# ON TRIVIALITY OF THE FUNCTOR Coker$(K_1(F) \to K_1(D))$ FOR DIVISION ALGEBRAS

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ABSTRACT. Let $D$ be a cyclic division algebra over its centre $F$ of index $n$. Consider the group $CK_1(D) = D^*/F^*D'$ where $D^*$ is the group of invertible elements of $D$ and $D'$ is its commutator subgroup. In this note we shall show that the group $CK_1(D)$ is trivial if and only if $D$ is an ordinary quaternion division algebra over a real Pythagorean field $F$. We construct a division algebra $D$ and a division subalgebra $A \subset D$ such that $CK_1(A) \cong CK_1(D)$. Using valuation theory, the group $CK_1(D)$ is computed for some valued division algebras.

## 1. INTRODUCTION

Let $A$ be an Azumaya algebra over a commutative ring $R$. Consider the functor $CK_1(A) = \text{Coker}(K_1(R) \xrightarrow{i} K_1(A))$ where $i$ is the inclusion map. If $R$ is semilocal, since the stable rank of $R$ and $A$ are one, $CK_1(A) = A^*/R^*A'$ where $A^*$ and $R^*$ are the group of invertible elements of $A$ and $R$ respectively and $A'$ is the derived subgroup of $A^*$. A study of this group in the case of central simple algebras is initiated in [6] and further in [5]. It has been established that despite of a "different nature" of this group from the reduced Whitehead group $SK_1$, these two groups have the same functorial properties. In [6] this functor is determined for totally ramified division algebras and in particular for any finite abelian group $G$, a division algebra $D$ such that $CK_1(D) = G \times G$ is constructed. Further in [5], this functor is studied in more cases and examples of cyclic $CK_1$ (even over non local fields) constructed. It was conjectured in [6] that if the functor $CK_1$ is trivial then the index of the division algebra is 2. In this note we characterise quaternion division algebras $D$ such that $CK_1(D)$ is trivial. In fact triviality of $CK_1$ characterises the ground field. We shall show that if $CK_1(D)$, for a quaternion division algebra $D$, is trivial then $D$ is an ordinary quaternion and the centre of $D$ is a real Pythagorean field. In fact, this is the only class of cyclic division algebras such that $CK_1$ is trivial (Theorem 2.3). We shall show that some of the notions from the theory of quadratic forms, like rigidity of an element, can be formulated as a property of the group $CK_1$. It seems that this group is highly sensitive to the arithmetic property of the ground field. For example we will observe that $CK_1(\frac{x,x}{\mathbb{R}((x))}) \cong \mathbb{Z}_2$ whereas $CK_1(\frac{x,x}{\mathbb{F}_3((x))}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

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## 2. Triviality of $CK_1$

Let $F$ be a field with characteristic not 2. Let $(\frac{a,b}{F})$ be a quaternion division algebra over $F$. The elements of $D = (\frac{a,b}{F})$ have the form $c_0 + c_1 i + c_2 j + c_3 ij$ and $Nrd_D(c_0 + c_1 i + c_2 j + c_3 ij) = c_0^2 - ac_1^2 - bc_2^2 + abc_3^2$ where $Nrd_D$ is the reduced norm map for $D$. Note that $CK_1(D) = D^*/F^*D'$ where $D^*$ and $F^*$ are the group of invertible elements of $D$ and $F$ respectively and $D'$ is the commutator subgroup of $D^*$. If $CK_1(D) = 1$ then $Nrd_D(D) = F^{*2}$. Thus $Nrd_D(i) = -a \in F^2$ and $Nrd_D(j) = -b \in F^2$. Therefore $(\frac{a,b}{F}) \cong (\frac{-1,-1}{F})$.

Our first theorem shows that triviality of $(\frac{a,b}{F})$ forces that $F$ be a real Pythagorean field. In fact this property characterises this field. Recall that $F$ is a real Pythagorean if $-1 \notin F^{*2}$ and sum of any two square elements is a square in $F$. It follows immediately that $F$ is an ordered field. $F$ is called Euclidean if $F^{*2}$ is an ordering of $F$

**Theorem 2.1.** *Let $F$ be a field of characteristic not 2. Then the following are equivalent.*

*1) $(\frac{-1,-1}{F})$ is a division ring and $CK_1(\frac{-1,-1}{F})$ is trivial.*

*2) $F$ is the real Pythagorean field.*

*3) $(\frac{-1,-1}{F})$ is a division ring and every maximal subfield of $(\frac{-1,-1}{F})$ is $F$-isomorphic to $F(\sqrt{-1})$.*

*Proof.* We shall show that 1 and 2 are equivalent. The equivalency of 2 and 3 are known (see [2]).

1) $\Rightarrow$ 2) Since $D = (\frac{-1,-1}{F})$ is a division ring, $-1 \notin F^{*2}$. By induction, if $-1 = {f_1}^2 + \cdots + {f_n}^2$, then considering ${f_1}^2 + {f_2}^2 = Nrd_D(f_1 + f_2 i) = f^2$, a contradiction follows. This shows that $-1$ is not sum of squares plus sum of any two squares is a square

2) $\Rightarrow$ 1) Suppose $F$ is a real Pythagorean. It is easy to see that $D = (\frac{-1,-1}{F})$ is a division ring. Now for any $x \in D^*$, $Nrd_D(x)$ is a sum of four squares, thus $Nrd_D(D^*) = F^{*2}$. Since $SK_1(D)$ is trivial, this forces that $CK_1$ be trivial. □

Over Euclidean fields, the only quaternion division algebra is the ordinary quaternion division algebra, and from the above Theorem it follows that its $CK_1$ is trivial.

We shall show that an ordinary quaternion division algebra over a real Pythagorean field is in fact the only type of cyclic division algebras such that its $CK_1$ is trivial. Before we state the theorem, note that there are examples of infinite dimensional division algebras $D$ such that $D^*$ coincide with $D'$ [7]. In the finite dimensional case, it is not hard to see that $D^* \neq D'$, infact $K_1(D) = D^*/D'$ is not torsion. In the case of algebraic division algebras, everything remains as a mystery.

We need the following theorem which says that the functor $CK_1$ has a decomposition property similar to one for the reduced Whitehead group.

**Theorem 2.2.** [5] *Let $A$ and $B$ be division algebras with centre $F$ such that $(i(A), i(B)) = 1$. Then $CK_1(A \otimes_F B) \cong CK_1(A) \times CK_1(B)$.*

**Theorem 2.3.** *Let $D$ be a cyclic division algebra finite dimension over its centre $F$. Then $CK_1(D) = 1$ if and only if $D$ is an ordinary quaternion division algebra over a real Pythagorean field $F$.*

*Proof.* The only if part is already proved in Theorem 2.1. For the if part, we shall show that the index of $D$ is 2. Then the claim follows again from Theorem 2.1. First suppose that $i(D) = n$ is odd. Let $\alpha \in D$ such that $F(\alpha)$ is a maximal cyclic subfield of $D$ with a mininal polynomial of the form $x^n - f$ where $f \in F$. Since $D^* = F^*D'$, we can choose $\alpha \in D'$. Now $\alpha^n = N_{F(\alpha)/F}(\alpha) = Nrd_D(\alpha) = 1$. From this it follows that the degree of the minimal polynomial of $\alpha$ is less than $n$ which can not happen. Since $CK_1(D)$ has a decomposition property (Theorem 2.2), the only case which remains, is $n = 2^k$, for some integer $k$. We shall show that $k = 1$. Again consider $\alpha$ as above. This time $-\alpha^{2^k} = N_{F(\alpha)/F}(\alpha) = Nrd_D(\alpha) = 1$. On the other hand since $1 + N_{F(\alpha)/F}(\alpha) = N_{F(\alpha)/F}(\alpha + 1) \in F^{*2^k}$, it follows that $\sqrt{2} \in F$. Thus if $k > 1$ then $\alpha^{2^k} + 1 = (\alpha^{2^{k-1}} + 1)^2 - 2\alpha^{2^{k-1}}$ can be decomposed further which leads to a contradiction that the minimal polynomial of $\alpha$ has degree less than $n$. Therefore $n = 2$ and the claim follows from Theorem 2.1 □

Note that in the above theorem we could replace $CK_1$ with a smaller group $D^*/F^*D^{(1)}$, where $D^{(1)}$ is set of elements with reduced norm 1.

The above theorem states that if in the case of division algebra $D$ of odd prime index, $CK_1(D)$ happens to be trivial, then this forces the well-known conjecture that any division algebra of prime index is cyclic would be false!

It is a celeberated result of Wedderburn that a division algebra of index 3 is cyclic. This together with Theorem 2.2 and Theorem 2.3 implies that

**Theorem 2.4.** *If $D$ be a division algebra of index $3n$ where $3 \nmid n$, then $CK_1(D) \neq 1$.*

*Remark* 2.5. Let $D$ be a finite dimensional division algebra with centre $F$, of index $n$. Consider the sequence

$$K_1(D) \xrightarrow{Nrd} K_1(F) \xrightarrow{i} K_1(D) \tag{1}$$

Where $Nrd_D$ is the reduced norm map and $i$ is the inclusion map. One can see that the composition $i\,Nrd_{D/F} = \eta_n$ where $\eta_n(a) = a^n$. (See for example the proof of Lemma 4, p. 157 [1]). From this the formula $a^n = Nrd_{D/F}(a)c_a$ where $a \in D^*$ and $c_a \in D'$ follows. In particular this implies that the exponent of the abelian group $CK_1(D)$ divides $n$, the index of $D$.

It is not known if $exp(CK_1(D)) < i(D)$ what would be imposed on the algebraic structure of $D$. We mention that if $D$ is a totally ramified division algebra then $exp(CK_1(D)) = i(D)$ if and only if $D$ is cyclic [5]. In fact from the above theorem it follows that if $D$ is a cyclic division algebra of index $p$, an odd prime, then the exponent of $CK_1(D)$ is exactly $p$. The converse is not true, as the following example shows. This is an example of a cyclic decomposable $F$-division algebra $D$ of index $2p$, $p$ an odd prime, such that it has a proper $F$-division subalgebra $A \subset D$, where $CK_1(A) \cong CK_1(D)$. This in particular shows that the exponent of $CK_1(D)$ is less than the index of $D$, but $D$ is cyclic, thats is, the phenomena of cyclicity of $D$ and $exp(CK_1(D))$

does not follow the same pattern as in the case of totally ramified. For this we need the Fein-Schacher-Wadsworth example of a division algebra of index $2p$ over a Pythagorean field $F$ [3]. We briefly recall the construction. Let $p$ be an odd prime and $K/F$ be a pair of real Pythagorean fields such that $K$ is cyclic over $F$ with the generating automorphism $\sigma$. Now consider

$$D = \Big(\frac{-1,-1}{F((x))}\Big) \otimes_{F((x))} \Big(K((x))/F((x)), \sigma, x\Big).$$

$K((x))/F((x))$ is a cyclic extension where $K((x))$ and $F((x))$ are Laurant power series fields of $K$ and $F$ respectively. $D$ is a division algebra of index $2p$. Since $F$ is real Pythagorean, so does $F((x))$. Now by Theorem 2.2 the primary decomposition of the division algebra $D$ induces a corresponding decomposition for $CK_1(D)$. Thus $CK_1(D) \cong CK_1(\frac{-1,-1}{F((x))}) \times CK_1(A)$ where $A = \big(K((x))/F((x)), \sigma, x\big)$. By Theorem 2.1, $CK_1(\frac{-1,-1}{F((x))}) = 1$, thus $CK_1(D) \cong CK_1(A)$.

**Question.** If $exp(CK_1(D))$ is less than the index $i(D)$, what can be said about the algebraic structure of $D$.

Recall that $a \in F$ is called *rigid* if $a \not\in \pm F^{*2}$ and $F^{*2} + aF^{*2} = F^{*2} \cup aF^{*2}$. The rigidity concept plays a role in the study of the extension of Pythagorean fields. In particular if $K = F(\sqrt{a})$ is a quadratic extension of $F$, then $K$ is real Pythagorean if and only if $F$ is real Pythagorean and $a$ is rigid [8].

**Theorem 2.6.** *Let $F$ be a real Pythagorean field. Let $a \not\in \pm F^{*2}$. Then $a$ is rigid if and only if $CK_1(\frac{-1,-a}{F}) = \mathbb{Z}_2$.*

*Proof.* Suppose $a$ is rigid. Since $-a \not\in F^{*2}$, then $D = (\frac{-1,-a}{F})$ is a division ring. Since $F$ is Pythagorean, $Nrd_D(D^*) = c_0^2 + ac_1^2$, where $c_0, c_1 \in F^*$. From rigidity of $a$ it follows that $Nrd_D(D^*) = F^{*2} \cup aF^{*2}$. Because $a \not\in F^{*2}$, then $F^{*2}$ and $F^{*2}a$ are two distinct elements in the group $Nrd_D(D^*)/F^{*2}$. Since $SK_1(D)$ is trivial, $CK_1(D) \cong Nrd_D(D^*)/F^{*2} \cong \mathbb{Z}_2$. The if part follows the same way. $\square$

*Remark* 2.7. Let $F$ be a real Pythagorean field and $a \in F^*$ such that $-a \not\in F^{*2}$. Then $CK_1(\frac{-1,-a}{F}) \cong L^*/F^*L^{*2}$ where $L = F(\sqrt{-a})$. For consider the square class exact sequence

$$1 \longrightarrow \{F^{*2}, F^{*2}a\} \longrightarrow F^*/F^{*2} \longrightarrow L^*/L^{*2} \xrightarrow{N} N_{L/F}(L^*)/F^{*2} \longrightarrow 1.$$

In this setting, it is easy to see that $N_{L/F}(L^*) = Nrd_{D/F}(D^*)$ thus the claim.

## 3. $CK_1$ for valued quaternion division algebras

In this section we study the valued quaternion division algebras. Valuation theory for division algebras is the main tool for computation of $CK_1$. Recall that if $D$ is equipped with a valuation, then $\bar{D}$ and $\bar{F}$ denote the residue division algebra and the residue field of $D$ and $F$ respectively. $\Gamma_D$ and $\Gamma_F$ are the value groups of the valuation.

A tame valued quaternion division algebra could be unramified, semiramified or totally ramified. This follows from the fact that $D$ is defectless. Recall that $D$ is *tame* if $char(\bar{F})$ does not divide the index of $D$. Thus $D$

is *defectless*, namely $[\bar{D} : \bar{F}][\Gamma_D : \Gamma_F] = [D : F]$. $D$ is called *unramified* if $[\bar{D} : \bar{F}] = [D : F]$ and *totally ramified* if $[\Gamma_D : \Gamma_F] = [D : F]$. $D$ is called *semiramified* if $\bar{D}$ is a field and $[\bar{D} : \bar{F}] = [\Gamma_D : \Gamma_F] = i(D)$. For an excellent survey of the valuation theory of division algebras see [10].

We will see that there are quaternion division algebras of any type mentioned above.

- $D = (\frac{-1,-1}{\mathbb{R}((x))})$ is unramified, and $CK_1(D) = 1$ (see Theorem 2.1).
- $D = (\frac{-1,x}{\mathbb{R}((x))})$ is semiramified, and $CK_1(D) = \mathbb{Z}_2$ (see Theorem 3.2).
- $D = (\frac{x,y}{\mathbb{R}((x,y))})$ is totally ramified, and $CK_1(D) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (see Remark 3.3).

Although all the examples that we are dealing with here have finite $CK_1$, let us mention that $CK_1$ of a quaternion division algebra could be infinite. It is known that if $p$ is a rational prime such that $p \equiv 3(mod 4)$ then $(\frac{-1,p}{\mathbb{Q}})$ is a division algebra. It is not hard to see that $CK_1(\frac{-1,p}{\mathbb{Q}}) = \bigoplus \mathbb{Z}_2$ is infinite.

In order to compute the functor $CK_1$ in the case of a valued division algebra, we need the following Theorem.

**Theorem 3.1.** [5] *Let $D$ be a Henselian division algebra tame over its centre with index $n$. Then*

*1) If $D$ is unramified, then $CK_1(D) \cong CK_1(\bar{D})$.*

*2) If $D$ is totally ramified, then $CK_1(D) = \Gamma_D/\Gamma_F$.*

*3) If $D$ is semiramified, and $\bar{D}$ is cyclic over $\bar{F}$, then there is an exact sequence*

$$1 \longrightarrow N_{\bar{D}/\bar{F}}(\bar{D}^*)/\bar{F}^{*n} \longrightarrow CK_1(D) \longrightarrow \Gamma_D/\Gamma_F \longrightarrow 1.$$

It is known that if $Z(\bar{D})$ is separable over $\bar{F}$, then the fundamental homomorphism $\Phi : \Gamma_D/\Gamma_F \to Gal(Z(\bar{D})/\bar{F})$ is epimorphism [10]. Since in the above theorem, part 3) $D$ is semiramified over $F$, $\Phi$ is an isomorphism. On the other hand $\bar{D}$ is cyclic over $\bar{F}$, Thus $\Gamma_D/\Gamma_F \cong \mathbb{Z}_n$. So we can rewrite the above exact sequence as

*3′) If $D$ is semiramified, and $\bar{D}$ is cyclic over $\bar{F}$, then there is an exact sequence*

$$1 \longrightarrow N_{\bar{D}/\bar{F}}(\bar{D}^*)/\bar{F}^{*n} \longrightarrow CK_1(D) \longrightarrow \mathbb{Z}_n \longrightarrow 1.$$

Theorem 3.1 is our main tool to compute the functor $CK_1$ for a valued quaternion division algebra.

Let $F$ be a real Pythagorean field. Let $N$ be the set of non rigid elements of $F$. Recall that if $N = \pm F^{*2}$ then $F$ is called *super-Pythagorean* [8]. The following theorem is a demonstration of how valuation theory enables us to compute $CK_1$.

**Theorem 3.2.** *Let $F$ be a real Pythagorean field and $N$ the set of non rigid elements of $F$. Then for any $t \in F((x)) \backslash \pm F((x))^2 \cup N$, $CK_1(\frac{-1,-t}{F((x))}) = \mathbb{Z}_2$.*

*Proof.* The valuation of $F((x))$ extends to $D = (\frac{-1,-1}{F((x))})$ as follows; $v(x) = \frac{1}{2}v(Nrd_D(x))$ where $v$ denotes the valuation map for both $F$ and $D$. Suppose first that $t \in F$. It is easy to see that

$$(\frac{-1,-t}{F((x))}) \cong (\frac{-1,-t}{F}) \otimes_F F((x))$$

is unramified. Thus by 3.1 1), $CK_1(\frac{-1,-t}{F((x))}) \cong CK_1(\frac{-1,-t}{F})$. Since $t$ is rigid in $F$ by Theorem 2.6, $CK_1(\frac{-1,-t}{F}) \cong \mathbb{Z}_2$ and we are done.

Now if $t \in F((x))\backslash F$, then it is not hard to see that $D$ is semiramified, $\bar{D} = F(\sqrt{-1})$ and $N_{\bar{D}}(F(\sqrt{-1})) = F^{*2}$. Now from Theorem 3.1 3′) it follows that $CK_1(D) = \mathbb{Z}_2$. □

Theorem above is another way to observe that if $F$ is super-Pythagorean then so is $F((x))$.

*Remark* 3.3. Consider the division algebra $D = (\frac{x,y}{\mathbb{R}((x,y))})$. Since $Nrd_D(i) = -x$ and $Nrd_D(j) = -y$, it can be seen that the extension of the valuation from $\mathbb{R}((x,y))$ to $D$ is totally ramified. From Theorem 3.1 2) follows that $CK_1(D) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

3.1. $CK_1$ **for local quaternion division algebras.** Let $F$ be a local field and $D$ a quaternion division algebra over $D$. We shall use Theorem 3.1 to show that $CK_1(D) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Recall that characteristic of $F$ is not 2. We further assume that $char(\bar{F})$ is not 2.

**Theorem 3.4.** *Let $D$ be a quaternion division algebra over a local field $F$. Then $CK_1(D) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

*Proof.* The valuation of $F$ extends to $D$. Furthermore one can see that with this valuation $D$ is semiramified over $F$ and $\bar{D}$ is cyclic over $\bar{F}$. Since $\bar{D}$ is finite, $N_{\bar{D}}(\bar{D}^*) = \bar{F}^*$ and the exact sequence of Theorem 3.1 3′ has the look

$$1 \longrightarrow \bar{F}^*/\bar{F}^{2*} \longrightarrow CK_1(D) \longrightarrow \mathbb{Z}_2 \longrightarrow 1.$$

Since $char(\bar{F})$ is not 2, the quotient $\bar{F}^*/\bar{F}^{2*}$ is not trivial. Because $\bar{F}^*$ is a cyclic group therefore $\bar{F}^*/\bar{F}^{2*} \cong \mathbb{Z}_2$. Since $exp(CK_1(D))$ divides 2, from the above exact sequence it follows that $CK_1(D) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. □

The above theorem shows that $CK_1(\frac{-1,x}{\mathbb{F}_3((x))}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ as mentioned before.

We shall mention that this result was not unknown. In fact, since $SK_1(D)$ is trivial and the reduced norm is surjective for division algebras over local fields, $CK_1(D) \cong F^*/F^{*2}$. From the theory of quadratic forms for local fields we can thus obtain the same result. Also if $char\bar{F} = 2$, it is known that $F^*/F^{*2} \cong \bigoplus_{2^{n+1}} \mathbb{Z}_2$ (See P.217 [9])

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