

Abstract

Let D be a division ring of degree m over its centre F . Herstein has shown that any finite normal subgroup of $D^* := GL_1(D)$ is central. Here, as a generalization of this result, it is shown that any finitely generated normal subgroup of D^* is central. This also solves a problem raised in [1] for finite dimensional division rings. The structure of maximal multiplicative subgroups of an arbitrary division ring D is then investigated. Given a maximal subgroup M of D^* whose centre is algebraic over F , it is proved that if M satisfies a multilinear polynomial identity over F , then $[D : F] < \infty$.

Let D be a division ring of degree m over its centre F . Denote by D' the commutator subgroups of the multiplicative group $D^* = D - \{0\}$. Since each element $d \in D^*$ is contained in a maximal subfield of D , a modification of the proof of Corollary 1 in [9] shows that $d^m = RN_{D/F}(d)c_d$ for some $c_d \in D'$, where $RN_{D/F}$ is the reduced norm of D to F . Thus the abelian group $G(D) := D^*/RN(D^*)D'$ is torsion of bounded exponent dividing the degree m of D over F , where $RN(D^*)$ is the image of D^* under the reduced norm of D to F . This group is not trivial in general. For example, if D is the real quaternions then $G(D)$ is trivial whereas for rational quaternions $G(D)$ is isomorphic to a direct product of copies of Z_2 , as it is easily checked. Assume that $G(D)$ is not trivial, by Prufer-Baer's Theorem (cf. [11, p. 105]), we conclude that $G(D)$ is isomorphic to a direct product of Z_{r_i} , where r_i divides m . In this way we may obtain a maximal normal subgroup of finite index in D^* . So, if $G(D)$ is not trivial, then D^* contains maximal subgroups. The aim of this note is to investigate the structure of maximal subgroups of D^* using general linear groups in the finite dimensional case and some results of skew linear groups for the general case. Given a finite dimensional division ring D over its centre F , it is shown in [1] that if F is an algebraic extension of the field of rational numbers, then any finitely generated normal subgroup of D^* is central. It is also conjectured there that whether the same result holds for an arbitrary division ring. Here we show that if D is of finite dimension over F , then the result remains true without any condition on F . We then investigate the structure of maximal multiplicative subgroups M of an arbitrary division

ring D . In 1978, Herstein [5] conjectured that given a normal subgroup N of D^* , if for any $x \in N$ there exists a positive integer $n(x)$ such that $x^{n(x)} \in F$, then N is central. He showed the conjecture is true in the finite dimensional case, but in general it is still open. Here, we replace “normal” by “maximal” in the above conjecture and show that the resulting conjecture is also true in the finite dimensional case. Given a maximal subgroup M of D^* whose centre is algebraic over F , it is also proved that if M satisfies a multilinear polynomial identity over F , then $[D : F] < \infty$. As a consequence, it is shown that if D is algebraic over F and D^* contains a soluble maximal subgroup, then D is locally finite. For some further (and recent) results on subgroups of division rings see [2], [9] and [10]. We begin our material with

PROPOSITION 1. *Let D be a division ring with centre F , and assume that M is a maximal subgroup of D^* . Then we have*

- (i) M contains either F^* or D' .
- (ii) Either $D = F(M)$ or $M \cup \{0\}$ is a division ring, where $F(M)$ is the division ring generated by M and F .

PROOF. (i) Assume that M does not contain F^* . Then we must have $D^* = F^*M$ and consequently $D' = M' \subset M$.

(ii) Consider the division ring $F(M)$ generated by F and M . By maximality of M , we have either $D^* = F(M)^*$ or $M = F(M)^*$. In the first case we obtain $D = F(M)$ and the other case implies that $M \cup \{0\}$ is a division ring.

We observe that, by Proposition 1, we have either $D = F(M)$ or $M = F(M)^*$. In the first case, one can easily see that $Z(M) = M \cap F$. For the second case, put $K = Z(M) \cup \{0\}$ so that $F \subset K$ is a field extension. For any element $a \in K \setminus F$, we have $M \subset C_D(a)$. Since a is not in F and M is maximal in D^* , we conclude that $M = C_D(a)$. Now, if D is algebraic over F , then $C_D(M) = C_D C_D(a) = F(a)$ which really means that there are no proper subfields between F and K . In any case, to pose the following conjecture is natural.

CONJECTURE 1. *Let D be a division ring with centre F and M is a maximal subgroup of D^* . Then we have $Z(M) = M \cap F$.*

REMARK. Let D be a finite dimensional division algebra over its centre

F . If N is a non-central finitely generated normal subgroup of D^* , we claim that there exists a finite set $\Lambda \subset F$ such that $F = P(\Lambda)$, where P is the prime subfield of F . To see this, assume that L is the division algebra generated by all elements of N . Since L is invariant under all inner automorphisms of D , by Cartan-Brauer-Hua's Theorem, $L = D$ or L is central. If L is central, then N is central, a contradiction. Thus, we may assume that $L = D$. Suppose that $[D : F] = n$ and consider the regular matrix representation of D in $GL_n(F)$. Since N is finitely generated, there exist matrices $A_1, \dots, A_k \in GL_n(F)$, such that $N = \langle A_1, \dots, A_k \rangle$. Let Λ be the set of all elements in F occurring as the entries of A_i and A_i^{-1} , $i = 1, \dots, k$. If H is the subring generated by N , then we have $H \subset GL_n(P(\Lambda))$, where $P(\Lambda)$ is the subfield of F generated by Λ . Now, since $L \subset GL_n(P(\Lambda))$ we have $aI \in GL_n(P(\Lambda))$, for any $a \in F^*$ and so $a \in P(\Lambda)$. Hence, $F = P(\Lambda)$ and the claim is established.

It is shown in [1] that if F is an algebraic extension of the field of rational numbers, then D^* contains no non-central finitely generated normal subgroups. The next theorem shows that the result remains true without any conditions on F .

THEOREM 2. *Let D be a finite dimensional division ring with centre F . Then any finitely generated normal subgroup of D^* is central.*

PROOF. If N is a non-central finitely generated normal subgroup of D^* , then by the remark made above, there exist elements $r_1, \dots, r_s \in F$ such that $F = P(r_1, \dots, r_s)$, where P is the prime subfield of F . If F is algebraic over P , we may consider two cases. If $\text{Char} D = p > 0$, then D is algebraic over a finite field and consequently D is commutative which is in contradiction with the fact that N is non-central. If $\text{Char} D = 0$, then the result follows from Theorem 4 of [1]. Finally, we may assume that F is not algebraic over P . Put $K_i = P(r_1, \dots, r_i)$, where $1 \leq i \leq s$ and consider the chain of subfields $P = K_0 \subset K_1 \subset K_2 \subset \dots \subset K_s = F$. By our assumption, there exists $j, 1 \leq j \leq s$ such that K_j is not algebraic over K_{j-1} . Now, consider the least such j so that $F = K_s$ is of finite dimension over K_j . Put $K = K_{j-1}$, $y = r_j$, and $L = K(y)$. Thus, we have $[F : L] < \infty$ and y is transcendental over K . Since $n = [D : F] < \infty$ we conclude that $k = [D : L] < \infty$.

Therefore, D^* has a matrix representation in $GL_k(L)$. Let $1, \alpha_2, \dots, \alpha_n$ be an F -basis of D . Let us consider an element $a \in N \setminus F$. Since a is not in the centre we conclude that a does not commute with all α_i 's. Without loss of generality, assume that $a\alpha_2 \neq \alpha_2a$. Assume also that $A, B \in GL_k(L)$ are the matrix representations of a, α_2 , respectively. It is clear that for each $x \in L$, the matrix representation of $x + \alpha_2$ is $B_x = xI + B$. Since N is finitely generated, by the argument used in the remark, we conclude that there is a set $\Lambda = \{m_1/n_1, \dots, m_t/n_t\} \subset L$, where $m_i, n_i \in K[y]$ such that each element of N has a matrix representation in $GL_k(K[y][\Lambda])$, and $K[y][\Lambda]$ is the subring of L generated by Λ over $K[y]$. On the other hand, $N \triangleleft D^*$ and so for each element $x \in L$ we have $B_x AB_x^{-1} \in GL_k(K[y][\Lambda])$. Since $\det B_x$ is a polynomial in x of degree k , and for each $1 \leq i, j \leq k$, the (i, j) -th entry of B_x^{-1} is of the form $f_{ij}(x)/g(x) \in L(x)$, where $\deg g(x) = k$, $\deg f_{ij}(x) \leq k - 1$, we conclude that the (i, j) -th entry of the matrix $B_x AB_x^{-1}$ is of the form $f_{ij}(x)/g(x)$, where for each $1 \leq i, j \leq k$, we have $\deg f_{ij}(x) \leq k$. If for each $1 \leq i, j \leq k$, there are elements $q_{ij} \in L$ such that for any $x \in L$, $f_{ij}(x)/g(x) = q_{ij}$, then for any $x_1, x_2 \in L$ with $x_1 \neq x_2$, we have $(x_1 + \alpha_2)a(x_1 + \alpha_2)^{-1} = (x_2 + \alpha_2)a(x_2 + \alpha_2)^{-1}$. This implies that $x_1(a\alpha_2 - \alpha_2a) = x_2(a\alpha_2 - \alpha_2a)$ and since $a\alpha_2 - \alpha_2a \neq 0$ we conclude that $x_1 = x_2$, a contradiction. Thus, there exists an entry of $B_x AB_x^{-1}$, say (i, j) -th which depends on x . Put $f_{ij}(x) = \sum_{i=0}^k a_i x^i$, $g(x) = x^k + \sum_{i=0}^{k-1} b_i x^i$. Thus for each $x \in L$ we have $f_{ij}(x)/g(x) \in K[y][\Lambda]$. If $a_k = m_{t+1}/n_{t+1}$, then for each $x \in L$ we obtain $f_{ij}(x)/g(x) - a_k \in K[y][\Lambda \cup \{m_{t+1}/n_{t+1}\}]$. So there exists a nonzero polynomial $f(x) \in L[x]$ such that $\deg f(x) \leq k - 1$ and for each $x \in L$ we have $f(x)/g(x) \in K[y][\Lambda \cup \{m_{t+1}/n_{t+1}\}]$. Multiplying $f(x)$ and $g(x)$ by suitable elements of $K[y]$, we may assume that $f(x), g(x) \in K[y][x]$. Put $f(x) = \sum_{i=0}^{k-1} a'_i x^i$, $g(x) = \sum_{i=0}^k b'_i x^i$. Since $\det B \neq 0$, we may assume that $b'_0 \neq 0$. Now, change the variable x to $b'_0 x$ to obtain $f_1(x), g_1(x) \in K[y][x]$, such that $\deg g_1 = k$, $\deg f_1 \leq k - 1$, and the constant term of $g_1(x)$ is 1. Further, for each $x \in L$ we have $f_1(x)/g_1(x) \in K[y][\Lambda \cup \{m_{t+1}/n_{t+1}\}]$. Assume that $S = \{p_1, \dots, p_l\}$ be the set of all irreducible polynomials occurring in the factorizations of n_1, \dots, n_{t+1} into irreducible polynomials. For each natural number r , put $x_r = (p_1 p_2 \dots p_l)^r$. Since $\deg f_1 < \deg g_1$, for a large enough number r , the degree of the denominator of $f_1(x_r)/g_1(x_r)$ with respect to y

is greater than that of the nominator. On the other hand, for each $r \geq 1$, and each $1 \leq i \leq l$, $g_1(x_r)$ and p_i are coprime, that is, $(g_1(x_r), p_i) = 1$. It is not hard to see that if $u/v \in K[y][m_1/n_1, \dots, m_{t+1}/n_{t+1}]$ with $(u, v) = 1$, then each irreducible factor of v belongs to S . Now since $f_1(x_r)/g_1(x_r) \in K[y][m_1/n_1, \dots, m_{t+1}/n_{t+1}]$ and for each $1 \leq i \leq l$, $r \geq 1$, $(g_1(x_r), p_i) = 1$, we arrive at a contradiction, and so the result follows.

As a consequence of the above theorem, we have the following

COROLLARY 3. *Let D be an infinite division ring with centre F such that $[D : F] < \infty$. Then D^* contains no finitely generated maximal subgroups.*

COROLLARY 4. *Let D be a division ring with centre F , and assume that M is a maximal subgroup of D^* containing F^* . If $[M : F^*] < \infty$, then $D = F$.*

PROOF. First assume that $[D : F] < \infty$. Let x_1, \dots, x_t be the representatives for cosets of F^* in M , i.e., $M = F^*x_1 \cup \dots \cup F^*x_t$. Then, we have $M = \langle x_1, \dots, x_t \rangle F^*$, where $\langle x_1, \dots, x_t \rangle$ is the group generated by x_1, \dots, x_t . Assume that $x \in D \setminus M$. By maximality of M , we obtain $D^* = \langle x_1, \dots, x_t, x \rangle F^*$. Put $H = \langle x_1, \dots, x_t, x \rangle$. Thus, $D^* = HF^*$ and consequently we have $D' = H' \subset H$, i.e., H is normal in D^* . Now, by Theorem 2, we conclude that $H \subset F^*$, i.e., $D^* = F^*$ which implies that $D = F$.

Now consider the case $[D : F] = \infty$. As in the above case we may assume $M = F^*x_1 \cup \dots \cup F^*x_t$. Put $A = \{\sum_{i=1}^n f_i x_i; f_i \in F\}$. It is clearly seen that A is a division ring of finite dimension over F and we have $M \subset A^*$. Since A is of finite dimension over F we conclude that $A \neq D$ and so $M = A^*$ by the maximality of M in D^* . Thus we have $[A^* : F^*] < \infty$. If A is infinite, then, by a result of Faith (cf. [8, p. 225]), $A = F$ and so $M \subset F$ which implies that $D = F$. So, we may assume that A is finite. Now, Wedderburn's Theorem implies that A is a finite field. So there exists an element $a \in D^*$ such that $A^* = \langle a \rangle$, i.e., $a^n = 1$ for some positive integer n . Since a is non-central in D , by Herstein's Lemma (cf. [8]), there is an element $b \in D^*$ such that $bab^{-1} = a^i \neq a$. Thus, $b \in N_{D^*}(A^*)$ and so $\langle M, b \rangle \subset N_{D^*}(A^*)$. Now, by maximality of M , we conclude that $N_{D^*}(A^*) = D^*$. Therefore, by Cartan-Brauer-Hua's Theorem, we have either $A \subset F$ or $A = D$, and it is

clear that none of these cases can occur. This completes the proof.

We shall also need the following result of Kolchin-Maltsev in the proof of our main theorem, for a proof see [7, p. 146].

THEOREM A. *Every soluble linear group has a normal subgroup of finite index with unipotent commutator subgroup.*

THEOREM 5. *Let D be a finite dimensional division ring with centre F whose characteristic is different from the degree of D over F . If M is a maximal subgroup of D^* and for each element $x \in M$ there exists a positive integer $n(x)$, depending on x , such that $x^{n(x)} \in F$, then $D = F$. Furthermore, if D is algebraic over F and M is torsion, then D is commutative.*

PROOF. By Proposition 1, either $D' \subset M$ or $F^* \subset M$. If the first case occurs, then we conclude that $D = F$, by Lemma 2 of [9]. Thus we may assume that M does not contain D' . We now divide the rest of the proof into two cases:

Case 1. $Char F = 0$. We may view M as a linear group in $GL_n(F)$, where $[D : F] = n$. By Theorem 1 of [16], either M contains a non-abelian free subgroup or it contains a soluble subgroup S of finite index. The first case can not occur since M/F^* is torsion. Thus, there is a soluble subgroup S in M with $[M : S] < \infty$. Now, using Theorem A, we conclude that S contains a subgroup T of finite index such that T' is unipotent. Since the only unipotent element in a division ring is the identity, we obtain $T' = \{1\}$. Thus S contains an abelian group of finite index and consequently M contains an abelian normal subgroup A of finite index. Put $K = F(A)$. Then we have $\langle K^*, M \rangle \subset N_{D^*}(K^*)$. If $K^* \not\subset M$, then $N_{D^*}(K^*) = D^*$ and so, by Cartan-Brauer-Hua's Theorem, we conclude that $K = F$, and consequently $[M : F^*] < \infty$. Now, using Corollary 4 establishes the result. Otherwise, assume that $K^* \subset M$. If $K = F$, the result follows from Corollary 4. Otherwise, K is radical over F . Thus, using Kaplansky's Lemma, we obtain $Char F = p > 0$ which is a contradiction. This completes the proof for the zero characteristic case.

Case 2. $Char F = p > 0$. Consider the group $G = D' \cap M$ and assume that $x \in G$. We know that $x^{n(x)} = a \in F^*$. Taking the reduced norm of D to F from both sides of the last equation we conclude that $1 = RN_{D/F}(x)^{n(x)} = a^m$,

where m is the degree of D over F . This means that G is a torsion group. Thus, $M' \subset G$ is torsion and consequently M' is locally finite by Burnside's Theorem. If $x, y \in M'$, then the subgroup $\langle x, y \rangle$ generated by x and y is finite. Since we are in characteristic p we conclude that $\langle x, y \rangle$ is cyclic, and so M' is abelian. Thus M is a maximal soluble subgroup of D^* . By Proposition 1, two cases may arise: (a) $D = F(M)$, (b) $D_1 = M \cup \{0\}$ is a division ring. If $D = F(M)$, then M is a maximal soluble irreducible subgroup of D^* . By a result of Suprunenko (cf. [14]), there exists a subfield K which is Galois over F such that $[M : K^*] < \infty$. If $K^* = F^*$, then the result follows from Corollary 4. Otherwise, $F^* \subset K^* \subset M$ which means that K is radical over F . By Kaplansky's Lemma, we conclude that either K is purely inseparable over F or K is algebraic over the prime subfield P . The first case can not happen since K is Galois over F . If the second case occurs, then D will be algebraic over a finite subfield and so by a result of Jacobson we conclude that $D = F$. In the case (b), by Kaplansky's Theorem (cf. [8]), we conclude that $D_1 = Z(D_1)$. If $Z(D_1) = F$, then we obtain $M = F^*$ which in turn implies that $D = F$, by maximality of M . Otherwise, $Z(D_1)$ is radical over F and so either $Z(D_1)$ is purely inseparable over F or $Z(D_1)$ is algebraic over the prime subfield. The second case reduces to $D = F$ as above. Thus, $D_1 = Z(D_1) := L$ is purely inseparable over F . Since $L^* = M$ is maximal in D^* one can easily conclude that L is a maximal subfield of D . Now, take an element $a \in L$ not in F . We then have $L \subset C_D(F(a))^*$ and so by maximality of L^* we obtain $C_D(F(a)) = L$. Thus, by the double centralizer theorem, we have $F(a) = C_D(C_D(F(a))) = C_D(L) = L$, i.e., L is a simple extension of F . Now, take t minimal such that $a^{p^t} \in F$ and put $b = a^{p^{t-1}}$. Then we have $L = F(b)$ and b is of degree p over F . Consequently, we obtain $[D : F] = p^2$ which is a contradiction. This completes the first part of the proof.

Furthermore, if $D' \subset M$, then D' is torsion. Thus, by Lemma 2 of [9], we obtain the result. Otherwise, $F^* \subset M$ and so F^* is torsion. This implies that $\text{Char} D = p > 0$ and D is algebraic over the prime subfield. Therefore, by a theorem of Jacobson (cf. [6]), the result follows.

The following result may be viewed as a generalization of the Noether-Jacobson Theorem (cf. [8]).

COROLLARY 6. *Let D be a non-commutative division ring of degree m over its centre F , and assume that M is a maximal subgroup of D^* . If the characteristic of D does not divide m , then there is an element in $M \setminus F$ which is separable over F .*

To prove our next theorem, we need the following results from [13, p. 215] and [4, p. 114], respectively.

THEOREM B. *Let H be a locally nilpotent normal subgroup of the absolutely irreducible subgroup G of $GL_n(D)$. Then H is centre by locally finite and $G/C_G(H)$ is periodic.*

THEOREM C. *Let G be a completely reducible linear group. If G is nilpotent, then $[G : Z(G)] < \infty$.*

THEOREM 7. *Let D be a non-commutative division ring with centre F , and assume that M is a nilpotent maximal subgroup of D^* . Then we have $F^* \subset M$. Also, there exists a maximal subfield K of D such that either $M = K^*$ or M/F^* is locally finite. Furthermore, if $[D : F] < \infty$, then the second case can not occur, i.e., M is the multiplicative group of a maximal subfield of D .*

PROOF. M is completely reducible since $F(M)$ is a division ring. By Proposition 1, we have either $F(M)^* = M$ or $D = F(M)$. In the first case, by Hua's Theorem (cf. [8]), we conclude that M is abelian. Now, put $K = F(M)$ so that $K^* = M$ and the result follows. Thus, we may assume that $D = F(M)$. By Theorem B, with $H = G = M$, we conclude that M/F^* is locally finite.

Furthermore, suppose $[D : F] < \infty$ so that M is a linear group. The case where $F(M)^* = M$ is treated as above. Finally, we claim that the case $D = F(M)$ leads to a contradiction. Suppose $D = F(M)$. We now show that $Z(M) = F^*$. We note that if D' is nilpotent, then D^* is soluble. Thus, by Hua's Theorem, D is commutative which is a contradiction. Therefore, D' is not contained in M and so, by Proposition 1, $F^* \subset M$ and consequently $Z(M) = F^*$. Now, as noted above, M is completely reducible since $F(M) = D$ is a division ring. Thus, by Theorem C, we conclude that $[M : F^*] < \infty$. We may now use Corollary 4 to obtain the contradiction $D = F$, and so the result follows.

The next result essentially says that Theorem 7 may be generalized to soluble groups. To present it, we need the following result which is due to Snider, (cf. [13, p. 207]).

THEOREM D. *A soluble absolutely irreducible skew linear group is abelian by locally-finite.*

THEOREM 8. *Let D be a non-commutative division ring with centre F , and assume that M is a soluble maximal subgroup of D^* . Then $F^* \subset M$. Also, either M/F^* is locally finite or there exists a maximal subfield K of D such that K^* is normal in M and M/K^* is locally finite. Furthermore, if $[D : F] < \infty$, then either M is the multiplicative group of a maximal subfield of D or there is a maximal subfield K of D such that K^* is normal in M such that K/F is Galois and $[M : K^*] < \infty$.*

PROOF. By Proposition 1, we have two cases to consider. If $F(M)^* = M$, as in the proof of Theorem 7 we conclude that $F^* \subset M$ and M is commutative and so $K = M \cup \{0\}$ is our required maximal subfield of D . Otherwise, assume that $F(M) = D$ and so M is absolutely irreducible. By Theorem D, there exists an abelian group A of M such that A is normal in M and M/A is locally finite. Put $D_1 = C_D(K)$, where $K = F(A)$. We have $\langle D_1^*, M \rangle \subset N_{D^*}(K^*)$. If D_1^* is not contained in M , then $D^* = N_{D^*}(K^*)$ and consequently either $K = D$ or $K \subset F$. The first case can not occur since D is non-commutative, and the second case implies that M/F^* is locally finite. Thus, we may assume that $D_1^* \subset M$. Then D_1 is commutative since M is soluble and so K is a maximal subfield of D . Since A is normal in M we conclude that $K^* = F(A)^*$ is also normal in M . Finally, since M/A is locally finite we obtain that M/K^* is locally finite and this completes the first part of the proof.

Furthermore, if $[D : F] < \infty$, as above we are faced with two cases, i.e., either $F(M)^* = M$ or $D = F(M)$. The first case is treated as above. In the second case we conclude that M is an irreducible maximal soluble linear subgroup of D^* . By a result of Suprunenko (cf. [14]), there is a subfield K of D such that K/F is Galois, K^* is normal in M and $[M : K^*] < \infty$. It remains to show that K is a maximal subfield of D . To see this, assume that $C_{D^*}(K^*)$ is not contained in M . Then $\langle C_D(K^*), M \rangle \subset N_{D^*}(K^*)$ and consequently

$D = N_{D^*}(K^*)$ which implies that $K \subset F$, i.e., $[M : F^*] < \infty$. Now, by Corollary 4, we conclude that D is commutative which is a contradiction. Thus, we must have $C_{D^*}(K^*) \subset M$ and since M is soluble we obtain $C_{D^*}(K^*)$ is soluble. Now, by Hua's Theorem, we conclude that $C_D(K)$ is commutative. This implies that K is maximal in D and the result follows.

Considering Theorem 7 and Theorem 8, one is tempted to pose the following conjectures:

CONJECTURE 2. *Let D be a division ring and M a nilpotent maximal subgroup of D^* . Then D is commutative.*

CONJECTURE 3. *Let D be a division ring and M a soluble maximal subgroup of D^* . Then D is commutative.*

To present our next result, we need the following result from [3, p. 391].

THEOREM E. *Let D be a division ring, Σ a multiplicative subset of D and K its centralizer in D . If for all $c \in D^*$, $Kc\Sigma$ is infinite-dimensional as left K -space, then any non-zero multilinear element of $D_K \langle X \rangle$ has a non-zero value for some choice of values of X in Σ .*

We recall that given a multiplicative set Σ of D and a non-zero polynomial $p(x_1, \dots, x_n) \in F[x_1, \dots, x_n]$, we say that Σ satisfies $p(X)$ if $p(X) = 0$ for any $X = (x_1, \dots, x_n) \in \Sigma \times \dots \times \Sigma$. Let D be a division ring with centre F and G a subgroup of D^* . We shall say that G is *algebraic* if each element of G is algebraic over F .

THEOREM 9. *Let D be a division ring with centre F , and assume that M is a maximal subgroup of D^* such that $Z(M)$ is algebraic over F . If M satisfies a non-zero multilinear polynomial $p(X) \in F[X]$, then $[D : F] < \infty$.*

PROOF. By Proposition 1, we have either $F(M) = D$ or $F(M)^* = M$. If $F(M) = D$, it is clearly seen that $C_D(M) = F$. Thus, by Theorem E, there exists an element $c \in D^*$ such that FcM and consequently $F[M]$ is of finite dimension over F . Therefore, we obtain $D = F(M) = F[M]$ and so $[D : F] < \infty$. Thus, we may assume that $F(M)^* = M$ and put $D_1 = F(M)$. Since $D_1^* = M$ satisfies a multilinear non-zero polynomial identity, by Theorem E, we conclude that $[D_1 : F_1] < \infty$, where $F_1 = Z(D_1)$. Now,

we may consider two cases: (i) $C_D(D_1) = F$ or (ii) $C_D(D_1) \neq F$. In the first case we have $[D_1 : F] < \infty$. Thus $D_1 = C_D C_D(D_1) = C_D(F) = D$, by the double centralizer theorem, and consequently $[D : F] < \infty$. Finally, assume $C_D(D_1) \neq F$ and consider $a \in C_D(D_1) \setminus F$. By maximality of M , it is easily checked that $a \in Z(M)$ and so a is algebraic over F . Since D_1^* is maximal in D^* we have $D_1 \subset C_D(a)$ and thus $D_1 = C_D(a)$. Now, we have $D \otimes_F F(a) \cong M_r(C_D(a)) = M_r(D_1)$, where $r = [F(a) : F]$. Since $[D_1 : F_1] = s < \infty$ we may embed D_1 in $M_s(F_1)$ and so D is embeddable in $M_{rs}(F_1)$. Now, the Capelli polynomial is a polynomial identity for $M_s(F_1)$ and so is for D (cf. [12, p.441]). Thus, by Theorem E, we obtain the result.

It is believed that the condition on $Z(M)$ being algebraic over F is superfluous.

COROLLARY 10. *Let D be a division ring algebraic over its centre F such that $[D : F] = \infty$. Then D^* contains no commutative maximal subgroups.*

To prove our final result, we need the following theorem (cf. [15, p. 175] or [17]).

THEOREM F. *Let A be an algebraic algebra over a field F and G a locally solvable subgroup of A^* . Then $F(G)$ is locally finite dimensional.*

COROLLARY 11. *Let D be a division ring algebraic over its centre F . If D^* contains a soluble maximal subgroup, then D is locally finite.*

PROOF. If D is of finite dimension over F , there is nothing to prove. Thus, assume that $[D : F] = \infty$ and M is a maximal subgroup of D^* which is soluble. By Proposition 1, we have either $D = F(M)$ or $M = F(M)^*$. If the second case occurs, by Hua's Theorem we conclude that M is commutative. This contradicts Corollary 10. Therefore, we may assume that $D = F(M)$. Now, use Theorem F to complete the proof.

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