THE DISCRIMINANT OF A DECOMPOSABLE SYMPLECTIC INVOLUTION

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Abstract. A formula is given for the discriminant of the tensor product of the canonical involution on a quaternion algebra and an orthogonal involution on a central simple algebra of degree divisible by 4. As an application, an alternative proof of Shapiro's "Pfister Factor Conjecture" is given for tensor products of at most five quaternion algebras.

Throughout this paper, the characteristic of the base field F is supposed to be different from 2. Recall from [4, (2.5)] that a symplectic (resp. orthogonal) involution on a central simple F-algebra A is a map $A \to A$ which after scalar extension to a splitting field K can be identified with the adjoint involution of a nonsingular alternating (resp. symmetric) bilinear form over a K-vector space. The discriminant of an orthogonal involution ρ on a central simple F-algebra of even degree was defined by Jacobson, Tits, and Knus–Parimala–Sridharan, see [4, §7]. It is an element of the Galois cohomology group $H^1(F, \mu_2) \simeq F^\times/F^{\times 2}$ which we denote by $\operatorname{disc} \rho$.

For symplectic involutions σ, σ_0 on a central simple F-algebra A of degree divisible by 4, a relative discriminant $\Delta_{\sigma_0}(\sigma) \in H^3(F, \mu_2)$ related to the Rost invariant of symplectic groups is defined in [3]. (The definition is recalled at the beginning of Section 1.) If the Schur index $\operatorname{ind} A$ divides $\frac{1}{2} \deg A$, then A carries a hyperbolic involution σ_0, and we write simply $\Delta(\sigma)$ for $\Delta_{\sigma_0}(\sigma)$. Our main result relates the discriminants of symplectic and orthogonal involutions as follows:

Main Theorem. *Suppose B is a central simple F-algebra of degree divisible by 4 with orthogonal involution ρ and Q is a quaternion F-algebra with canonical (symplectic) involution γ. Let $A = B \otimes_F Q$ and $\sigma = \rho \otimes \gamma$, a symplectic involution on A. If $\operatorname{ind} A$ divides $\frac{1}{2} \deg A$, then, denoting by $[Q] \in H^2(F, \mu_2)$ the cohomology class associated with Q,*

$$\Delta(\sigma) = (\operatorname{disc} \rho) \cup [Q].$$

In the particular case where A is a tensor product of three quaternion algebras and σ is the tensor product of the canonical involutions, it follows that $\Delta(\sigma) = 0$. We then use a theorem of Berhuy–Monsurrò–Tignol to derive a decomposition of σ where one of the factors is an orthogonal involution on a split algebra of degree 2, see Section 1.

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In [9, (9.17)], Shapiro proposes the following conjecture:

PC(m) Let (Q_1, σ_1), ..., (Q_m, σ_m) be quaternion F-algebras with involutions of orthogonal or symplectic type such that $\sigma_1 \otimes \cdots \otimes \sigma_m$ is orthogonal. If $Q_1 \otimes \cdots \otimes Q_m$ is split, then $\sigma_1 \otimes \cdots \otimes \sigma_m$ is adjoint to an m-fold Pfister form.

As shown in [9, p. 166], this conjecture is equivalent to the "Pfister factor conjecture" on spaces of similarities discussed by Shapiro in [8, §7] and proved (by techniques from the algebraic theory of quadratic forms) for $m \leq 5$ and over number fields by Wadsworth–Shapiro in [11]. In the final section, we show how the decomposition result of Section 1 for involutions on triquaternion algebras yields another proof of PC(m) for $m \leq 5$ and also for all m over some special fields. A different proof for $m \leq 4$ using central simple algebras with involution was found by Bayer-Fluckiger–Parimala [1].

1. Discriminants

Let A be a central simple algebra over a field F of characteristic different from 2. For $u \in A^\times$, we denote by $\mathrm{Int}(u)$ the inner automorphism induced by u,

$$\mathrm{Int}(u)(x) = uxu^{-1} \qquad \text{for } x \in A.$$

If σ, σ_0 are symplectic involutions on A, then $\sigma = \mathrm{Int}(u) \circ \sigma_0$ for some $u \in A^\times$ such that $\sigma_0(u) = u$ (see [4, (2.7)]). We may then consider the pfaffian norm $\mathrm{Nrp}_{\sigma_0} u \in F^\times$ (which is a square root of the reduced norm $\mathrm{Nrd}_A u$, see [4, p. 19]) and the corresponding cohomology class $(\mathrm{Nrp}_{\sigma_0} u) \in H^1(F, \mu_2)$ under the canonical isomorphism $F^\times/F^{\times 2} \simeq H^1(F, \mu_2)$. If $\deg A \equiv 0 \bmod 4$, the cup product

$$(\mathrm{Nrp}_{\sigma_0} u) \cup [A] \in H^3(F, \mu_2)$$

depends only on σ and σ_0, and not on the choice of u. It is called the *discriminant of σ relative to σ_0* and denoted by $\Delta_{\sigma_0}(\sigma)$, see [3].

Proof of the Main Theorem. The main point is to establish the existence of an orthogonal involution ρ_0 on B such that $\operatorname{disc} \rho_0 = 1$ and $\rho_0 \otimes \gamma$ is hyperbolic. We then have

$$\rho = \mathrm{Int}(u) \circ \rho_0 \qquad \text{for some } u \in B^\times, \tag{1}$$

hence $\sigma = \mathrm{Int}(u \otimes 1) \circ (\rho_0 \otimes \gamma)$ and therefore

$$\Delta(\sigma) = \Delta_{\rho_0 \otimes \gamma}(\sigma) = \big(\mathrm{Nrp}_{\rho_0 \otimes \gamma}(u \otimes 1)\big) \cup [A]. \tag{2}$$

We have $\mathrm{Nrp}_{\rho_0 \otimes \gamma}(u \otimes 1) = \mathrm{Nrd}_B u$ by [3, Lemma 9] and $(\mathrm{Nrd}_B u) \cup [B] = 0$, hence (2) yields

$$\Delta(\sigma) = (\mathrm{Nrd}_B u) \cup [Q].$$

On the other hand, (1) implies that $\operatorname{disc} \rho = (\mathrm{Nrd}_B u)$ since $\operatorname{disc} \rho_0 = 1$ (see [4, (7.3)]), and the theorem follows.

To prove the existence of ρ_0, we consider two cases:

Case 1: If $\operatorname{ind} B$ divides $\frac{1}{2} \deg B$, then we may take for ρ_0 an orthogonal hyperbolic involution.

Case 2: If $\operatorname{ind} B$ does not divide $\frac{1}{2} \deg B$, then we may identify $B = M_r(F) \otimes D$ for some central division F-algebra D and some odd integer r. Since $\operatorname{ind} A$ divides $\frac{1}{2} \deg A$, the tensor product $D \otimes Q$ is not a division algebra. By a theorem of Risman [6], it follows that D contains a splitting field L of Q. The extension

L/F is separable since the exponent of D is 2 and the characteristic is not 2. Since $\deg B$, hence also $\deg D$, is divisible by 4, a theorem of Parimala–Sridharan–Suresh [5, Theorem 2.1] shows that D carries an involution ρ_1 of discriminant 1 which leaves invariant a generating element of L. Since L splits Q, the involution $\mathrm{Id}_L \otimes \gamma$ on $L \otimes Q$ is hyperbolic, hence there is an idempotent $e \in L \otimes Q$ such that $(\mathrm{Id}_L \otimes \gamma)(e) = 1 - e$, see [2, Theorem 2.1] or [4, (6.7)]. Since $L \otimes Q \subset D \otimes Q$, we also have $e \in D \otimes Q$ and $(\rho_1 \otimes \gamma)(e) = 1 - e$, hence $\rho_1 \otimes \gamma$ is hyperbolic. If t denotes the transpose involution on $M_r(F)$, the involution $\rho_0 = t \otimes \rho_1$ on $B = M_r(F) \otimes D$ satisfies the required conditions and the proof is complete. □

Remark. The formula for $\Delta(\sigma)$ in the main theorem does not hold if $\deg B$ is not divisible by 4. For instance, if B is a quaternion algebra, $B = (a, b)_F$ and $Q = (a, c)_F$ for some a, b, $c \in F^\times$, then for any orthogonal involution ρ on B,

$$\Delta(\rho \otimes \gamma) = (b \operatorname{disc} \rho) \cup [Q].$$

The proof is omitted.

Corollary. *Let A be a central simple F-algebra of degree 8 and let σ be a symplectic involution on A. If A is not a division algebra, the following conditions are equivalent:*

(a) $\Delta(\sigma) = 0$;

(b) *there are quaternion F-algebras with involution (Q_1, σ_1), (Q_2, σ_2), (Q_3, σ_3) such that*

$$(A, \sigma) = (Q_1, \sigma_1) \otimes (Q_2, \sigma_2) \otimes (Q_3, \sigma_3);$$

(c) *there is a 2-dimensional quadratic form π (which, upon scaling, may be assumed to be a 1-fold Pfister form) with adjoint involution ad_π on $M_2(F)$, and quaternion algebras B, C with an orthogonal involution ρ and canonical (symplectic) involution γ respectively, such that*

$$(A, \sigma) = (M_2(F), \mathrm{ad}_\pi) \otimes (B, \rho) \otimes (C, \gamma).$$

Proof. The equivalence (a) $\iff$ (c) is proved in [3, Theorem 8], and (c) $\Rightarrow$ (b) is clear. To complete the proof, it suffices to show (b) $\Rightarrow$ (a).

Suppose (b) holds. Since σ is symplectic, at least one of σ_1, σ_2, σ_3 is symplectic. Renumbering, we may assume σ_3 is symplectic. The main theorem yields

$$\Delta(\sigma) = (\operatorname{disc} \sigma_1 \otimes \sigma_2) \cup [Q_3].$$

Since $\operatorname{disc} \sigma_1 \otimes \sigma_2 = 1$ by [4, (7.3)], condition (a) follows. □

Note that in degree 4 the special decomposition in (c) of the preceding corollary always holds (independently of the value of Δ), as we now show:

Proposition. *Let A be a central simple F-algebra of degree 4 and let σ be a symplectic involution on A. If A is not a division algebra, then there is a 2-dimensional quadratic form π and a quaternion algebra with canonical involution (Q, γ) such that*

$$(A, \sigma) = (M_2(F), \mathrm{ad}_\pi) \otimes (Q, \gamma).$$

Proof. Since A is not a division algebra, it is Brauer-equivalent to a quaternion algebra Q, and it can be represented as $A = \operatorname{End}_Q V$ for some free Q-module V of rank 2. The involution σ is adjoint to a hermitian form h on V with respect to the canonical involution γ on Q. Let e_1, e_2 be an orthogonal basis of V. We have

$h(e_1, e_1)$, $h(e_2, e_2) \in F$, and the F-span of e_1, e_2 is an F-vector space V_0 on which $\pi(x) = h(x, x)$ is a quadratic map. We have

$$A = \operatorname{End}_Q V = (\operatorname{End}_F V_0) \otimes Q,$$

and the involution σ has a corresponding decomposition $\sigma = \operatorname{ad}_\pi \otimes \gamma$. □

2. Pfister factor conjecture

If ρ_1, ρ_2 are orthogonal involutions on quaternion F-algebras Q_1, Q_2, then there are quaternion F-algebras Q'_1, Q'_2 with canonical involutions γ_1, γ_2 such that

$$(Q_1, \rho_1) \otimes (Q_2, \rho_2) \simeq (Q'_1, \gamma_1) \otimes (Q'_2, \gamma_2),$$

see [4, (15.12)]. (The algebras Q'_1 and Q'_2 are the direct factors of the Clifford algebra $C(Q_1 \otimes Q_2, \rho_1 \otimes \rho_2)$. If i_α, $j_\alpha \in Q_\alpha^\times$ are such that $\rho_\alpha(i_\alpha) = -i_\alpha = j_\alpha i_\alpha j_\alpha^{-1}$ for $\alpha = 1$, 2, then the algebras Q'_1 and Q'_2 are generated by $i_1 \otimes 1$, $j_1 \otimes i_2$ and $1 \otimes i_2$, $i_1 \otimes j_2$ respectively.) Therefore, as noticed by Bayer-Fluckiger–Parimala [1], in discussing Conjecture PC(m) we may assume that all the involutions σ_i, except at most one, are canonical.

For any quaternion F-algebra Q with canonical involution γ, there is an isomorphism $Q \otimes Q \xrightarrow{\sim} \operatorname{End}_F Q$ which maps $x \otimes y$ to the linear transformation $z \mapsto xz\gamma(y)$. Under that isomorphism, $\gamma \otimes \gamma$ corresponds to the adjoint involution with respect to the norm form of Q, see [4, (11.1)]. This proves PC(2) since the norm form is a 2-fold Pfister form. Alternatively, from an isomorphism

$$(Q_1, \sigma_1) \otimes (Q_2, \sigma_2) \simeq (M_4(F), \operatorname{ad}_q)$$

it follows that $\operatorname{disc} q = 1$, hence q is a 2-fold Pfister form up to a scalar.

Lemma 1. PC(3) *holds.*

Proof. Suppose (Q_1, σ_1), (Q_2, σ_2), (Q_3, σ_3) are quaternion algebras with involution such that $Q_1 \otimes Q_2 \otimes Q_3$ is split and $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$ is orthogonal. As observed at the beginning of this section, when aiming to prove that $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$ is adjoint to a Pfister form, we may assume that $\sigma_1 = \gamma_1$ and $\sigma_2 = \gamma_2$ are symplectic, hence σ_3 is orthogonal. Then $Q_2 \otimes Q_3 \simeq M_2(Q_1)$ and $\sigma_2 \otimes \sigma_3$ is symplectic, hence the proposition at the end of the preceding section yields a decomposition

$$(Q_2, \sigma_2) \otimes (Q_3, \sigma_3) = (M_2(F), \operatorname{ad}_\pi) \otimes (Q_1, \gamma_1)$$

for some 2-dimensional form π. Substituting in $Q_1 \otimes Q_2 \otimes Q_3$, we get

$$(Q_1, \sigma_1) \otimes (Q_2, \sigma_2) \otimes (Q_3, \sigma_3) \simeq (M_2(F), \operatorname{ad}_\pi) \otimes (Q_1, \gamma_1) \otimes (Q_1, \gamma_1).$$

Now, as observed in the proof of PC(2) above, $\gamma_1 \otimes \gamma_1$ is adjoint to the reduced norm form Nrd_{Q_1} of Q_1, which is a 2-fold Pfister form, hence

$$\sigma_1 \otimes \sigma_2 \otimes \sigma_3 \simeq \operatorname{ad}_\pi \otimes \operatorname{ad}_{\operatorname{Nrd}_{Q_1}} \simeq \operatorname{ad}_{\pi \otimes \operatorname{Nrd}_{Q_1}}$$

and the proof is complete. □

Remark. A different proof, using triality, was found by David Tao [10].

To prove PC(m) for $m \geq 4$, the following lemma is useful.

Lemma 2. *Let* $m \geq 4$ *and suppose* PC($m-1$) *holds over* F. *Let* (Q_1, σ_1), ..., (Q_m, σ_m) *be quaternion* F*-algebras with involution such that* $Q_1 \otimes \cdots \otimes Q_m$ *is split and* $\sigma_1 \otimes \cdots \otimes \sigma_m$ *is orthogonal. If* $Q_1 \otimes Q_2 \otimes Q_3$ *is not a division algebra and* $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$ *is symplectic, then* $\sigma_1 \otimes \cdots \otimes \sigma_m$ *is adjoint to an* m*-fold Pfister form.*

Proof. By the corollary of Section 1, there exist quaternion F-algebras with involution (B,ρ), (C,γ) and a 2-dimensional quadratic form π with adjoint involution ad_π on $M_2(F)$ such that

$$(Q_1,\sigma_1)\otimes(Q_2,\sigma_2)\otimes(Q_3,\sigma_3)\simeq(M_2(F),\mathrm{ad}_\pi)\otimes(B,\rho)\otimes(C,\gamma).$$

Since multiplying π by a scalar does not change the adjoint involution, we may assume π is a 1-fold Pfister form.

Now

$$(Q_1,\sigma_1)\otimes\cdots\otimes(Q_m,\sigma_m)\simeq \\ (M_2(F),\mathrm{ad}_\pi)\otimes(B,\rho)\otimes(C,\gamma)\otimes(Q_4,\sigma_4)\otimes\cdots\otimes(Q_m,\sigma_m).$$

The product $B\otimes C\otimes Q_4\otimes\cdots\otimes Q_m$ is split since $Q_1\otimes\cdots\otimes Q_m$ is split, and by $\mathrm{PC}(m-1)$ the product $\rho\otimes\gamma\otimes\sigma_4\otimes\cdots\otimes\sigma_m$ is adjoint to an $(m-1)$-fold Pfister form π'. Therefore, $\sigma_1\otimes\cdots\otimes\sigma_m$ is adjoint to the m-fold Pfister form $\pi\otimes\pi'$. □

Theorem. $\mathrm{PC}(m)$ *holds for* $m\leq 5$. *It holds for all* m *over fields* F *which have the following property:*

(∗) *Every tensor product of three quaternion* F*-algebras contains zero-divisors.*

Proof. We have already seen that $\mathrm{PC}(m)$ holds for $m=2$ and $m=3$. As observed at the beginning of this section, when proving $\mathrm{PC}(m)$ we may assume that all the quaternion involutions $\sigma_1, \dots, \sigma_m$ are symplectic, except at most one. Therefore, for $m\geq 4$ we may assume

$\sigma_1\otimes\sigma_2\otimes\sigma_3$ is symplectic. If

$m\leq 5$, the condition that $Q_1\otimes\cdots\otimes Q_m$ is split implies that $\mathrm{ind}(Q_1\otimes Q_2\otimes Q_3)\leq 4$. Since PC(3) holds, the lemma successively yields PC(4) and PC(5).

If F satisfies (∗), then $\mathrm{ind}(Q_1\otimes Q_2\otimes Q_3)\leq 4$ for all tensor products $Q_1\otimes\cdots\otimes Q_m$, and the lemma inductively yields $\mathrm{PC}(m)$ for all $m\geq 4$. □

Number fields and, more generally, linked fields in the sense of Elman–Lam (see [9, (9.14)]), satisfy (∗). Indeed, these fields satisfy the more restrictive condition that no tensor product of two quaternion algebras is a division algebra, hence we may inductively use the arguments in the proof of Lemma 1 instead of applying the theorem.

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