# WEDDERBURN'S FACTORIZATION THEOREM APPLICATION TO REDUCED $K$-THEORY

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Abstract. This article provides a short and elementary proof of the key theorem of reduced $K$-theory, namely Platonov's Congruence theorem. Our proof is based on Wedderburn's factorization theorem.

Let $D$ be a division algebra with center $F$. If $a \in D$ is algebraic over $F$ of degree $m$, then by Wedderburn's factorization theorem, one can find $m$ conjugates of $a$ such that the sum and the product of them are in $F$. This observation has been used in many different circumstances to give a short proof of known theorems of central simple algebras. (See [8] for a list of these theorems.) Here we will use this fact to prove Platonov's congruence theorem.

The non-triviality of the reduced Whitehead group $SK_1(D)$ was first shown by V. P. Platonov who developed a so-called *reduced K-theory* to compute $SK_1(D)$ for certain division algebras. The key step in his theory is the "congruence theorem" which is used to connect $SK_1(\overline{D})$ where $\overline{D}$ is a residue division algebra of $D$ to $SK_1(D)$. This in effect enables one to compute the group $SK_1(D)$ for certain division algebras. (See [5] and [6].)

Before we describe the congruence theorem, we employ Wedderburn's factorization theorem to obtain a result regarding normal subgroups of division algebras.

Suppose that $D$ has index $n$. Let $N$ be a normal subgroup of the group of units $D^*$ of $D$. Let $a \in N$ with the minimal polynomial $f(x) \in F[x]$ of degree $m$. Then from the theory of central simple algebras we have the following equality,

$$f(x)^{n/m} = x^n - Trd_{D/F}(a)x^{n-1} + \cdots + (-1)^n Nrd_{D/F}(a), \tag{1}$$

where $Nrd_{D/F} : D^* \to F^*$ is the reduced norm, $Trd_{D/F} : D \to F$ is the reduced trace and the right hand side of the equality is the reduced characteristic polynomial of $a$. (See [7], §9.)

*Key words and phrases.* Division algebra, Reduced $K$-theory, Congruence theorem, Reduced Whitehead group.

Using Wedderburn's factorization theorem for the minimal polynomial $f(x)$ of $a$, one obtains $f(x) = (x - d_1 a d_1^{-1}) \cdots (x - d_m a d_m^{-1})$ for certain $d_i \in D$. Now from the equality (1), it follows that $Nrd_{D/F}(a)$ is the product of $n$ conjugates of $a$. Since $N$ is a normal subgroup of $D^*$, it follows that $Nrd_{D/F}(a) \in N$. Therefore $Nrd|_N : N \longrightarrow Z(N)$ is well defined, where $Z(N) = F^* \cap N$ is the center of the group $N$.

Before stating the main Lemma, we fix some notation.

Let $\mu_n(F)$ denote the group of $n-th$ roots of unity in $F$ and let $Z(D')$ denote the center of the commutator subgroup $D'$ of $D^*$. Let $D^{(1)}$ stand for the kernel of the reduced norm. Observe that $\mu_n(F) = F^* \cap D^{(1)}$ and $Z(D') = F^* \cap D'$. If $G$ is a group, denote by $G^n$ the subgroup of $G$ generated by all $n-th$ powers of elements of $G$. If $H$ and $K$ are subgroups of $G$, denote by $[H, K]$ the subgroup of $G$ generated by all mixed-commutators $[h, k] = hkh^{-1}k^{-1}$, where $h \in H$ and $k \in K$.

We are now in a position to state our main lemma which is interesting in its own right.

**Lemma 1.** *Let $D$ be a division algebra with center $F$, of index $n$. Let $N$ be a normal subgroup of $D^*$. Then $N^n \subseteq (F^* \cap N)[D^*, N]$.*

*Proof.* Let $a \in N$. As stated above, using Wedderburn's factorization theorem, $Nrd_{D/F}(a) = d_1 a {d_1}^{-1} \cdots d_n a {d_n}^{-1}$. But

$$d_1 a {d_1}^{-1} \cdots d_n a {d_n}^{-1} = [d_1, a]a[d_2, a]a \cdots [d_n, a]a = a^n d_a$$

for some $d_a \in [D^*, N]$. This implies that $a^n = Nrd_{D/F}(a){d_a}^{-1}$. Therefore $N^n \subseteq (F^* \cap N)[D^*, N]$. $\square$

Let $N = D^*$. Then by above Lemma, for any $x \in D^*$, $x^n = Nrd_{D/F}(x)d_x$ where $d_x \in D'$. This shows that the group $G(D) = D^*/F^*D'$ is a torsion group of bounded exponent $n$. Some algebraic properties of this group are studied in [4].

In order to describe Platonov's congruence theorem, we need to recall some concepts from valued division algebras. Let $D$ be a finite dimensional division algebra with center a Henselian field $F$. Recall that a valuation $v$ on a field $F$ is called *Henselian* if and only if $v$ has a unique extension to each field algebraic over $F$. Therefore $v$ has a unique extension denoted also by $v$ to $D$ ([11]). Denote by $V_D, V_F$ the valuation rings of $v$ on $D$ and $F$ respectively and let $M_D, M_F$ denote their maximal ideals and $\overline{D}, \overline{F}$ their residue division algebra and residue field, respectively. We let $\Gamma_D, \Gamma_F$ denote the value groups of $v$ on $D$ and $F$, respectively and $U_D, U_F$ the groups of units of $V_D, V_F$ respectively. Furthermore, we assume that $D$ is a *tame* division algebra, i.e., $Char\overline{F}$ does not divide $i(D)$, the index of $D$.

Platonov's congruence theorem asserts that if $D$ is a tame division algebra over a Henselian field $F$ then $(1 + M_D) \cap D^{(1)} \subseteq D'$. This is the crucial theorem of reduced $K$-theory which is proved in [5] (Note that [5] provides a lengthy and complicated proof for the special case of a complete discrete valuation of rank 1, and [3] notes that the same proof works for general case of tame Henselian valued division algebras). Here we give a short and elementary proof of this fact.

**Theorem 2** *(Congruence Theorem). Let $D$ be a tame division algebra over a Henselian field $F = Z(D)$, of index $n$. Then $(1 + M_D) \cap D^{(1)} = [D^*, 1 + M_D]$.*

*Proof.* First we show that $(1+M_F)\cap D^{(1)} = 1$. Let $1+f \in 1+M_F$. If $1+f \in D^{(1)}$, then $(1+f)^n = 1$. But $v((1+f)^n - 1) = v(f)$. This shows that $f = 0$ and so our claim. Now take $N = 1 + M_D$. By Lemma 1,

$$(1 + M_D)^n \subseteq \Big((1 + M_D) \cap F^*\Big)\Big[D^*, (1 + M_D)\Big].$$

Since the valuation is tame and Henselian, Hensel's lemma shows that $(1+M_D)^n = 1 + M_D$. Therefore $1 + M_D = (1 + M_F)\Big[D^*, (1 + M_D)\Big]$. Now using the fact that $(1 + M_F) \cap D^{(1)} = 1$, the theorem follows. □

*Remark.* There is an elegant proof of the congruence theorem by A. Suslin in [9], in the case of a discrete valuation of rank 1. This proof uses substantial results from valuation theory and the fact that the group $SK_1(D)$ is torsion of bounded exponent $n = i(D)$. Using results of Ershov in [3], Suslin's proof can be written for arbitrary tame Henselian division algebras.

Having the congruence theorem, it is easy to see, in the case of discrete valuation of rank 1, that the sequence,

$$SK_1(\overline{D}) \to SK_1(D) \to L_1/L_{\sigma^{-1}} \to 1,$$

is exact where $L = Nrd(\overline{D})$, $L_1 = L \cap N^{-1}_{Z(\overline{D})/\overline{F}}(1)$ and $L_{\sigma^{-1}} =$ the image of $L$ under the homomorphism $a \mapsto \sigma(a)a^{-1}$, where $\langle\sigma\rangle = Gal(Z(\overline{D})/\overline{F})$. This leads to computations of $SK_1(D)$ for certain division algebras. (See [5], [6] and [9].)

Another look at the proof of Theorem 2 shows that $1 + M_D \subseteq (1 + M_F)D'$ and therefore $1 + M_D \subseteq U_F D'$. Put $G(\overline{D}) = \overline{D}^*/\overline{F}^*\overline{D'}$. In many applications, it is easy to obtain information about the residue data of division algebras. The following theorem gives an explicit formula for the group $SK_1(D)$ when the group $G(\overline{D})$ is trivial.

**Theorem 3.** *Let $D$ be a tame division algebra over a Henselian field $F = Z(D)$, of index $n$. If $G(\overline{D}) = 1$ then $SK_1(D) = \mu_n(F)/Z(D')$.*

*Proof.* The reduction map $U_D \longrightarrow \overline{D}^*$ induces an isomorphism $\overline{D}^* \longrightarrow U_D/1{+}M_D$, $\overline{a} \mapsto (1 + M_D)a$. Since $1 + M_D \subseteq U_F D'$, it follows that

$$\overline{D}^*/\overline{F}^*\overline{D'} \xrightarrow{\simeq} U_D/U_F D'.$$

Now if $G(\overline{D}) = \overline{D}^*/\overline{F}^*\overline{D'} = 1$ then $U_D = U_F D'$. But $D^{(1)} \subseteq U_D$. This shows that $D^{(1)} = \mu_n(F)D'$. Using the fact that $\mu_n(F) \cap D' = Z(D')$, the theorem follows. □

Note that Hensel's lemma implies that $\mu_n(F) \simeq \mu_n(\overline{F})$. In particular if $D$ is a totally ramified division algebra, i.e. $\overline{D} = \overline{F}$, then $G(\overline{D}) = 1$.

**Example 4.** Let $\mathbb{C}$ be the field of complex numbers and $r$ be a nonnegative integer. Let $D_1 = \mathbb{C}((x_1))$ and define $\sigma_1 : D_1 \to D_1$ by the rule $\sigma_1(x_1) = -x_1$. Now let $D_2 = D_1((x_2, \sigma_1))$ and set $D_3 = D_2((x_3))$. Again define $\sigma_3 : D_3 \to D_3$ by $\sigma_3(x_3) = -x_3$. In general, if $i$ is even, set $D_{i+1} = D_i((x_{i+1}))$ and if $i$ is odd define $\sigma_i : D_i \to D_i$ by $\sigma_i(x_i) = -x_i$ and $D_{i+1} = D_i((x_{i+1}, \sigma_i))$. By Hilbert's construction (see [1], §1 and §24), $D = D_{2r} = \mathbb{C}((x_1, \cdots, x_{2r}, \sigma_1, \cdots, \sigma_{2r-1}))$ is a division algebra with center $F = \mathbb{C}((x_1^2, x_2^2, \cdots, x_{2r-1}^2, x_{2r}^2))$ and $n = i(D) = 2^r$. Finally define $v : D^* \to \Gamma_D = Z^{2r}$ by the rule $v(\sum c_i x_1^{i_1} \cdots x_{2r}^{i_{2r}}) = (i_1, \cdots, i_{2r})$ where $i_1, \cdots, i_{2r}$ are the smallest powers of the $x_i$'s in the lexicographic order. It can be observed that $v$ is a tame valuation and $\overline{D} = \mathbb{C}$ and $\overline{F} = \mathbb{C}$. Therefore $G(\overline{D}) = 1$. Theorem 3 implies that $SK_1(D) = \mu_n(F)/Z(D')$. From the multiplication rule in $D$, it follows that

$$D' \subseteq \Big\{ \pm 1 + \sum_{i>0} c_i x_1^{i_1} \cdots x_{2r}^{i_{2r}} \Big\}.$$

Since $Z(D') \subseteq \mu_n(F)$, it follows that $Z(D') = \{1, -1\}$. But $\mu_n(F) = \mu_n(\overline{F}) = \mathbb{Z}_{2^r}$, hence $SK_1(D) = \mathbb{Z}_{2^{r-1}}$.

In [4], as another application of Lemma 1, we obtain theorems of reduced $K$-theory which previously required heavy machinery, as simple consequence of this approach.

*Acknowledgments.* I wish to thank Anthony Bak who read the note and made numerous corrections.

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