphic over some *subfield* of F, notably when σ and σ' are not the identity on F. This question will not be investigated here.)

The classification results we obtain are for conjugacy of a pair of involutions σ and σ' on a given central simple algebra A. The results are analogues of the above-mentioned results for forms. An extra feature in the classification of involutions is that we need the extra hypothesis that F is a SAP field [14]. Examples are given to show that this hypothesis cannot be deleted. The basic strategy for the proofs is to look at the hermitian form associated to the algebra with involution and modify by a scalar multiple so that the corresponding form invariants all coincide. Then appealing to [2] yields the results. In one case not covered by [2] we require the appendix provided by Parimala.

Recall from [17, Ch. 8, §7], [11, (2.5)] that an involution on a central simple algebra is called *orthogonal* (resp. *symplectic*, resp. *unitary*) if it is adjoint to a symmetric (resp. skew-symmetric, resp. hermitian) form after scalar extension to a splitting field of A.

Theorem A. *Suppose $I^3F = 0$. All the symplectic involutions on a central simple F-algebra are conjugate. Orthogonal involutions on a central simple F-algebra are conjugate if and only if their Clifford algebras are F-isomorphic.*

Suppose that the cohomological dimension of F is at most 2. All the unitary involutions on a central simple algebra of odd degree over a quadratic extension of F are conjugate. Unitary involutions on a central simple algebra of even degree over a quadratic extension of F are conjugate if and only if their discriminant algebras are isomorphic.

The proof is given in Section 3. Examples of fields of cohomological dimension at most 2 (hence such that $I^3F = 0$, see [1]) include fields of transcendence degree at most 2 over an algebraically closed field, p-adic fields and non formally real global fields.

Using signatures of involutions, we also treat the case where F is ordered. Examples show that the Strong Approximation Property (SAP) is an indispensable tool in most of the cases. (See [14] for a discussion of SAP fields.) We will often require not just that F is SAP but that also each quadratic extension of F is SAP. The notion of a field F having the Effective Diagonalization Property (ED) was introduced in [15] and is precisely the condition we require. There are a number of equivalent conditions which may be used as the definition of ED. For our purposes it suffices to know that a field F is ED if and only if F is SAP and every quadratic extension field of F is SAP. For instance, fields with finite Hasse number $\tilde{u}$, i.e., over which the dimensions of totally indefinite anisotropic forms are bounded, are ED by [8, Corollary 2.6].

The virtual cohomological dimension of F, denoted by $\mathrm{vcd}(F)$, is the cohomological dimension of $F(\sqrt{-1})$. If $\mathrm{vcd}(F) \leq 2$, then I^3F is torsion-free. (See for instance [3, Lemma 5.1].)

Theorem B. *Let F be a formally real field.*

Suppose F is SAP and I^3F is torsion-free. Symplectic involutions on a central simple F-algebra are conjugate if and only if they have the same signature.

Suppose I^3F is torsion-free. Orthogonal involutions on a central simple F-algebra of odd degree are conjugate if and only if they have the same signature and their Clifford algebras are isomorphic.

Suppose I^3F is torsion-free and F is ED. Orthogonal involutions on a central simple F-algebra of even degree are conjugate if and only if they have the same signature and their Clifford algebras are F-isomorphic.

Suppose F is SAP and $\mathrm{vcd}(F) \leq 2$. *Unitary involutions on a central simple algebra of odd degree over a quadratic extension of F are conjugate if and only if they have the same signature.*

Suppose F is ED and $\mathrm{vcd}(F) \leq 2$. *Unitary involutions on a central simple algebra of even degree over a quadratic extension of F are conjugate if and only if they have the same signature and their discriminant algebras are isomorphic.*

The proof is given in Section 4. Formally real number fields are examples of fields to which all the statements in Theorem B apply, and also function fields in one variable over a real closed field.

Our method for proving Theorems A and B consists in a reduction to hermitian or skew-hermitian forms, where we can apply the classification results of Bayer-Fluckiger and Parimala [2], [3]. We first explain this reduction.

Let A be a (finite-dimensional) simple F-algebra and suppose σ, σ' are involutions on A which have the same restriction to the center of A. We assume F is the subfield of σ- (or σ'-) invariant elements in the center of A. Then $\sigma' \circ \sigma$ is an automorphism which leaves the center fixed, hence $\sigma' = \mathrm{Int}(s) \circ \sigma$ for some unit $s \in A^\times$. Assume moreover that σ and σ' have the same type; then we may assume $\sigma(s) = s$. For any $\lambda \in F^\times$, we define an involution θ_λ on $M_2(A)$ by

$$\theta_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sigma(a) & -\lambda^{-1}\sigma(c)s^{-1} \\ -\lambda s\sigma(b) & s\sigma(d)s^{-1} \end{pmatrix}.$$

The algebra $\big(M_2(A), \theta_\lambda\big)$ is an "orthogonal sum" of (A,σ) and (A,σ'), see Dejaiffe [4]. Under the usual identification $M_2(A) = \mathrm{End}_A(A^2)$, the involution θ_λ is adjoint to the hermitian form $\langle 1, -\lambda^{-1}s^{-1}\rangle$ on A^2 (with respect to σ).

Proposition 1. *The following are equivalent:*

(a) *σ and σ' are conjugate, i.e., $\sigma' = \mathrm{Int}(a) \circ \sigma \circ \mathrm{Int}(a)^{-1}$ for some $a \in A^\times$;*
(b) *$s = \mu a \sigma(a)$ for some $\mu \in F^\times$, $a \in A^\times$;*
(c) *the hermitian form $\langle 1, -\lambda^{-1}s^{-1}\rangle$ is hyperbolic for some $\lambda \in F^\times$;*
(d) *the algebra with involution $\big(M_2(A), \theta_\lambda\big)$ is hyperbolic for some $\lambda \in F^\times$.*

Proof. The equivalence of (a) and (b) follows from an easy computation, and (d) is equivalent to (c) by definition of hyperbolic involutions. Finally, the equivalence of (a) and (d) is proved in [4, Proposition 2.4]. □

The above proposition is also a main tool in Wadsworth's work on isomorphic involutions [19].

2. Low degree cases

The degree of a central simple algebra is the square root of its dimension. In this section, we consider central simple algebras of degree at most 4 over an arbitrary field of characteristic different from 2.

Proposition 2. *Let A be a central simple F-algebra of degree $deg\,A \leq 4$. Orthogonal involutions σ, σ' on A are conjugate if and only if their Clifford algebras $C(A,\sigma)$, $C(A,\sigma')$ are F-isomorphic.*

Proof. If $deg\,A = 2$, the Clifford algebras $C(A,\sigma)$, $C(A,\sigma')$ are quadratic F-algebras given by the discriminants of σ and σ', and the proposition follows from [11, (7.4)].

If $deg\,A = 3$, the algebra A is necessarily split, since its Schur index is a power of 2 which divides the degree (see for instance [11, (2.8)]). Therefore, σ and σ' are adjoint to 3-dimensional quadratic forms q, q'. The Clifford algebras $C(A,\sigma)$, $C(A,\sigma')$ are quaternion algebras:

$$C(A,\sigma) = C_0(q), \qquad C(A,\sigma') = C_0(q').$$

Actually, by [11, §15.A], we may take for q (resp. q') the squaring map on the vector space of pure quaternions in $C(A,\sigma)$ (resp. $C(A,\sigma')$). Therefore, σ and σ' are conjugate if $C(A,\sigma) \simeq C(A,\sigma')$.

Finally, assume $deg\,A = 4$. The Clifford algebra $C(A,\sigma)$ is a quaternion algebra over a quadratic extension Z of F, and its norm (or corestriction) $N_{Z/F}\big(C(A,\sigma)\big)$ is canonically isomorphic to A. More precisely, there is a canonical isomorphism of algebras with involution

$$N_{Z/F}\big(C(A,\sigma),\gamma\big) \simeq (A,\sigma)$$

where γ is the conjugation involution on $C(A,\sigma)$, see [11, (15.7)].

If $C(A,\sigma) \simeq C(A,\sigma')$, the centers Z, Z' are isomorphic, and there is an isomorphism of algebras with involution $(A,\sigma) \simeq (A,\sigma')$. The involutions σ and σ' are therefore conjugate. □

Symplectic involutions exist only on central simple algebras of even degree, since skew-symmetric bilinear forms of odd dimension are singular (see [11, (2.8)]). In degree 2, the classification problem does not arise, since the conjugation involution is the only symplectic involution on a quaternion algebra. For algebras of degree 4, the problem is discussed in Knus-Lam-Shapiro-Tignol [10] (see also [11, §16.B]). If σ, σ' are symplectic involutions on a central simple F-algebra of degree 4, a 3-fold Pfister form $j_\sigma(\sigma')$ is defined, with the property that σ, σ' are conjugate if and only if $j_\sigma(\sigma')$ is hyperbolic.

Finally, we consider unitary involutions.

Proposition 3. *Let A be a quaternion algebra over a quadratic extension K of F, and let σ, σ' be unitary involutions on A which restrict to the identity on F. The involutions σ and σ' are conjugate if and only if their discriminant algebras $D(A,\sigma)$, $D(A,\sigma')$ are isomorphic.*

Proof. By [11, p. 129], the discriminant algebra $D(A,\sigma)$ is a quaternion F-subalgebra of A on which σ restricts to the conjugation involution γ, hence there is a decomposition of algebras with involution

$$(A,\sigma) = \big(D(A,\sigma),\gamma\big) \otimes (K,{}^{-})$$

where ${}^{-}$ is the nontrivial automorphism of K over F. Therefore, every isomorphism $D(A,\sigma) \to D(A,\sigma')$ extends to an inner automorphism of A which conjugates σ into σ'. □

The classification of unitary involutions on central simple algebras of degree 3 is discussed in Haile-Knus-Rost-Tignol [9] (see also [11, §19.B]). By means of the trace form, a 3-fold Pfister form $\pi(\sigma)$ is associated to any unitary involution σ on a central simple algebra A over a quadratic extension of F, and it is shown that unitary involutions are conjugate if and only if the corresponding Pfister forms are isometric.

On central simple algebras of degree 4, the discriminant algebra is not sufficient to classify unitary involutions, as the following example shows.

Example 1. Let F be a field over which there exists an anisotropic 3-fold Pfister form $\langle 1,-\alpha\rangle \otimes \langle 1,-\beta\rangle \otimes \langle 1,-\gamma\rangle$, and let $K = F(\sqrt{\alpha})$. On the split algebra $A = M_4(K)$, consider the involution σ adjoint to the hermitian form $\langle 1,-\beta,-\gamma,\beta\gamma\rangle_K$ and the involution σ' adjoint to the hyperbolic hermitian form. The discriminant algebras $D(A,\sigma)$ and $D(A,\sigma')$ are both split, by [11, (10.35)], but σ and σ' are not conjugate since σ is anisotropic whereas σ' is hyperbolic (see [17, Ch. 10, §1]).

Using the correspondence between central simple algebras of degree 4 with unitary involution and central simple algebras of degree 6 with orthogonal involution [11, §15.D], we may derive an example of orthogonal involutions on a (split) central simple algebra of degree 6 which are not conjugate, even though their Clifford algebras are isomorphic (and split, with center isomorphic to K).

Example 2. Let F be as in the above example, let $A = M_6(F)$, and consider the adjoint involutions of the quadratic forms $\langle 1,-\alpha\rangle \otimes \langle 1,-\beta,-\gamma\rangle$ and $\langle 1,-\alpha\rangle \otimes \langle 1,-1,1\rangle$. These involutions cannot be conjugate since the first one is anisotropic but the second is not. However their Clifford algebras are isomorphic because each is split. (The two forms each have discriminant α and become hyperbolic on $F(\sqrt{\alpha})$).

3. Classification over non formally real fields

This section is devoted to the proof of Theorem A, considering separately the various types of involutions. Throughout the section, F denotes a field of characteristic different from 2.

Proposition 4. *If $I^3F = 0$, all the symplectic involutions on a given central simple F-algebra are conjugate.*

Proof. Let σ be a symplectic involution on a central simple F-algebra A. By [2, Theorem 4.3.1], every nonsingular hermitian form of even rank over (A,σ) is hyperbolic, hence $\langle 1,-s^{-1}\rangle$ is hyperbolic for every σ-symmetric unit s. By Proposition 1, it follows that all the symplectic involutions on A are conjugate to σ. □

Proposition 5. *If $I^3F = 0$, orthogonal involutions on a given central simple F-algebra of odd degree are conjugate if and only if their Clifford algebras are isomorphic.*

Proof. Recall that every central simple algebra of odd degree with orthogonal involution is split, see [11, (2.8)]. Every orthogonal involution on a split algebra $A = \mathrm{End}_F(V)$ of odd degree is adjoint to a quadratic form on V which may be assumed of discriminant 1, upon multiplication by a suitable scalar. Therefore, it suffices to see that quadratic forms of trivial discriminant on V are isometric if and only if their even Clifford algebras are isomorphic. This follows from [6], since for quadratic forms of odd dimension the Witt invariant is the Brauer class of the even Clifford algebra. □

In order to treat the case of orthogonal involutions on central simple F-algebras of even degree, we start with a few general observations on Clifford algebras of orthogonal sums. Let σ, σ' be orthogonal involutions on a central simple F-algebra A of even degree. As in the introduction, we may write

$$\sigma' = \mathrm{Int}(s) \circ \sigma$$

for some σ-symmetric unit s. Let Z (resp. Z') denote the center of the Clifford algebra $C = C(A,\sigma)$ (resp. $C' = C(A,\sigma')$), which is a quadratic étale F-algebra. Assume that σ and σ' have the same discriminant; then Z and Z' are isomorphic (see [11, (8.10)]). We fix an isomorphism $\phi\colon Z \to Z'$ and use it to view C' as a Z-algebra. Choose z in $Z \setminus F$ such that $z^2 \in F$ and write $z' = \phi(z)$ and $\mu = z^2 \in F$. Now consider the involution θ_λ on $M_2(A)$ defined in the introduction and write $C_\lambda = C(M_2(A),\theta_\lambda)$. As observed in [4] there is a canonical embedding $C \otimes_F C' \hookrightarrow C_\lambda$, which we use to identify $C \otimes_F C'$ with a subalgebra of C_λ. From [4] the center of C_λ is $F(1\otimes 1) + F(z \otimes z')$. The center of C_λ is thus spanned by the idempotents $e_+ = \frac{1}{2}(1\otimes 1 + \mu^{-1} z\otimes z')$ and $e_- = \frac{1}{2}(1\otimes 1 - \mu^{-1} z\otimes z')$, and we have

$$C_\lambda = C_\lambda^+ \oplus C_\lambda^-,$$

where $C_\lambda^+ = C_\lambda e_+$ and $C_\lambda^- = C_\lambda e_-$ are each central simple F-algebras. Note also that C_λ^+, (resp. C_λ^-), contains $Z_\lambda^+ = (Z \otimes_F Z')e_+$, (resp. $Z_\lambda^- = (Z \otimes_F Z')e_-$), and Z_λ^+, (resp. Z_λ^-), is isomorphic to Z.

Lemma 1. *Using the above notation, the centralizer of Z_λ^+ in C_λ^+ is canonically isomorphic to $C \otimes_Z C'$. The centralizer of Z_λ^- in C_λ^- is canonically isomorphic to $C \otimes_Z \overline{C'}$ where $\overline{C'}$ is the conjugate Z-algebra where scalar multiplication is twisted by the non-trivial F-automorphism $^-$ of Z over F.*

Proof. Since $C \otimes_F C'$ is the centralizer of $Z \otimes_F Z'$ in C_λ we see that $(C \otimes_F C')e_+$ is the centralizer of Z_λ^+ in C_λ^+. The canonical epimorphism $C \otimes_F C' \longrightarrow (C \otimes_Z C')e_+$ sends e_- to zero since $z \otimes 1 = 1 \otimes z'$ in $C \otimes_Z C'$. Hence it induces an epimorphism $(C \otimes_F C')e_+ \longrightarrow C \otimes_Z C'$ and, by dimension count, it is an isomorphism. The conjugate Z-algebra $\overline{C'}$ is viewed as a Z-algebra via the homomorphism $\overline{\phi}\colon Z \longrightarrow \overline{C'}$ where $\overline{\phi}(z) = -z'$. By a similar argument to the above $(C \otimes_F C')e_-$ is the centralizer of Z_λ^- in C_λ^-, and the canonical epimorphism $C \otimes_F C' \longrightarrow C \otimes_Z \overline{C'}$ sends e_+ to zero (since $z\otimes 1 = 1 \otimes(-z')$ in $C \otimes_Z \overline{C'}$.) Hence we have an isomorphism of $(C \otimes_F C')e_-$ with $C \otimes_Z \overline{C'}$. This proves the lemma. □

Lemma 2. *With the same notation as above, if $C(A,\sigma)$ and $C(A,\sigma')$ are isomorphic as F-algebras, then there exists $\lambda \in F^\times$ such that C_λ^+ or C_λ^- is split.*

Proof. If the F-algebras $C(A,\sigma)$ and $C(A,\sigma')$ are isomorphic, we have either $C(A,\sigma) \simeq C(A,\sigma')$ or $C(A,\sigma) \simeq \overline{C(A,\sigma')}$ as Z-algebras. Assume first $C(A,\sigma) \simeq C(A,\sigma')$. If the canonical involution $\underline{\sigma}$ on $C(A,\sigma)$ is the identity on Z, then $C(A,\sigma) \otimes_Z C(A,\sigma')$ is split. If $\underline{\sigma}$ restricts to $^-$ on Z, then it induces an isomorphism between $\overline{C(A,\sigma)}$ and the opposite algebra of $C(A,\sigma)$, hence $C(A,\sigma) \otimes_Z \overline{C(A,\sigma')}$ is split. Similarly, if $C(A,\sigma) \simeq \overline{C(A,\sigma')}$, then (at least) one of the algebras $C(A,\sigma) \otimes_Z C(A,\sigma')$, $C(A,\sigma) \otimes_Z \overline{C(A,\sigma')}$ is split. Thus in all cases the centralizer of Z in C_λ^+ or C_λ^- is split.

Assume for instance that the centralizer of Z in C_λ^+ is split. If $Z \simeq F \times F$, it follows that C_λ^+ is split (for any $\lambda \in F^\times$). If Z is a field, say $Z \simeq F(\sqrt{\delta})$ for some $\delta \in F^\times$, it follows that C_λ^+ is Brauer-equivalent to a quaternion algebra $(\delta,\varepsilon)_F$ for some $\varepsilon \in F^\times$. By [4, Proposition 3.8], we have a Brauer-equivalence

$$C^\pm(\lambda\lambda') \sim C^\pm(\lambda) \otimes_F (\delta,\lambda')_F$$

for all λ, $\lambda' \in F^\times$, hence $C_{\lambda\varepsilon}^+$ is split. □

Proposition 6. *Suppose $I^3F = 0$. Orthogonal involutions σ, σ' on a central simple F-algebra A of even degree are conjugate if and only if their Clifford algebras $C(A,\sigma)$, $C(A,\sigma')$ are isomorphic as F-algebras.*

Proof. The "only if" part is clear. If $C(A,\sigma)$ and $C(A,\sigma')$ are isomorphic, then by Lemma 2 we may find $\lambda \in F^\times$ such that C^+_λ or C^-_λ is split. Therefore, the Clifford invariant of the hermitian form $\langle 1, -\lambda^{-1}s^{-1}\rangle$, as defined by Bayer-Fluckiger and Parimala [2, Lemma 2.1.3], is trivial. (Recall that this Clifford invariant is the element of $\mathrm{Br}_2(F)/(A)$ determined by C^+_λ, or equivalently by C^-_λ.) By [2, Theorem 4.4.1], it follows that $\langle 1, -\lambda^{-1}s^{-1}\rangle$ is hyperbolic, hence σ and σ' are conjugate, by Proposition 1. □

We finally turn to unitary involutions. In the rest of this section, we denote by A a central simple algebra over a quadratic extension $K = F(\sqrt{\alpha})$ of F and by σ a unitary involution on A which is the identity on F. Recall the notion of discriminant defined by Bayer-Fluckiger and Parimala [2, §2.2]: for the hermitian form $\langle 1, -\lambda^{-1}s^{-1}\rangle$ over A (with respect to σ), the discriminant is

$$d\big(\langle 1, -\lambda^{-1}s^{-1}\rangle\big) = \lambda^{-\,deg\,A}\,\mathrm{Nrd}_A(s)^{-1}N(K/F) \in F^\times/N(K/F),$$

where $N(K/F) \subset F^\times$ is the group of norms from K/F.

Proposition 7. *Suppose the cohomological dimension of F is at most 2. If $deg\,A$ is odd, all the unitary involutions on A are conjugate. If $deg\,A$ is even, unitary involutions on A are conjugate if and only if their discriminant algebras are isomorphic.*

Proof. If $deg\,A$ is odd, the formula above shows that the discriminant of $\langle 1, -\lambda^{-1}s^{-1}\rangle$ is trivial for $\lambda = \mathrm{Nrd}_A(s)$. Therefore, this form is hyperbolic, by [2, Theorem 4.1.1], and Proposition 1 shows that σ and $\sigma' = \mathrm{Int}(s)\circ\sigma$ are conjugate.

If $deg\,A$ is even, the discriminant algebras of σ and σ' are related by

$$D(A,\sigma') \sim D(A,\sigma) \otimes_F \big(\alpha, \mathrm{Nrd}_A(s)\big)_F$$

(see [11, (10.36)]), hence the condition that $D(A,\sigma)$ and $D(A,\sigma')$ are isomorphic implies that $\mathrm{Nrd}_A(s) \in N(K/F)$. Therefore, for all $\lambda \in F^\times$ the discriminant of the form $\langle 1, -\lambda^{-1}s^{-1}\rangle$ is trivial and we may conclude as above. □

4. Classification over formally real fields

Throughout this section, the base field F is assumed to be formally real. We denote by X_F the space of orderings of F. For $P \in X_F$, we let F_P denote a real closure of F with respect to the ordering P.

The signature of involutions on simple algebras has been defined by Lewis-Tignol [12] in the orthogonal and symplectic cases, and by Quéguiner [16] in the unitary case (see also [11, §11]). In all cases, the signature is defined by means of the quadratic form $\phi^\sigma\colon A \to F$ defined by

$$\phi^\sigma(x) = \mathrm{Trd}_A\big(\sigma(x)x\big):$$

for $P \in X_F$ one sets

$$\mathrm{sig}_P\,\sigma = \begin{cases} \sqrt{\mathrm{sig}_P\,\phi^\sigma} & \text{if } \sigma \text{ is orthogonal or symplectic,}\\ \sqrt{\frac{1}{2}\,\mathrm{sig}_P\,\phi^\sigma} & \text{if } \sigma \text{ is unitary.}\end{cases}$$

Thus, the signature of an involution which is adjoint to a symmetric bilinear form or to a hermitian form is the absolute value of the signature of the form.

Recall that the field F is called SAP (or Pasch) if the following equivalent conditions hold:

(a) every closed and open subset of X_F has the form $\{P \in X_F \mid a > 0 \text{ at } P\}$ for some $a \in F$;

(b) for all $a, b \in F^\times$, the quadratic form $\langle 1, a, b, -ab\rangle$ is weakly isotropic, i.e. there exists an integer n such that $n\langle 1, a, b, -ab\rangle$ is isotropic.

(See [14].) The following examples show that the "classical" invariants are not sufficient to classify involutions if the base field is not SAP.

Examples. Suppose F is not SAP, and let $a, b \in F^\times$ be such that the quadratic form $4\langle 1, a, b, -ab\rangle$ is anisotropic. Denote by H the quaternion algebra $H = (-1,-1)_F$ and let $A = M_4(H)$. This central simple F-algebra carries a symplectic involution σ adjoint to the hermitian form with diagonalization $\langle 1, a, b, -ab\rangle_H$ over H (with respect to the conjugation involution) and a symplectic involution σ' adjoint to the hermitian form $\langle 1, 1, 1, -1\rangle_H$. By [12, Corollary 2], the signatures of the involutions σ, σ' are given by

$$\begin{aligned} \operatorname{sig}_P \sigma &= 2|\operatorname{sig}_P\langle 1, a, b, -ab\rangle| = 4 \qquad \text{for all } P \in X_F, \\ \operatorname{sig}_P \sigma' &= 2|\operatorname{sig}_P\langle 1, 1, 1, -1\rangle| = 4 \qquad \text{for all } P \in X_F. \end{aligned}$$

The involutions σ, σ' thus have the same signature. They are not conjugate however, since σ' is isotropic whereas σ is not. Indeed, the hermitian form $\langle 1, a, b, -ab\rangle_H$ is anisotropic since the quadratic form $4\langle 1, a, b, -ab\rangle$ is anisotropic over F.

Similar examples can be constructed for orthogonal or unitary involutions: with a, b, F as above, let $K = F(\sqrt{-1})$ and $B = M_4(K)$. Let τ be the unitary involution on B adjoint to the hermitian form $\langle 1, a, b, -ab\rangle_K$ (with respect to the non trivial automorphism of K/F) and let τ' be the unitary involution on B adjoint to $\langle 1, 1, 1, -1\rangle_K$. By [16], the signatures of τ, τ' are as follows:

$$\begin{aligned} \operatorname{sig}_P \tau &= |\operatorname{sig}_P\langle 1, a, b, -ab\rangle| = 2 \qquad \text{for all } P \in X_F, \\ \operatorname{sig}_P \tau' &= |\operatorname{sig}_P\langle 1, 1, 1, -1\rangle| = 2 \qquad \text{for all } P \in X_F. \end{aligned}$$

Moreover, the Brauer classes of the discriminant algebras can be determined from [11, (10.35)]:

$$\begin{aligned} D(B,\tau) &\sim \big(-1, \operatorname{disc}\langle 1, a, b, -ab\rangle\big)_F = (-1,-1)_F, \\ D(B,\tau') &\sim \big(-1, \operatorname{disc}\langle 1, 1, 1, -1\rangle\big)_F = (-1,-1)_F. \end{aligned}$$

However, τ and τ' are not conjugate since τ' is isotropic whereas τ is not. Using the correspondence between central simple algebras of degree 6 with orthogonal involution and central simple algebras of degree 4 with unitary involution (see [11, §15.D]), we may derive an example of orthogonal involutions on a central simple F-algebra of degree 6 which are not conjugate, even though their signatures are the same and their Clifford algebras are isomorphic. Explicitly, we may choose $A = M_3(H)$ where $H = (-1,-1)_F$, and take for σ and σ' the adjoint involutions of the skew-hermitian forms with diagonalizations $\langle i, ia, ib\rangle_H$ and $\langle i, i, -i\rangle_H$ respectively (with respect to the conjugation involution), where $i \in H$ is a pure quaternion with $i^2 = -1$. Since A does not split over any real closure of F, the signatures of σ and σ' are trivial. It may be seen that $C(A,\sigma) \simeq M_4(K) \simeq C(A,\sigma')$, and that the canonical involutions on $C(A,\sigma)$ and $C(A,\sigma')$ are the involutions τ and τ' above. Therefore, σ and σ' are not conjugate. Details of this computation are left to the reader.

These examples explain why our results require the base field to be SAP. (See however Proposition 10.) Under this hypothesis, we relate as follows involutions with the same signature:

Proposition 8. *Suppose F is a SAP field. Let σ be an involution on a simple F-algebra A. If σ is orthogonal or symplectic, assume F is the center of A; if σ is unitary, assume F is the subfield of σ-symmetric elements in the center of A. Let $s \in A^\times$ be a symmetric unit* *and let* $\sigma' = \operatorname{Int}(s) \circ \sigma$ *be an involution of the same type as* σ*. If* $\operatorname{sig}_P \sigma = \operatorname{sig}_P \sigma'$ *for all* $P \in X_F$*, then there exists* $\lambda \in F^\times$ *such that the involution* θ_λ *on* $M_2(A)$ *defined in the introduction (see Proposition 1) has trivial signature, i.e.*

$$\operatorname{sig}_P \theta_\lambda = 0 \qquad \text{for all } P \in X_F.$$

Proof. Let $\phi_s^\sigma \colon A \to F$ be the quadratic form defined by

$$\phi_s^\sigma(x) = \operatorname{Trd}_A\big(\sigma(x)sx\big).$$

Computation yields an orthogonal decomposition

$$\phi^{\theta_\lambda} \simeq \phi^\sigma \perp \langle -\lambda^{-1}\rangle \phi_{s^{-1}}^\sigma \perp \langle -\lambda\rangle \phi_s^{\sigma'} \perp \phi^{\sigma'}$$

(compare [5, Proposition 1]). On the other hand, we have

$$\phi_{s^{-1}}^\sigma(x) = \phi_s^\sigma(s^{-1}x) \qquad \text{and} \qquad \phi_s^{\sigma'}(x) = \phi_s^\sigma\big(\sigma(x)\big)$$

for all $x \in A$, hence $\phi_{s^{-1}}^\sigma \simeq \phi_s^{\sigma'} \simeq \phi_s^\sigma$ and

$$\phi^{\theta_\lambda} \simeq \phi^\sigma \perp 2\langle -\lambda\rangle \phi_s^\sigma \perp \phi^{\sigma'}. \tag{1}$$

If σ is orthogonal or symplectic, we may identify $A \otimes_F A$ with $\operatorname{End}_F(A)$ by mapping $a \otimes b$ to the endomorphism $x \mapsto ax\sigma(b)$. The involution $\sigma' \otimes \sigma$ on $A \otimes A$ then corresponds to the adjoint involution with respect to $\phi_{s^{-1}}^\sigma$, hence

$$\operatorname{sig}_P \sigma' \cdot \operatorname{sig}_P \sigma = |\operatorname{sig}_P \phi_{s^{-1}}^\sigma| = |\operatorname{sig}_P \phi_s^\sigma| \quad \text{for all } P \in X_F.$$

From equation (1), it follows that

$$\operatorname{sig}_P \theta_\lambda = |\operatorname{sig}_P \sigma - \varepsilon_P \operatorname{sig}_P\langle\lambda\rangle \operatorname{sig}_P \sigma'|$$

where $\varepsilon_P = +1$ if $\operatorname{sig}_P \phi_s^\sigma \geq 0$ and $\varepsilon_P = -1$ if $\operatorname{sig}_P \phi_s^\sigma < 0$. Since F is SAP, we may find $\lambda \in F^\times$ such that $\operatorname{sig}_P\langle\lambda\rangle = \varepsilon_P$ for all $P \in X_F$, hence $\operatorname{sig}_P \theta_\lambda = |\operatorname{sig}_P \sigma - \operatorname{sig}_P \sigma'| = 0$.

If σ is unitary, its restriction to the center K of A is the non trivial automorphism $^-$ of K/F. The same arguments apply, identifying $A \otimes_K \overline{A}$ to $\operatorname{End}_K A$ as above. ☐

We now begin the proof of Theorem B, starting with the symplectic case.

Proposition 9. *Assume* F *is SAP and* I^3F *is torsion-free. Symplectic involutions on a given central simple* F*-algebra are conjugate if and only if they have the same signature.*

Proof. Let σ, σ' be symplectic involutions on a central simple F-algebra A, and let $\sigma' = \operatorname{Int}(s) \circ \sigma$ for some σ-symmetric unit $s \in A^\times$. By Proposition 8, we may find $\lambda \in F^\times$ such that $\operatorname{sig}_P \theta_\lambda = 0$ for all $P \in X_F$. It follows from [3, Theorem 6.2] that the hermitian form $\langle 1, -\lambda^{-1}s^{-1}\rangle$ is hyperbolic, hence σ and σ' are conjugate, by Proposition 1. ☐

Proposition 10. *Assume* I^3F *is torsion-free. Orthogonal involutions on a given central simple* F*-algebra of odd degree are conjugate if and only if their Clifford algebras are isomorphic and their signatures are the same.*

Proof. Arguing as in the proof of Proposition 5, we are reduced to proving that quadratic forms q, q' of the same odd dimension and of trivial discriminant are isometric if and only if $C_0(q) \simeq C_0(q')$ and $|\operatorname{sig}_P q| = |\operatorname{sig}_P q'|$ for all $P \in X_F$. Since the signature of a quadratic form of odd dimension and trivial discriminant is $1 \bmod 4$, the conditions $\operatorname{disc} q = \operatorname{disc} q' = 1$ and $|\operatorname{sig}_P q| = |\operatorname{sig}_P q'|$ imply $\operatorname{sig}_P q = \operatorname{sig}_P q'$. The result then follows from the fact that over a field F with I^3F torsion-free, quadratic forms are classified by dimension, discriminant, Witt invariant and signature [7, Classification theorem 3]. ☐

Note that Proposition 10 holds without the hypothesis that F is SAP.

We now turn to the case of orthogonal involutions on a central simple algebra of even degree. The following example shows that this case requires more stringent hypotheses.

Example 3. Let F_0 be a maximal element in the inductive set of subfields of $\mathbb{R}$ which do not contain $\sqrt{2}$. The field F_0 has four square classes, represented by 1, -1, 2 and -2. Let $F = F_0((t))$, the field of Laurent series in one indeterminate over F_0. The unique ordering of F_0 extends to two orderings of F, and it is easy to see that F is SAP. Moreover, I^2F_0 is torsion-free, hence I^3F is torsion-free. Consider the following quadratic forms over F:

$$q = \langle 1, -2, t, t, t, t\rangle, \qquad q' = \langle 1, -1, t, t, t, 2t\rangle.$$

Clearly, q and q' have the same signature at each ordering of F. Computation shows that $C_0(q) \simeq C_0(q') \sim (-1,-1)_{F(\sqrt{2})}$. However, q and q' are not similar since q' is isotropic whereas q is not. Therefore, on the split algebra $A = M_6(F)$, the adjoint involutions $\sigma = \sigma_q$ and $\sigma' = \sigma_{q'}$ are not conjugate, even though $\operatorname{sig}\sigma = \operatorname{sig}\sigma'$ and $C(A,\sigma) \simeq C(A,\sigma')$.

In order to obtain a classification result for orthogonal involutions in the even degree case, we shall require that F is an ED field [15]. The particular property of ED fields that matters to us is that F is ED if and only if F is SAP and every quadratic extension of F is SAP.

Let σ, σ' be orthogonal involutions on a central simple algebra A of even degree over an arbitrary formally real field F. As observed in section 2, the condition that $\operatorname{disc}\sigma = \operatorname{disc}\sigma'$ implies that $\operatorname{disc}\theta_\lambda = 1$, hence

$$C\big(M_2(A),\theta_\lambda\big) = C_\lambda^+ \oplus C_\lambda^-$$

for some central simple F-algebras C_λ^+, C_λ^-.

Lemma 3. *Suppose F is ED. If $C(A,\sigma) \simeq C(A,\sigma')$ (as F-algebras) and $\operatorname{sig}_P\sigma = \operatorname{sig}_P\sigma'$ for all $P \in X_F$, then there exists $\lambda \in F^\times$ such that $\operatorname{sig}_P\theta_\lambda = 0$ for all $P \in X_F$ and (at least) one of C_λ^+, C_λ^- is split.*

Proof. Since F is SAP, Proposition 8 yields $\lambda_0 \in F^\times$ such that $\operatorname{sig}_P\theta_{\lambda_0} = 0$ for all $P \in X_F$. On the other hand, since $C(A,\sigma) \simeq C(A,\sigma')$, one of $C_{\lambda_0}^+$, $C_{\lambda_0}^-$ is Brauer-equivalent to a quaternion algebra $(\delta,\gamma)_F$, where $\delta \in F^\times$ is a representative of $\operatorname{disc}\sigma$ (see the proof of Lemma 2). Say $C_{\lambda_0}^+ \sim (\delta,\gamma)_F$. If $\delta \in F^{\times 2}$ we may choose $\lambda = \lambda_0$. For the rest of the proof we assume $\delta \notin F^{\times 2}$.

Writing $K = F(\sqrt{\delta})$ we will put $\lambda = \lambda_0\gamma N_{K/F}(z)$ for a suitably chosen element z of K. Then $C_\lambda^+ \sim (\delta,\gamma) \otimes_F (\delta, \gamma N_{K/F}(z))$ is split. To ensure $\operatorname{sig}_P\theta_\lambda = 0$ we require that $\gamma N_{K/F}(z) > 0$ at all orderings P of F where A_P splits. (Whenever A_P is not split $\operatorname{sig}_P\theta_\lambda = 0$ for all λ.)

Define an open subset U of the space of orderings X_K of K as follows;

$$U = \{P \in X_K \mid \gamma > 0 \text{ in } P\} \cup \{P \in X_K \mid \gamma < 0 \text{ and } \sqrt{\delta} < 0 \text{ in } P\}$$

Since F is ED we know that K is SAP and so we may choose $z \in K$ such that $z > 0$ for all P in U, and $z < 0$ for all P not in U.

Now let P be an ordering of F. If $\delta < 0$ then P does not extend to K and $N_{K/F}(z) > 0$.

If $\delta > 0$ and $\gamma > 0$ then $z > 0$ in each of the two orderings of K which extend P and so $N_{K/F}(z) > 0$.

If $\delta > 0$ and $\gamma < 0$ then $z > 0$ in exactly one of the two orderings of K which extend P and $N_{K/F}(z) < 0$.

So whenever $\delta > 0$ we have $\gamma N_{K/F}(z) > 0$, and also $\gamma N_{K/F}(z) > 0$ if $\delta < 0$ and $\gamma > 0$. The remaining case when $\delta < 0$ and $\gamma < 0$ implies $\gamma N_{K/F}(z) < 0$ but need not be considered because A_P is not split. (If A_P were split then $(M_2(A_P), (\theta_{\lambda_0})_P)$ would split and hence both components $C^{\pm}(M_2(A_P), (\theta_{\lambda_0})_P)$ would split. However

$$C^+(M_2(A_P), (\theta_{\lambda_0})_P) \sim (\gamma, \delta)_P$$

which cannot split because $\delta < 0, \gamma < 0$ in P.) Thus λ satisfies the desired properties and the lemma is proved. □

Proposition 11. *Suppose F is ED and I^3F is torsion-free. Orthogonal involutions σ, σ' on a central simple F-algebra A of even degree are conjugate if and only if* $\operatorname{sig}_P \sigma = \operatorname{sig}_P \sigma'$ *for all $P \in X_F$ and $C(A,\sigma) \simeq C(A,\sigma')$ (as F-algebras).*

Proof. The "only if" part is clear. For the converse, Lemma 3 yields an element $\lambda \in F^\times$ such that C^+_λ or C^-_λ is split and θ_λ is hyperbolic over every real closure of F. By Parimala's refinement of [3, Corollary 7.5] given in the Appendix, it follows that the form $\langle 1, -\lambda^{-1}s^{-1}\rangle$ is hyperbolic, hence σ and σ' are conjugate, by Proposition 1. □

This proposition applies in particular to algebraic number fields.

To complete the proof of Theorem B, we finally consider the case of unitary involutions. In the rest of this section, we denote by A a central simple algebra over a quadratic extension $K = F(\sqrt{\alpha})$ of F and by σ a unitary involution on A which is the identity on F. Let $\sigma' = \operatorname{Int}(s) \circ \sigma$ for some symmetric unit $s \in A^\times$ and let θ_λ be the involution on $M_2(A)$ defined in the introduction. In order to establish that the hermitian form $\langle 1, -\lambda^{-1}s^{-1}\rangle$ is hyperbolic for some $\lambda \in F^\times$, we shall invoke the results of Bayer-Fluckiger and Parimala in [3, §4]. These results use a notion of Discriminant (a refinement of the usual discriminant) such that

$$\begin{aligned}\operatorname{Disc}\bigl(\langle 1, -\lambda^{-1}s^{-1}\rangle\bigr) &= \lambda^{-\deg A}\operatorname{Nrd}_A(s)^{-1}\cdot N(K^{+(A)})\\ &\in F^\times/N(K^{+(A)})\end{aligned}$$

where $N(K^{+(A)})$ is the group of elements of $K^\times$ which are reduced norms from $A\otimes_K K_Q$ for every ordering $Q \in X_K$.

Lemma 4. *Let $P \in X_F$ be an ordering such that $\alpha < 0$. If $\lambda \in F^\times$ is such that* $\operatorname{sig}_P \theta_\lambda = 0$*, then* $\operatorname{Nrd}_A(\lambda s) > 0$.

Proof. Fix an isomorphism $A \otimes_F F_P \simeq M_n\bigl(F_P(\sqrt{-1})\bigr)$. Let $*$ be the standard involution on $M_n\bigl(F_P(\sqrt{-1})\bigr)$ defined by

$$(a_{ij})^* = (\overline{a_{ij}})^t,$$

where $^-$ is the non trivial automorphism of $F_P(\sqrt{-1})/F_P$. We have

$$\sigma = \operatorname{Int}(h^{-1}) \circ *, \qquad \sigma' = \operatorname{Int}({h'}^{-1}) \circ *$$

for some $*$-symmetric matrices h, h'. Then $s = \mu {h'}^{-1} h$ for some $\mu \in F^\times$, and

$$\theta_\lambda = \operatorname{Int}\begin{pmatrix} -\lambda^{-1}\mu^{-1}h^{-1} & 0 \\ 0 & {h'}^{-1}\end{pmatrix} \circ *.$$

Therefore, θ_λ is the adjoint involution of the hermitian form $\langle -\lambda\mu h, h'\rangle$ on A^2 (with respect to $*$) or, equivalently, of the hermitian form $-\lambda\mu h \perp h'$ on $F_P(\sqrt{-1})^{2n}$. The equality $\operatorname{sig}_P \theta_\lambda = 0$ implies $\operatorname{sig}(\lambda\mu h) = \operatorname{sig} h'$, hence the hermitian forms $\lambda\mu h$ and h' are isometric and $\det(\lambda\mu h) \equiv \det h' \bmod F_P^2$. The lemma follows, since $\operatorname{Nrd}_{A\otimes F_P}(\lambda s) = \det(\lambda\mu {h'}^{-1} h)$. □

This lemma is sufficient to settle the odd-degree case:

Proposition 12. *If F is SAP and $\operatorname{vcd}(F) \leq 2$, unitary involutions on an odd-degree central simple algebra over a quadratic extension of F are classified by their signature.*

Proof. We use the same notation as above. If $\operatorname{sig}_P \sigma' = \operatorname{sig}_P \sigma$ for all $P \in X_F$, we may find $\lambda_0 \in F^\times$ such that $\operatorname{sig}_P \theta_{\lambda_0} = 0$ for all $P \in X_F$, by Proposition 8. Let $\lambda = \lambda_0 \operatorname{Nrd}_A(\lambda_0 s)$. By Lemma 4 we have $\operatorname{sig}_P \theta_\lambda = 0$ for all $P \in X_F$, since by definition the signature of a unitary involution is 0 at the orderings for which $\alpha > 0$. Moreover, we have

$$\begin{aligned} \lambda^{deg\, A} \operatorname{Nrd}_A(s) &= \operatorname{Nrd}_A(\lambda_0 s)^{1+deg\, A} \\ &= N_{K/F}\big(\operatorname{Nrd}_A(\lambda_0 s)^{(1+deg\, A)/2}\big) \in N_{K/F}\big(\operatorname{Nrd}_A(A^\times)\big) \end{aligned}$$

hence $\operatorname{Disc}\big(\langle 1, -\lambda^{-1}s^{-1}\rangle\big)$ is trivial. It follows from [3, Theorem 4.8] that $\langle 1, -\lambda^{-1}s^{-1}\rangle$ is hyperbolic. □

To treat the even degree case, we need the following description of the group $N(K^{+(A)})$:

Lemma 5. *Suppose $deg\, A$ is even and its center K is SAP, and let $Y_A = \{Q \in X_K \mid A \otimes_K K_Q$ is not split$\}$. We have*

$$K^{+(A)} = \{x \in K^\times \mid x > 0 \textit{ for all } Q \in Y_A\}$$

and $N(K^{+(A)}) = N(K/F) \cap K^{+(A)}$.

Proof. If $A \otimes_K K_Q$ is not split, then $\operatorname{Nrd}(A \otimes_K K_Q) = K_Q^{\times 2}$. Therefore,

$$K^{+(A)} = \{x \in K^\times \mid x > 0 \text{ for all } Q \in Y_A\}.$$

Observe that the set Y_A is preserved by the non trivial automorphism $\bar{\ }$ of K/F, since $A \otimes_K K_Q$ and $A \otimes_K K_{\overline{Q}}$ are isomorphic as F-algebras. Therefore, $K^{+(A)}$ is preserved under $\bar{\ }$, hence

$$N(K^{+(A)}) \subset N(K/F) \cap K^{+(A)}.$$

To prove the reverse inclusion, pick $y \in K^\times$ such that $N_{K/F}(y) \in K^{+(A)}$. We shall find $u \in K^\times$ such that $yu\overline{u}^{-1} \in K^{+(A)}$, hence

$$N_{K/F}(y) = N_{K/F}(yu\overline{u}^{-1}) \in N(K^{+(A)}).$$

Let $\xi \in K^\times$ be such that $\xi^2 = \alpha$ and consider the set

$$Z = \{Q \in Y_A \mid (y > 0) \text{ or } (y < 0 \text{ and } \xi > 0) \text{ at } Q\}.$$

Since K is SAP, we may find $u \in K^\times$ which is positive exactly at the orderings in Z. If $Q \in Y_A$ is such that $y > 0$, then $\overline{y} > 0$ since $N_{K/F}(y) \in K^{+(A)}$, hence $\overline{Q} \in Z$ and therefore $\overline{u} > 0$ at Q. By contrast, if $Q \in Y_A$ is such that $y < 0$, then exactly one of $Q, \overline{Q}$ is in Z, hence $u\overline{u}^{-1} < 0$ at Q. Thus, $yu\overline{u}^{-1} > 0$ for all $Q \in Y_A$, hence $yu\overline{u}^{-1} \in K^{+(A)}$, as required. □

Proposition 13. *Suppose F and K are SAP, and $\operatorname{vcd}(F) \leq 2$. Unitary involutions on a central simple K-algebra A of even degree are classified up to conjugation by their signature and discriminant algebra.*

Proof. As above, let σ be a unitary involution on A and $\sigma' = \operatorname{Int}(s) \circ \sigma$. Assume $D(A,\sigma) \simeq D(A,\sigma')$ and $\operatorname{sig}_P \sigma = \operatorname{sig}_P \sigma'$ for all $P \in X_F$. Since F is SAP, Proposition 8 yields an element $\lambda \in F^\times$ such that $\operatorname{sig}_P \theta_\lambda = 0$ for all $P \in X_F$, hence the signature of the hermitian form $\langle 1, -\lambda^{-1}s^{-1}\rangle$ is trivial. We next compute its Discriminant. As observed in the non formally real case (see the proof of Proposition 7), the condition $D(A,\sigma) \simeq D(A,\sigma')$ implies that $\operatorname{Nrd}_A(s) \in N(K/F)$. On the other hand, it is clear from the definition of $K^{+(A)}$ that $\operatorname{Nrd}_A(s) \in K^{+(A)}$. Since $deg\,A$ is even, $\lambda^{deg\,A} \in F^{\times 2}$, hence

$$\lambda^{deg\,A} \operatorname{Nrd}_A(s) \in N(K/F) \cap K^{+(A)} = N(K^{+(A)}).$$

This shows that $\operatorname{Disc}\big(\langle 1, -\lambda^{-1}s^{-1}\rangle\big)$ is trivial. It follows from [3, Theorem 4.8] that $\langle 1, -\lambda^{-1}s^{-1}\rangle$ is hyperbolic, hence σ and σ' are conjugate. □

Proposition 14. *If F is ED and $\operatorname{vcd}(F) \leq 2$, then unitary involutions on a central simple algebra over a quadratic extension of F are classified up to conjugation by their signature and discriminant algebra.*

Proof. Follows at once from Proposition 13 since if F is ED then F is SAP and every quadratic extension of F is SAP. □

Remark. The field F of Example 3 is SAP and may be checked to satisfy $\operatorname{vcd}(F) = 2$. Using the correspondence between central simple algebras of degree 6 with orthogonal involution and central simple algebras of degree 4 with unitary involution (see [11, §15.D]), one can give an example of two unitary involutions on $M_2\big((-1,-1)_{F(\sqrt{2})}\big)$ which have the same signature and the same discriminant algebra, but which are not conjugate. Explicitly, we let ν be the tensor product of the conjugation involutions on $H = (-1,-1)_F \otimes F(\sqrt{2}) = (-1,-1)_{F(\sqrt{2})}$ and take for τ and τ' the adjoint involutions of the hermitian forms with diagonalizations $\langle 1, t\rangle_H$ and $\langle 1, -1\rangle_H$ respectively (with respect to ν). Therefore, Proposition 14 does not hold under the weaker hypothesis that F is SAP and $\operatorname{vcd}(F) \leq 2$.

APPENDIX BY R. PARIMALA

Theorem. *Let F be a formally real SAP field such that I^3F is torsion free. Let (D,σ) be a central division algebra over F with an involution σ of orthogonal type. Then nondegenerate hermitian forms over (D,σ) are classified up to isomorphism by dimension, discriminant, Clifford invariant and signatures.*

Proof. Let h, h' be two hermitian forms over (D,σ) with the same dimension, discriminant, Clifford invariant and signatures. Then $h \perp -h'$ has even dimension $2n$ and has trivial discriminant, Clifford invariant and signatures. Let $\mathbf{U}_{2n}$, $\mathbf{SU}_{2n}$ and $\mathbf{Spin}_{2n}$ denote respectively the unitary, special unitary and spin group of the standard hyperbolic form of dimension $2n$ over (D,σ). Then the class of $h \perp -h'$ in $H^1(F,\mathbf{U}_{2n})$ is the image of some $\xi \in H^1(F,\mathbf{Spin}_{2n})$ under the composite map

$$H^1(F,\mathbf{Spin}_{2n}) \xrightarrow{i} H^1(F,\mathbf{SU}_{2n}) \xrightarrow{j} H^1(F,\mathbf{U}_{2n})$$

induced by the maps $\mathbf{Spin}_{2n} \xrightarrow{i} \mathbf{SU}_{2n} \overset{j}{\hookrightarrow} \mathbf{U}_{2n}$. The map j restricted to $i(H^1(F,\mathbf{Spin}_{2n}))$ has trivial kernel, by [3, Lemma 7.11]. Since $h \perp -h'$ is hyperbolic over F_P for all $P \in X_F$, $i(\xi)_P \in H^1(F_P,\mathbf{SU}_{2n})$ maps to zero in $H^1(F_P,\mathbf{U}_{2n})$ and hence $i(\xi)_P = 0$ for all $P \in X_F$. We have the following commutative diagram with exact columns:

$$\begin{array}{ccc}
F^\times / \operatorname{Sn} \mathbf{SU}_{2n}(F) & \longrightarrow & \prod_{P\in X_F} F_P^\times / \operatorname{Sn} \mathbf{SU}_{2n}(F_P) \\
\big\downarrow c & & \big\downarrow c \\
H^1(F, \mathbf{Spin}_{2n}) & \longrightarrow & \prod_{P\in X_F} H^1(F_P, \mathbf{Spin}_{2n}) \\
\big\downarrow i & & \big\downarrow i \\
H^1(F, \mathbf{SU}_{2n}) & \longrightarrow & \prod_{P\in X_F} H^1(F_P, \mathbf{SU}_{2n}).
\end{array}$$

Here $\operatorname{Sn}\colon \mathbf{SU}_{2n}(F) \to F^\times/F^{\times 2}$ denotes the spinor map. There exists $u_P \in F_P^\times$ such that $c(\overline{u_P}) = \xi_P$, $P \in X_F$. The set $\{P \in X_F \mid \xi_P = 0\}$ is an open and closed subset of X_F (see [18, Corollary 2.2]) and F being SAP, there exists $\theta \in F^\times$ such that $\{P \in X_F \mid \xi_P = 0\} = \{P \in X_F \mid \theta > 0 \text{ at } P\}$. Since $\operatorname{Sn} \mathbf{SU}_{2n}(F_P)$ is the subgroup of $F_P^\times$ generated by norms from extensions of F_P where D_P is split ([13]), $\operatorname{Sn} \mathbf{SU}_{2n}(F_P) = \operatorname{Nrd} D_P^\times$. Thus if $P \in X_F$ is such that D_P is split, then $F_P^\times = \operatorname{Sn} \mathbf{SU}_{2n}(F_P)$ and in this case $\xi_P = 0$ and $\theta > 0$ at P. If $P \in X_F$ is such that D_P is not split, $u_P \in F_P^\times$ belongs to $\operatorname{Sn} \mathbf{SU}_{2n}(F_P)$ modulo squares if and only if $u_P > 0$. Thus if $P \in X_F$ is such that $\xi_P \neq 0$, then D_P is non split and $u_P < 0$ and $\theta < 0$. Thus $\theta u_P \in \operatorname{Sn} \mathbf{SU}_{2n}(F_P)$ modulo squares. If $\xi_P = 0$ at P, $\theta > 0$ at P and $u_P \in \operatorname{Sn} \mathbf{SU}_{2n}(F_P)$ and hence $\theta u_P \in \operatorname{Sn} \mathbf{SU}_{2n}(F_P)$ modulo squares and $[\overline{\theta}] = [\overline{u_P}]$ in $F_P^\times / \operatorname{Sn} \mathbf{SU}_{2n}(F_P)$ for all $P \in X_F$. Thus $c(\theta)$ and ξ in $H^1(F, \mathbf{Spin}_{2n})$ have the same image in $H^1(F_P, \mathbf{Spin}_{2n})$ for all $P \in X_F$ and in view of [3, Theorem 7.12], $c(\theta) = \xi$. Hence ξ maps to zero in $H^1(F, \mathbf{SU}_{2n})$ and this implies that $h \perp -h'$ is hyperbolic. This completes the proof of the theorem. □

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REFERENCES

[1] J.K. Arason, A proof of Merkurjev’s theorem, in *Quadratic and Hermitian Forms,* CMS Conference Proceedings **4** (eds. I.Hambleton, C.Riehm) (1984) 121–130.

[2] E. Bayer-Fluckiger, R. Parimala, Galois cohomology of the classical groups over fields of cohomological dimension ≤ 2, Invent. Math. **122,** (1995) 195–229.

[3] E. Bayer-Fluckiger, R. Parimala, Classical groups and Hasse principle, Annals of Math. **147,** (1998) 651–693.

[4] I. Dejaiffe, Somme orthogonale d’algèbres à involution et algèbre de Clifford, Commun. Algebra **26** (5), (1998) 1589–1612.

[5] I. Dejaiffe, D.W. Lewis, J.-P. Tignol, Witt equivalence of central simple algebras with involution, to appear in Rendiconti del Circolo Matematico di Palermo.

[6] R. Elman, T.Y. Lam, On the quaternion symbol homomorphism $g_F\colon k_2F \to B(F)$, *Proc. of Seattle Algebraic K-theory conference*, Lecture Notes in Math. 342 (Springer-Verlag, Berlin-Heidelberg-New York 1973) pp. 447–463.

[7] R. Elman, T.Y. Lam, Classification theorems for quadratic forms over fields, Comment. Math. Helv. **49,** (1974) 373–381.

[8] R. Elman, A. Prestel, Reduced stability of the Witt ring of a field and its Pythagorean closure, Amer. J. Math. **106,** (1984) 1237–1260.

[9] D. Haile, M.-A. Knus, M. Rost, J.-P. Tignol, Algebras of odd degree with involution, trace forms and dihedral extensions, Israel J. Math. **96,** (1996) 299–340.

[10] M.-A. Knus, T.Y. Lam, D.B. Shapiro, J.-P. Tignol, Discriminants of involutions on biquaternion algebras, in: *K-theory and algebraic geometry: connections with quadratic forms and division algebras* (B. Jacob and A. Rosenberg, eds), Proceedings of Symposia in Pure Mathematics vol. **58,** Part 2 (Amer. Math. Soc., Providence, RI, 1995) pp. 279–303.

[11] M.-A. Knus, A.S. Merkurjev, M. Rost, J.-P. Tignol, *The Book of Involutions*, Coll. Pub. 44 (Amer. Math. Soc., Providence, RI, 1998).

[12] D.W. Lewis, J.-P. Tignol, On the signature of an involution, Arch. Math. (Basel) **60,** (1993) 128–135.

[13] A.S. Merkurjev, Norm principle for algebraic groups, St. Petersburg J. Math. **7,** (1996) 243–264.

[14] A. Prestel, Lectures on Formally Real Fields, Lecture Notes in Mathematics 1093 (Springer-Verlag, Berlin Heidelberg New York, 1984).

[15] A. Prestel, R. Ware, Almost isotropic forms, J. London Math. Soc. **19,** (1979) 241–244.

[16] A. Quéguiner, Signature des involutions de deuxième espèce, Arch. Math. (Basel) **65,** (1995) 408–412.

[17] W. Scharlau, *Quadratic and Hermitian Forms*, Grundlehren math. Wiss. 270 (Springer-Verlag, Berlin, 1985).

[18] C. Scheiderer, Hasse principles and approximation theorems for homogeneous spaces over fields of virtual cohomological dimension one, Invent. Math. **125,** (1996) 307–365.

[19] A.R. Wadsworth, Making involutions isomorphic, unpublished notes, September 1993.

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