# DETECTION THEOREM FOR FINITE GROUP SCHEMES

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## Introduction

The classical detection theorem for finite groups (due to D. Quillen [Q] and B. Venkov [Q-V]) tells that if $\pi$ is a finite group and $\Lambda$ is a $\mathbb{Z}/p[\pi]$-algebra then a cohomology class $z \in H^*(\pi, \Lambda)$ is nilpotent iff for every elementary abelian $p$-subgroup $i : \pi_0 \hookrightarrow \pi$ the restriction $i^*(z) \in H^*(\pi_0, \Lambda)$ of $z$ to $\pi_0$ is nilpotent. This theorem is very useful for the identification of the support variety of the group $\pi$ and for the identification of support varieties of $\pi$-modules.

In [S-F-B] we proved a similar detection theorem for cohomology of infinitesimal group schemes $G$. The role of elementary abelian $p$-subgroups is played in this case by the so called one parameter subgroups of $G$, i.e. closed subgroup schemes $i : \mathbb{G}_{a(r)} \hookrightarrow G$. The analogy between elementary abelian $p$-groups and one parameter subgroups is emphasized by the fact that the corresponding cocommutative Hopf algebras (which happen to be commutative as well in this case) are isomorphic as algebras: $k[\mathbb{G}_{a(r)}]^{\#} \cong k[(\mathbb{Z}/p)^{\times r}]$ (but have quite different coproducts) and hence have isomorphic cohomology algebras. Note that above we use very similar notations $k[\mathbb{G}_{a(r)}]$ and $k[(\mathbb{Z}/p)^{\times r}]$ for two very different objects: on the left we have the coordinate algebra of the group scheme $\mathbb{G}_{a(r)}$, which is a commutative Hopf algebra for any group scheme, while on the right we have a group ring of a finite group. In this paper we are dealing with algebro-geometric objects so we will try to avoid the notation $k[\pi]$ for the group algebra. In particular if we want to consider the finite group $\pi$ as a discrete group scheme over $k$ then $k[\pi]$ will stand for the coordinate algebra of this discrete group scheme (which is dual to the group algebra of $\pi$), i.e. $k[\pi] = k^{\times \pi}$.

The main purpose of this paper is to prove the general detection theorem, which covers both the discrete and the infinitesimal cases and looks as follows.

**Theorem(Theorem 4.1 below).** *Let $G/k$ be a finite group scheme, let further $\Lambda$ be a unital associative rational $G$-algebra and let $z \in H^*(G, \Lambda)$ be a cohomology class. Assume that for any field extension $K/k$ and any closed subgroup scheme $i : \pi_0 \times \mathbb{G}_{a(r)} \hookrightarrow G$ ($\pi_0$ being an elementary abelian $p$-group) the restriction $i^*(z_K)$ of $z_K$ to $\pi_0 \times \mathbb{G}_{a(r)}$ is nilpotent then $z$ is nilpotent itself.*

To shorten the language we call group schemes like $\pi_0 \times \mathbb{G}_{a(r)}$ (with $\pi_0$ elementary abelian $p$-group ) elementary abelian group schemes.

To prove the above theorem we follow the general approach developed in [S-F-B]. Clearly one may assume that the base field $k$ is algebraically closed. In this case $G$ splits canonically as a semidirect product of a discrete group $\pi = G(k)$ and an infinitesimal group $G_0$ – the connected component of $G$. Moreover the usual transfer argument allows to reduce the general case to the special one when $\pi$ is a finite $p$-group - see [Be]. One has to work out first the case when $G_0$ is unipotent. Fortunately this step was taken already by Chris Bendel [Be]. One probably should note that the argument presented by Bendel contains a minor error, however this error is insignificant and can be easily straightened out (Julia Pevtsova suggested to me how this should be done). The unipotent case implies easily that the Detection Theorem works for finite group schemes of the form $\pi \ltimes B_{(r)}$, where $B$ is a Borel subgroup in a connected smooth group scheme $\mathcal{G}$ and $\pi$ is a finite $p$-subgroup in $B(k)$ (hence a subgroup in $U(k)$, where $U$ is the unipotent radical of $B$).

Next one has to consider the case of Frobenius kernels. Thus we take $G = \pi \ltimes \mathcal{G}_{(r)}$, where $\mathcal{G}$ is a smooth connected group scheme with Borel subgroup $B$ and $\pi$ is a $p$-subgroup in $B(k)$. The cases considered above allow to conclude that for any $g \in G(k)$ the restriction of $z$ to the subgroup $\pi_g \ltimes gB_{(r)}g^{-1}$ (where $\pi_g$ is the normalizer in $\pi$ of $gBg^{-1}$) is nilpotent. To be able to conclude that $z$ itself is nilpotent we need to have some kind of a spectral sequence relating cohomology of $\pi \ltimes \mathcal{G}_{(r)}$ to cohomology of a family of subgroups $\pi_g \ltimes gB_{(r)}g^{-1}$.

The first section of the present paper is devoted to the construction of a very general spectral sequence (which at first glance has nothing to do with the problem at hand). We start the second section with the proof of the Theorem which shows that in a certain special case the above constructed spectral sequence coincides with the spectral sequence introduced in [S-F-B] (which worked magnificently for the proof of the detection theorem in the infinitesimal case). We show next how to apply the above spectral sequence to the present situation. It's not surprising that the scheme whose cohomology appear in the above mentioned spectral sequence identifies with the scheme of double cosets $\pi \setminus G/B$. A thorough analysis of the information provided by the spectral sequence is made in § 3, which ends up with the proof of the detection theorem for Frobenius kernels. In section 4 we finally prove the general detection theorem. Once again this is done along the same lines as in [S-F-B]. One embeds $G = \pi \ltimes G_0$ into appropriate $GL_n$. Since $\pi$ is a finite $p$-group it's easy to see that there exists a Borel subgroup $B \subset GL_n$ such that $\pi \subset B(k)$. Denoting by $r$ the height of the infinitesimal group scheme $G_0$ we conclude that we get an embedding $G \subset \pi \ltimes GL_{n(r)}$ and for the ambient group the detection theorem is already known. Set $\Lambda' = Ind_G^{\pi \ltimes GL_{n(r)}} \Lambda$. Then $H^*(G, \Lambda)$ coincides as an algebra with $H^*(\pi \ltimes GL_{n(r)}, \Lambda')$ so the only thing remained to verify is that the cohomology class $z' \in H^*(\pi \ltimes GL_{n(r)}, \Lambda')$ corresponding to $z$ still has the same property, i.e. its restriction to any elementary abelian subgroup scheme of $\pi \ltimes GL_{n(r)}$ is nilpotent. This is a relatively easy exercise (see § 4 for details) however the proof uses essentially the following result.

**Theorem 4.3.** *Let $\mu' : H' \to G'$ be a homomorphism of finite group schemes over $k$. Let further $G \subset G'$ be a subgroup scheme and let $H$ denote its inverse image in*

*$H$. The canonical morphism of quotient schemes $\overline{\mu}' : H'/H \longrightarrow G'/G$ is a closed embedding.*

This result was mentioned as obvious and well-known in [S-F-B], however a more careful analysis shows that this is not the case: it's definitely not obvious and I was not able to find any reference to this fact in the literature. I still presume that it must be known but for the lack of reference we give in § 5 a detailed proof valid not only for finite group schemes but for any group schemes (for arbitrary group schemes one should replace closed embedding in the formulation of the above theorem by locally closed embedding). This proof, worked out jointly with Eric Friedlander, follows closely the argument presented in [Wa] for the proof of the theorem that states that any homomorphism of affine group schemes with trivial kernel is a closed embedding.

To finish this introduction I would like to express my sincere gratitude to Eric Friedlander who helped me a lot with the work on this paper, first and foremost by insisting that the paper should be written and also by being there when I needed an assistance or advice with my work over the present text.

All schemes throughout the paper are presumed to be of finite type over the base field $k$. We denote by $Sch/k$ the category of such schemes, sometimes we consider $Sch/k$ as a site in the fppf-topology.

## § 1. Construction of the spectral sequence

All through this section $T/k$ is an affine albebraic group, acting on the right on an affine scheme $Y = Spec\ A$, $M$ is an $A$-module on which $T$ acts (on the left) compatibly with its $A$-module structure. We assume further that we are given an affine $T$-invariant morphism $p : Y \to X$.

The main purpose of this section is to show that under the above circumstances there exist canonical quasicoherent sheaves $\mathcal{H}^q$ on $X$ and a spectral sequence

$$E_2^{pq} = H^p(X, \mathcal{H}^q) \Longrightarrow H^{p+q}(T, M) \tag{1.0}$$

Recall that to give an action of $T$ on a (not necessarily commutative) $k$-algebra $A$ means to make $A$ into a rational $T$-module in such a way that the multiplication map $A \otimes_k A \to A$ is a homomorphism of rational $T$-modules, where as always $T$ acts on the tensor product diagonally. In this case we say that $A$ is a rational $T$-algebra. The above condition concerning the action of $T$ on $A$ is easily seen to be equivalent to the requirement that the diagonal map $\Delta_A : A \to A \otimes_k k[T]$ is a $k$-algebra homomorphism. Note also that in case $A$ is a commutative $k$-algebra to give an action of $T$ on $A$ is the same as to give a right action of $T$ on the affine scheme $Y = Spec\ A$ (over $k$). In what follows we assume (if not specified otherwise) that the algebra $A$ is commutative. Assume now that $M$ is a rational $T$-module and simultaneously an $A$-module. We say that these two structures are compatible provided that the multiplication map $A \otimes_k M \to M$ is a homomorphism of $T$-modules, where on the left we take (as always) the diagonal module structure.

One checks immediately that this condition could be rephrased by saying that the diagonal map
$$\Delta_M : M \to M \otimes_k k[T]$$
is a homomorphism of $A$-modules, where the $A$-module structure on the right is defined by the $k$-algebra homomorphism $\Delta_A : A \to A \otimes_k k[T]$.

More generally assume that we are given a (right) action of the group scheme $T$ on an arbitrary scheme $Y$ $\quad \mu : Y \times_k T \to Y$. Let further $\mathcal{M}$ be a quasicoherent $\mathcal{O}_Y$ -module. We say that $T$ acts on $\mathcal{M}$ compatibly with its action on $Y$ provided that we are given a homomorphism of quasicoherent $\mathcal{O}_Y$ -modules $\Delta_{\mathcal{M}} : \mathcal{M} \to \mu_*(\mathcal{M} \boxtimes \mathcal{O}_T)$ (or what amounts to the same thing a homomorphism of quasicoherent $\mathcal{O}_{Y\times T}$-modules $\Delta^{\mathcal{M}} : \mu^*(\mathcal{M}) \to \mathcal{M} \boxtimes \mathcal{O}_T$), which satisfies the usual formal properties - see [J], ch. 1, §2. In case $Y = Spec\ A$ is affine and $\mathcal{M}$ corresponds to an $A$-module $M$ to give an action of $T$ on $\mathcal{M}$ compatible with its action on $Y$ is the same as to give a homomorphism of $A$-modules $\Delta_M : M \to M \otimes_k k[T]$ (satisfying the usual formal properties) i.e. is the same as to give $M$ a structure of a rational $T$-module compatible with its structure of an $A$-module.

**Lemma 1.1.** *Let $Y' \xrightarrow{f} Y$ be a $T$-equivariant morphism of schemes provided with the action of $T$. Assume further that $\mathcal{M}$ is a quasicoherent $\mathcal{O}_Y$-module provided with the action of $T$ compatible with the action of $T$ on $Y$. Then the quasicoherent $\mathcal{O}_{Y'}$-module $\mathcal{M}' = f^*(\mathcal{M})$ has a canonical $T$-module structure compatible with the action of $T$ on $Y'$.*

*Proof.* This follows immediately from the commutative diagram

$$\begin{array}{ccc}
Y' \times_k T & \xrightarrow{f\times 1_T} & Y \times_k T \\
{\scriptstyle \mu'}\downarrow & & {\scriptstyle \mu}\downarrow \\
Y' & \xrightarrow{f} & Y
\end{array}$$

Assume once again that $T$ acts on a scheme $Y$ (not necessarily affine). We say that an open subscheme $U \subset T$ is $T$-invariant in case $\mu(U \times_k T) \subset U$ (or what's the same if $U \times_k T \subset \mu^{-1}(U)$). In this case there exists a unique action of $T$ on $U$ for which the open embedding $U \hookrightarrow Y$ is $T$-equivariant. Lemma 1.1 applies to show that for any quasicoherent $\mathcal{O}_Y$-module $\mathcal{M}$ with action of $T$ (compatible with the action of $T$ on $Y$) the restriction sheaf $\mathcal{M}_{|U}$ inherits a canonical action of $T$ (compatible with the action of $T$ on $U$).

**Lemma 1.2.** *Assume that $T$ acts (compatibly) on a scheme $Y$ and on a quasicoherent $\mathcal{O}_Y$-module $\mathcal{M}$. Then for any open (not necessarily affine) $T$-invariant subscheme $U \subset Y$ we have compatible actions of $T$ on the $k$-algebra $k[U] = \Gamma(U, \mathcal{O}_Y)$ and on a $k[U]$-module $\Gamma(U, \mathcal{M})$.*

*Proof.* The previous remarks show that it suffices to consider the case $U = Y$. In this case we can proceed as follows. The structure homomorphism $\Delta_{\mathcal{M}} : \mathcal{M} \to$

$\mu_*(\mathcal{M} \boxtimes \mathcal{O}_T)$ defines a map on global sections $\Delta_{\mathcal{M}}(Y) : \Gamma(Y, \mathcal{M}) \to \Gamma(Y \times T, \mathcal{M} \boxtimes \mathcal{O}_T)$. The latter $k$-module may be identified (see Sublemma 1.2.1 below) with $\Gamma(Y, \mathcal{M}) \otimes_k k[T]$. Thus we get the desired diagonal map on the $k$-module $M = \Gamma(Y, \mathcal{M})$

$$\Delta_M = \Delta_{\mathcal{M}}(Y) : M \to M \otimes_k k[T]$$

The formal properties of $\Delta_{\mathcal{M}}$ immediately imply that $\Delta_M$ has the same formal properties, i.e. makes $M$ into a rational $T$-module. Various compatibilities are also easily checked, we leave details to the reader.

**Sublemma 1.2.1.** *Let $Y$ and $T$ be arbitrary schemes of finite type over a field $k$. Let further $\mathcal{M}$ and $\mathcal{F}$ be quasicoherent sheaves on $Y$ and $T$ respectively. Then the canonical map*

$$\Gamma(Y, \mathcal{M}) \otimes_k \Gamma(T, \mathcal{F}) \to \Gamma(Y \times_k T, \mathcal{M} \boxtimes \mathcal{F})$$

*is an isomorphism*

*Proof.* In case both $Y$ and $T$ are affine the statement is well-known (and trivial). Assume now that only $T$ is affine. Find an open affine covering $Y = \bigcup_{i=1}^n Y_i$ of $Y$ and next find open affine coverings $Y_i \cap Y_j = \bigcup_k Y_{ij}^k$. Taking products with an affine scheme $T$ we get induced open affine coverings $Y \times_k T = \bigcup_{i=1}^n Y_i \times_k T$ for $Y \times_k T$ and also affine open coverings for $(Y_i \times T) \cap (Y_j \times T) = (Y_i \cap Y_j) \times_k T$. Now our statement follows from the commutative diagram with exact rows

$$\begin{CD}
0 @>>> \Gamma(Y, \mathcal{M}) \otimes_k F @>>> \oplus_i \Gamma(Y_i, \mathcal{M}) \otimes_k F @>>> \oplus_{ij}^k \Gamma(Y_{ij}^k, \mathcal{M}) \otimes_k F \\
@. @VVV @VV{\cong}V @VV{\cong}V \\
0 @>>> \Gamma(Y \times T, \mathcal{E}) @>>> \oplus_i \Gamma(Y_i \times T, \mathcal{E}) @>>> \oplus_{ij}^k \Gamma(Y_{ij}^k \times T, \mathcal{E})
\end{CD}$$

where we abbreviated $\Gamma(T, \mathcal{F})$ to $F$ and $\mathcal{M} \boxtimes \mathcal{F}$ to $\mathcal{E}$. In our applications we need only the case when the scheme $T$ is affine so we stop here, however repeating the same trick again we immediately get the proof in complete generality.

We return back to the situation we described at the beginning of the section. So let $p : Y \to X$ be an affine $T$-invariant morphism. The following statement is obvious from definitions.

**Lemma 1.3.** *For any open $V \subset X$ its inverse image $U = p^{-1}(V)$ is $T$-invariant in $Y$. Moreover the image of the obvious homomorphism $p^* : k[V] \to k[U]$ consists of $T$-invariant functions.*

Let $\mathcal{M} = \widetilde{M}$ be the quasicoherent $\mathcal{O}_Y$-module defined by $M$. The action of $T$ on $M$ (compatible with the action of $T$ on $k[Y] = A$) defines in an obvious way an action of $T$ on the quasicoherent sheaf $\mathcal{M}$ compatible with the action of $T$ on $Y$. According to Lemmas 1.2 and 1.3 this implies that for any open subset $V \subset X$ we get a natural action of $T$ on $k[U]$ and a compatible action of $T$ on $\Gamma(U, \mathcal{M})$ (where $U = p^{-1}(V) \subset Y$). Since $p^*(k[V]) \subset k[U]^T$ we conclude further that the action of $T$ on $\Gamma(U, \mathcal{M})$ is $k[V]$-linear. This implies readily that all cohomology groups

$H^*(T, \Gamma(U, \mathcal{M}))$ have canonical structures of $k[V]$-modules. Finally if $V' \subset V$ is another open subset, then (setting $U' = p^{-1}(V')$) we see immediately that the restriction map $\Gamma(U, \mathcal{M}) \xrightarrow{res} \Gamma(U', \mathcal{M})$ is a homomorphism of $T$-modules and hence defines induced maps in cohomology $H^*(T, \Gamma(U, \mathcal{M})) \xrightarrow{res_*} H^*(T, \Gamma(U', \mathcal{M}))$. Moreover a straightforward verification shows that the map $res_*$ is a homomorphism of $k[V]$ modules, provided we make $H^*(T, \Gamma(U', \mathcal{M}))$ into a $k[V]$-module via the homomorphism $k[V] \xrightarrow{res} k[V']$.

The above discussion shows that setting

$$\mathcal{F}^q(V) = H^q(T, \Gamma(p^{-1}(V), \mathcal{M}))$$

we define a presheaf $\mathcal{F}^q$ of $\mathcal{O}_X$-modules on $X$. We define the sheaf $\mathcal{H}^q$ of $\mathcal{O}_X$-modules on $X$ as the sheaf associated to $\mathcal{F}^q$ in Zariski topology: $\mathcal{H}^q = \mathcal{F}^q_{Zar}$. To understand the properties of the sheaf $\mathcal{H}^q$ we need two more general Lemmas.

**Lemma 1.4.** *a) Let $\mathcal{M}$ be a quasicoherent sheaf on a scheme $X$, then for any open affine subsets $V' \subset V \subset X$ the canonical map*

$$k[V'] \otimes_{k[V]} \Gamma(V, \mathcal{M}) \to \Gamma(V', \mathcal{M})$$

*is an isomorphism.*
*b) Assume that $\mathcal{M}$ is a presheaf of $\mathcal{O}_X$-modules on $X$ with the property that for any open affine subsets $V' \subset V \subset X$ the canonical map*

$$k[V'] \otimes_{k[V]} \Gamma(V, \mathcal{M}) \to \Gamma(V', \mathcal{M})$$

*is an isomorphism. Then the associated Zariski sheaf $\mathcal{M}_{Zar}$ is quasicoherent, moreover the natural map $\mathcal{M}(V) \to \mathcal{M}_{Zar}(V)$ is an isomorphism for any affine open $V \subset X$.*

*Proof.* The point a) is well-known and trivial. To prove the point b) we note that the sheaf $\mathcal{M}_{Zar}$ may be obtained from $\mathcal{M}$ by applying twice the functor $\check{\mathcal{H}}^0$ (zero-dimensional Čech cohomology). Thus to show that the natural map $\mathcal{M}(V) \to \mathcal{M}_{Zar}(V)$ is an isomorphism for any open affine $V \subset X$ it suffices to show that for any such $V$ the canonical map

$$\mathcal{M}(V) \to \check{H}^0(V, \mathcal{M}) = \varinjlim_{\mathcal{V}\text{open covering of } V} \check{H}^0(\mathcal{V}, \mathcal{M})$$

is an isomorphism. Moreover computing the above direct limit we may replace the filtered poset of coverings of $V$ (up to equivalence) by any cofinal poset. In particular, since any open covering of $V$ contains a finite affine subcovering it suffices to show that for any finite affine covering $\mathcal{V}: \ V = \bigcup_{i=1}^n V_i$ the corresponding map $\mathcal{M}(V) \to \check{H}^0(\mathcal{V}, \mathcal{M})$ is an isomorphism. In other words it suffices to establish the exactness of the sequence

$$0 \to \mathcal{M}(V) \to \prod_i \mathcal{M}(V_i) \to \prod_{i,j} \mathcal{M}(V_{ij})$$

Since $V$ all $V_i$ and all $V_{ij} = V_i \cap V_j$ are affine our basic assumption on $\mathcal{M}$ yields the formulae

$$\mathcal{M}(V_i) = k[V_i] \otimes_{k[V]} \mathcal{M}(V) \quad \mathcal{M}(V_{ij}) = k[V_{ij}] \otimes_{k[V]} \mathcal{M}(V)$$

Thus the sequence under consideration takes the form

$$0 \to \mathcal{M}(V) \to A \otimes_{k[V]} \mathcal{M}(V) \to A \otimes_{k[V]} A \otimes_{k[V]} \mathcal{M}(V)$$

where $A = \prod_i k[V_i]$ is a faithfully flat $k[V]$-algebra. The exactness of the resulting sequence is well-known - see for example [M], ch. 1, 2.19.

Finally to prove that $\mathcal{M}_{Zar}$ is quasicoherent we use the following well-known criterion.

**Lemma 1.4.1.** *A sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$ on an affine scheme $X$ is quasicoherent iff for every $f \in k[X]$ the canonical homomorphism $\mathcal{M}(X)_f \to \mathcal{M}(X_f)$ is an isomorphism.*

We are going to apply Lemma 1.4 to the presheaf $\mathcal{F}^q$. To be able to do so we need one more elementary fact.

**Lemma 1.5.** *Let $A$ be a commutative $k$-algebra, let further $M$ be an $A$-module on which the group $T$ acts by $A$ linear transformations. For any commutative $A$-algebra $A'$ we get the induced $A'$-linear action of $T$ on $A' \otimes_A M$. Moreover in case the algebra $A'$ is $A$-flat we have the following canonical identifications:*

$$A' \otimes_A H^*(T, M) \xrightarrow{\sim} H^*(T, A' \otimes_A M)$$

*Proof.* The first statement is trivial and the second follows for example from the consideration of the corresponding Hochschild complexes (cf. also [J], ch. 1, 4.13).

Now we are prepaired to investigate the sheaves $\mathcal{H}^q$. Note that the sheaf $p_*(\mathcal{M})$ is a quasicoherent $\mathcal{O}_X$-module - see [Ha], ch.2, 5.8. Thus for any open affine subsets $V' \subset V \subset X$ we have according to (the trivial part of) Lemma 1.4 canonical identifications

$$k[V'] \otimes_{k[V]} \mathcal{M}(p^{-1}(V)) = \mathcal{M}(p^{-1}(V'))$$

Since $k[V']$ is a flat $k[V]$-algebra we conclude from Lemma 1.5 that the natural map

$$k[V'] \otimes H^*(T, \mathcal{M}(p^{-1}(V)) \to H^*(T, \mathcal{M}(p^{-1}(V')))$$

is an isomorphism. Applying once again Lemma 1.4 we get the following Corollary.

**Corollary 1.6.** *The sheaf $\mathcal{H}^q$ is quasicoherent. For any open affine $V \subset X$ the canonical map*

$$H^q(T, \mathcal{M}(p^{-1}(V)) = \mathcal{F}^q(V) \to \mathcal{H}^q(V)$$

*is an isomorphism.*

We now turn to the construction of the spectral sequence (1.0). This construction is based on the use of the following well-known elementary observation.

**Lemma 1.7.** *Let $\mathcal{M}$ be a quasicoherent sheaf on an affine scheme $Y$. Then for any affine open covering $\mathcal{U}: \quad Y = \bigcup_{i=1}^n U_i$ the corresponding Čech cohomology $\check{H}^*(\mathcal{U}, \mathcal{M})$ are given by the formula*

$$\check{H}^*(\mathcal{U}, \mathcal{M}) = \begin{cases} \mathcal{M}(Y), & \text{for } * = 0 \\ 0, & \text{for } * > 0. \end{cases}$$

*Proof.* Recall that Čech cohomology can be computed equally using either the alternating Čech complex or the total Čech complex - see [Se], ch.1, §3,n° 20. At this particular moment it's preferable to work with total Čech complex, which in view of Lemma 1.4 can be identified with

$$M \otimes_A B \to M \otimes_A B \otimes_A B \to \ \ldots$$

where $M = \mathcal{M}(Y)$, $A = k[Y]$, $B = \prod_i k[U_i]$. Thus our statement follows from the fact that $B$ is a faithfully flat $A$-algebra in view of [M], ch. 1, 2.19.

**Theorem 1.8.** *In the situation described at the beginning of this section there exists a natural spectral sequence*

$$E_2^{pq} = H^p(X, \mathcal{H}^q) \Longrightarrow H^{p+q}(T, M).$$

*Proof.* Choose an affine open covering $\mathcal{V}: \quad X = \bigcup_{i=1}^n V_i$ and set $U_i = p^{-1}(V_i) \subset Y$. Since the morphism $p$ is affine we conclude that $\mathcal{U}: \quad Y = \bigcup_{i=1}^p U_i$ is an affine open covering of $Y$. Consider the Čech complex $C^*(\mathcal{U}, \mathcal{M})$. Since this complex consists of $T$-modules and $T$-homomorphisms we get two hypercohomology spectral sequences. The second spectral sequence degenerates in view of Lemma 1.7 and yields the limit of the first spectral sequence:

$$\mathrm{I}_1^{pq} = H^q(T, C^p(\mathcal{U}, \mathcal{M})) \Longrightarrow H^{p+q}(T, M)$$

To compute the second term of this spectral sequence we note that $H^q(T, C^p(\mathcal{U}, \mathcal{M})) = C^p(\mathcal{V}, \mathcal{H}^q)$ and the differential $d_1$ is nothing but the usual differential of the Čech complex $C^*(\mathcal{V}, \mathcal{H}^q)$. Thus $\mathrm{I}_2^{pq} = H^p(\mathcal{V}, \mathcal{H}^q) = H^p(X, \mathcal{H}^q)$, where the last identification holds since $\mathcal{V}$ is an affine open covering of the scheme $X$ and the sheaf $\mathcal{H}^q$ is quasicoherent.

To be on the safe side we mention also the following fact.

**Lemma 1.8.1.** *The above spectral sequence is independent of the choice of the affine open covering $\mathcal{V}$.*

*Proof.* Let $\mathcal{V}': \quad X = \bigcup_{j=1}^m V'_j$ be another affine open covering of $X$. Assume first that $\mathcal{V}'$ is a refinement of $\mathcal{V}$. Fix a function $\tau : \{1, ..., m\} \to \{1, ..., n\}$ such that $V'_j \subset V_{\tau(j)}$ for all $j$. The choice of $\tau$ defines a $T$-equivariant homomorphism $C^*(\mathcal{U}, \mathcal{M}) \xrightarrow{r_\tau} C^*(\mathcal{U}', \mathcal{M})$ and hence defines a homomorphism of the corresponding hypercohomology spectral sequences. The resulting map on the $E_2$-terms coincides with the obvious map on the Čech cohomology groups $H^p(\mathcal{V}, \mathcal{H}^q) \longrightarrow H^p(\mathcal{V}', \mathcal{H}^q)$ and hence is an isomorphism. This implies that the map of the spectral sequences is an isomorphism from $E_2$-term on. The resulting isomorphism is independent of the choice of $\tau$. For a different choice of $\tau$ the corresponding homomorphisms of complexes $r_\tau,\ r_{\tau'}$ are canonically homotopic (see for example [Se], ch. 1, §3, n° 21) and moreover the explicit formula for the homotopy shows that it happens to be $T$-equivariant. Since the homomorphisms of the hypercohomology spectral sequences induced by homotopic homomorphisms of complexes coincide - [C-E], ch. 17, §2 (from $E_2$-term on) the statement follows. Finally for an arbitrary choice of $\mathcal{V}'$ we can find a common refinement $\mathcal{V}$" of $\mathcal{V}$ and $\mathcal{V}'$ and thus conclude that all three spectral sequences identify canonically.

To finish this section we discuss briefly the multiplicative properties of the above spectral sequence

To get products on the spectral sequence we assume that $M = \Lambda$ is a (not necessarily commutative) rational $T$-algebra which simultaneously is a $k[Y]$-algebra and that these two structures are compatible, i.e. the structure homomorphism $k[Y] \longrightarrow \Lambda$ is a homomorphism of rational $T$-algebras. Applying to $M = \Lambda$ the above construction we get quasicoherent sheaves $\mathcal{H}^q = \mathcal{H}^q(\Lambda)$ on $X$ and a spectral sequence

$$E_2^{pq} = H^p(X, \mathcal{H}^q) \Longrightarrow H^{p+q}(T, \Lambda) \tag{1.9.0}$$

**Theorem 1.9.** *a) For each $q, q' \geq 0$ we have canonical pairings of quasicoherent sheaves $\mathcal{H}^q \otimes_{\mathcal{O}_X} \mathcal{H}^{q'} \longrightarrow \mathcal{H}^{q+q'}$.*

*b) The spectral sequence (1.9.0) has canonical multiplicative structure. The product maps on the limit coincide with the obvious pairings in cohomology $H^n(T, \Lambda) \otimes_k H^m(T, \Lambda) \longrightarrow H^{n+m}(T, \Lambda)$ induced by the multiplication $\Lambda \otimes_k \Lambda \longrightarrow \Lambda$. The product maps on the $E_2$-term coincide with the pairings in sheaf cohomology*

$$H^p(X, \mathcal{H}^q) \otimes_k H^{p'}(X, \mathcal{H}^{q'}) \longrightarrow H^{p+p'}(X, \mathcal{H}^{q+q'})$$

*induced by the pairing of sheaves $\mathcal{H}^q \otimes_{\mathcal{O}_X} \mathcal{H}^{q'} \longrightarrow \mathcal{H}^{q+q'}$.*

*Proof.* We start by noting that for any rational $T$-algebra $\Lambda$ we have canonical pairings in group cohomology $H^*(T, \Lambda)$. These pairings can be described in several equivalent ways. The most convenient for our purposes way is to note that the Hochschild complex $C^*(T, \Lambda)$ is a differential graded algebra with respect to a product defined by the usual formula:

$$(f \cup f')(t_1, ..., t_{n+m}) = f(t_1, ..., t_n) \cdot {}^{t_1 \cdot ... \cdot t_n} f'(t_{n+1}, ..., t_{n+m})$$

(here we identify $C^n(G, \Lambda)$ with $Mor(G^n, \Lambda_a)$ as in [J], ch. 1, 4.14).

To prove point a) we note that $\mathcal{H}^q \otimes_{\mathcal{O}_X} \mathcal{H}^{q'}$ is a quasicoherent $\mathcal{O}_X$-module, whose sections over any open affine $V \subset X$ coincide with $H^q(T, \mathcal{L}(U)) \otimes_{k[V]} H^{q'}(T, \mathcal{L}(U))$, where $\mathcal{L}$ is a quasicoherent $\mathcal{O}_Y$-algebra defined by the $k[Y]$-algebra $\Lambda$ and $U = p^{-1}(V)$ is an open affine in $Y$. The rational $T$-module $\mathcal{L}(U) = k[U] \otimes_{k[Y]} \Lambda$ is clearly a rational $T$-algebra. Hence, according to the previous remark, we have canonical pairings

$$H^q(T, \mathcal{L}(U)) \otimes_k H^{q'}(T, \mathcal{L}(U)) \to H^{q+q'}(T, \mathcal{L}(U)) = \Gamma(V, \mathcal{H}^{q+q'})$$

Finally, since the action of $T$ on $\mathcal{L}(U)$ is $k[V]$-linear one checks easily that the Hochschild complex $C^*(T, \mathcal{L}(U))$ is actually a differential graded $k[V]$-algebra and hence we actually have products

$$\Gamma(V, \mathcal{H}^q \otimes_{\mathcal{O}_X} \mathcal{H}^{q'}) = H^q(T, \mathcal{L}(U)) \otimes_{k[V]} H^{q'}(T, \mathcal{L}(U)) \to H^{q+q'}(T, \mathcal{L}(U)) = \\ = \Gamma(V, \mathcal{H}^{q+q'})$$

Since these pairings for different $V$'s are compatible one with another we get the desired pairings of quasicoherent sheaves $\mathcal{H}^q \otimes_{\mathcal{O}_X} \mathcal{H}^{q'} \to \mathcal{H}^{q+q'}$.

To get products on the spectral sequence (1.9.0) we start with an arbitrary open affine covering $\mathcal{V} : X = \bigcup_{i=1}^n V_i$ and denote by $\mathcal{U} : Y = \bigcup_{i=1}^n p^{-1}(V_i) = \bigcup_{i=1}^n U_i$ the induced open affine covering of $Y$. The spectral sequence (1.9.0) appears as the hypercohomology spectral sequence defined by the Čech complex $C^*(\mathcal{U}, \mathcal{L})$ (considered as a complex of rational $T$-modules). Since $\mathcal{L}$ is a sheaf of $\mathcal{O}_Y$-algebras we conclude immediately that $C^*(\mathcal{U}, \mathcal{L})$ is a differential graded $k$-algebra with respect to the usual product of Čech cochains:

$$(f \cup f')(i_0, ..., i_{p+p'}) = f(i_0, ..., i_p)_{|U_{i_0} \cap ... \cap U_{i_{p+p'}}} \cdot f'(i_p, ..., i_{p+p'})_{|U_{i_0} \cap ... \cap U_{i_{p+p'}}}$$

Moreover the product maps $C^p(\mathcal{U}, \mathcal{L}) \otimes_k C^{p'}(\mathcal{U}, \mathcal{L}) \to C^{p+p'}(\mathcal{U}, \mathcal{L})$ are easily seen to be homomorphisms of rational $T$-modules.

The hypercohomology of $T$ with coefficients in the complex of rational $T$-modules $C^*(\mathcal{U}, \mathcal{L})$ coincides with the cohomology of the bicomplex $C^*(T, C^*(\mathcal{U}, \mathcal{L}))$. The latter bicomplex has a canonical structure of a differential bigraded algebra. Hence both spectral sequences of this bicomplex have canonical multiplicative structures. Looking on the second spectral sequence we conclude easily that the hypercohomology of $T$ with coefficients in $C^*(\mathcal{U}, \mathcal{L})$ identifies as an algebra with $H^*(T, \Lambda)$. Finally looking on the first spectral sequence we conclude easily that the pairing on $E_2$ coincides with pairing in Čech cohomology of $\mathcal{V}$ defined by the pairing of sheaves $\mathcal{H}^q \otimes_{\mathcal{O}_X} \mathcal{H}^{q'} \to \mathcal{H}^{q+q'}$ and hence coincides with pairing in cohomology of $X$ defined by the pairing of sheaves $\mathcal{H}^q \otimes_{\mathcal{O}_X} \mathcal{H}^{q'} \to \mathcal{H}^{q+q'}$.

## § 2. Examples and applications

Throughout this section $G$ will denote a connected smooth affine algebraic group over a field $k$ of positive characteristic $p$, and $B$ will denote its Borel subgroup (thus we assume that Borel subgroups in $G$ exist). Let $F^r : G \to G^{(r)}$ be the $r$-th power of the Frobenius map, where $G^{(r)}$ is the $r$-th Frobenius twist of $G$ (see, for example, [F-S], §1). The kernel $G_{(r)}$ of $F^r$ is an infinitesimal group scheme of height $r$ and $F^r$ induces an isomorphism $F^r : G/G_{(r)} \xrightarrow{\sim} G^{(r)}$. Our first goal in this section is to show how the mashinery developed in §1 can be used to recover the spectral sequence exibited in [S-F-B], Theorem 3.6, which played a crucial role in the proof of the detection theorem for infinitesimal group schemes.

So let $M$ be a rational $G_{(r)}$-module. Since the quotient $G/G_{(r)} = G^{(r)}$ is affine we conclude that $H^*(G_{(r)}, M) = H^*(G, Ind_{G_{(r)}}^G M)$ (see [J], 1.5.12). Moreover, according to the theorem of Clein, Parshall, Scott and van der Kallen (see [S-F-B], 3.1) the latter cohomology group identifies with $H^*(B, Ind_{G_{(r)}}^G M)$. Denote the induced module $Ind_{G_{(r)}}^G M$ by $I$. Thus $I = (k[G] \otimes_k M)^{G_{(r)}}$ and this formula implies readily that $I$ is a module over a commutative ring $k[G]^{G_{(r)}} = k[G^{(r)}]$. The $k[G^{(r)}]$-module structure and the $B$-module structure on $I$ are compatible in the following sence (which is somewhat different from the compatibility we required in § 1). The multiplication map $k[G^{(r)}] \otimes I \longrightarrow I$ is a homomorphism of rational $B$-modules provided one considers the left regular action of $B$ on $k[G^{(r)}]$. To achieve the desired compatibility one can either change (as it was done in [S-F-B]) the $k[G^{(r)}]$-module structure on $I$ using the automorphism of $k[G^{(r)}]$ induced by the automorphism $x \mapsto x^{-1}$ of the scheme $G^{(r)}$ or (what's equivalent) modify slightly the definition of the induced module. In what follows we use the following (modified) definition of the induced module:

$$Ind_H^G M = (M \otimes_k k[G])^H$$

where $H$ operates as given on $M$ and operates via left regular representation on $k[G]$, the action of $G$ on $Ind\ M$ is induced by the right regular representation of $G$ on $k[G]$.

We may apply now the construction of § 1 to the algebraic group $T = B$ acting on $Y = G^{(r)}$, $k[G^{(r)}]$-module $I$ on which $B$ acts compatibly with its module structure and the affine $B$-invariant morphism $p : Y = G^{(r)} \longrightarrow X = G^{(r)}/B^{(r)} = (G/B)^{(r)}$. Theorem 1.8 implies that we get quasicoherent sheaves $\mathcal{H}^q$ on $G^{(r)}/B^{(r)} = (G/B)^{(r)}$ and a spectral sequence

$$E_2^{pq} = H^p((G/B)^{(r)}, \mathcal{H}^q) \Longrightarrow H^{p+q}(B, I) = H^{p+q}(G_{(r)}, M) \tag{2.0}$$

**Proposition 2.1.** *The sheaf $\mathcal{H}^q$ coincides with the sheaf $\mathcal{H}^q(B_{(r)}, M)$ introduced in* [S-F-B] *3.5.1.*

*Proof.* Recall that the sheaf $\mathcal{H}^q(B_{(r)}, M)$ was defined using the descence theory starting with a quasicoherent $\mathcal{O}_{G^{(r)}}$-module defined by the $k[G^{(r)}]$-module $H^q(B_{(r)}, I)$, provided with the descence data which comes from the canonical action of $B^{(r)}$ on $H^q(B_{(r)}, I)$ - see [S-F-B] §3. Let $V \subset (G/B)^{(r)}$ be an open affine subset and let $U = p^{-1}(V)$ be its inverse image in $G^{(r)}$. The sections of $\mathcal{H}^q(B_{(r)}, M)$ over $V$ may be identified with the kernel of the map

$$k[U] \otimes_{k[G^{(r)}]} H^q(B_{(r)}, I) \xrightarrow{\Delta - Id \otimes_k 1_{k[B^{(r)}]}} (k[U] \otimes_{k[G^{(r)}]} H^q(B_{(r)}, I)) \otimes_k k[B^{(r)}]$$

i.e. with $H^q(B_{(r)}, k[U] \otimes_{k[G^{(r)}]} I)^{B^{(r)}}$. We show below in Corollary 2.3 that $H^q(B_{(r)}, k[U] \otimes_{k[G^{(r)}]} I)$ is an acyclic $B^{(r)}$-module. Hence the Hochschild-Serre spectral sequence degenerates providing natural identifications $H^q(B_{(r)}, k[U] \otimes_{k[G^{(r)}]} I)^{B^{(r)}} = H^q(B, k[U] \otimes_{k[G^{(r)}]} I)$. In other words we have canonical identifications of sections over any open affine subset $V \subset (G/B)^{(r)}$: $\mathcal{H}^q(B_{(r)}, M)(V) = \mathcal{H}^q(V)$ and hence the sheaf $\mathcal{H}^q(B_{(r)}, M)$ coincides with $\mathcal{H}^q$.

**Lemma 2.2.** *In notations of the proof of Proposition 2.1 for any rational $B^{(r)}$-module $P$ the tensor product module $k[U] \otimes_k P$ is acyclic.*

*Proof.* Note that the action of $B^{(r)}$ on $k[U] \otimes_k P$ is $k[V]$-linear and hence all cohomology groups $H^*(B^{(r)}, k[U] \otimes_k P)$ are $k[V]$-modules. Furthermore, since $k[U]$ is a faithfully flat $k[V]$-algebra the vanishing of $H^*(B^{(r)}, k[U] \otimes_k P)$ (for $* > 0$) would follow from the vanishing of $k[U] \otimes_{k[V]} H^*(B^{(r)}, k[U] \otimes_k P) = H^*(B^{(r)}, k[U]_{\mathrm{triv}} \otimes_{k[V]} k[U] \otimes_k P)$. The cartesian diagram

$$\begin{tikzcd} U \times B^{(r)} \arrow[r, "\mu"] \arrow[d, "pr_1"'] & U \arrow[d, "p"'] \\ U \arrow[r, "p"] & V \end{tikzcd}$$

shows that the $B^{(r)}$-module $k[U]_{\mathrm{triv}} \otimes_{k[V]} k[U]$ may be identified with $k[B^{(r)}] \otimes_k k[U]_{\mathrm{triv}}$. Thus $k[U]_{\mathrm{triv}} \otimes_{k[V]} k[U] \otimes_k P$ is isomorphic to $k[B^{(r)}] \otimes_k k[U]_{\mathrm{triv}} \otimes_k P$ and the latter module is injective (see [J], ch. 1, 3.10) and hence acyclic.

**Corollary 2.3.** *Let $N$ be any $k[U]$-module on which the group $B$ act compatibly with its action on $k[U]$ (given by the Frobenius homomorphism $F^r : B \longrightarrow B^{(r)}$ and the right regular action of $B^{(r)}$ on $k[U]$). Then the $B^{(r)}$-modules $H^q(B_{(r)}, N)$ are acyclic.*

*Proof.* Consider first the special case $N = k[U] \otimes_k S$, where $S$ is a certain rational $B$-module and $B$ acts on $N$ diagonally. Since the action of $B_{(r)}$ on $k[U]$ is trivial we conclude immediately that $H^q(B_{(r)}, N) = k[U] \otimes_k P$, where $P = H^q(B_{(r)}, S)$ is a rational $B^{(r)}$-module. Thus in this case our statement follows directly from Lemma 2.2.

In the general case we note that according to our assumptions the multiplication map $k[U] \otimes_k N \longrightarrow N$ is a homomorphism of rational $B$-modules, which moreover has a canonical $B$-equivariant section $s : N \longrightarrow k[U] \otimes_k N$, given by the formula $s(n) = 1 \otimes n$. Thus $B$-module $N$ is a direct summand in the $B$-module $k[U] \otimes_k N$ and the statement follows.

Proposition 2.1 shows that the second term and the limit of the spectral sequence (2.0) may be identified with the second term and limit of the spectral sequence exibited in [S-F-B], Therorem 3.6. Our next goal is to show that these two spectral sequences actually coincide.

**Theorem 2.4.** *The spectral sequence (2.0) coincides with the spectral sequence constructed in* [S-F-B] *§ 3.*

*Proof.* Recall that the spectral sequence exibited in [S-F-B], theorem 3.6 coincides with the Hochschild-Serre spectral sequence corresponding to the group extension

$$1 \longrightarrow B_{(r)} \longrightarrow B \xrightarrow{F^r} B^{(r)} \longrightarrow 1$$

and the rational $G$-module $I = Ind_{G_{(r)}}^{G} M$. The results of [S-F-B] §3 were needed to identify its second term with appropriate sheaf cohomology - see [S-F-B], Proposition 3.4. Moreover the Hochschild-Serre spectral sequence may be constructed in the following way. For any rational $B$-module $P$ denote by $J^*(B,P)$ the standard $B$-injective resolution of $P$. The complex $J^*(B,I)^{B_{(r)}}$ consists of injective $B^{(r)}$-modules and its homology coincides with $H^*(B_{(r)},I)$. Let $J^{*,*}$ be the Cartan-Eilenberg resolution of $J^*(B,I)^{B_{(r)}}$. Then the Hochschild-Serre spectral sequence is nothing but the first spectral sequence of the bicomplex $(J^{*,*})^{B^{(r)}}$.

Let $\mathcal{V} = \{V_i\}_{i=1}^n$ be an affine open covering of the scheme $(G/B)^{(r)}$ and let $\mathcal{U} = \{U_i = p^{-1}(V_i)\}_{i=1}^n$ be the induced open affine covering of $G^{(r)}$. Consider the Čech complex $C^*(\mathcal{U},\mathcal{I})$ defined by the quasicoherent $\mathcal{O}_{G^{(r)}}$-module $\mathcal{I} = I^{\sim}$ and the covering $\mathcal{U}$. Lemma 1.7 shows that complex $C^* = C^*(\mathcal{U},\mathcal{I})$ is a resolution of $I = \mathcal{I}(G^{(r)})$. Since the functor $J^*(B,-)$ is obviously exact we conclude further that $J^*(B,C^*)$ is a resolution of $J^*(B,I)$. Denote by $\mathcal{A}$ the abelian category of complexes of rational $B^{(r)}$-modules and consider the bicomplex $J^*(B,C^*)^{B_{(r)}}$ as a complex in $\mathcal{A}$ under $J^*(B,I)^{B_{(r)}}$. This complex has the following properties.

**Lemma 2.4.1.** *a)$J^*(B,C^*)^{B_{(r)}}$ is a resolution of $J^*(B,I)^{B_{(r)}}$.*
*b) For any $p$ taking $p$-th homology in the rows of the bicomplex $J^*(B,C^*)^{B_{(r)}}$ we get a resolution of $H^p(J^*(B,I)^{B_{(r)}}) = H^p(B_{(r)},I)$.*

*Proof.* The $p$-th term of the standard injective resolution $I^p(B,P)$ identifies with $P \otimes_k k[B]^{\otimes p} \otimes_k k[B]$ with the action of $B$ trivial on all factors except the last one on which it identifies with the right regular action. Thus $I^p(B,P)^{(B_{(r)})} = P \otimes_k k[B]^{\otimes p} \otimes_k k[B^{(r)}]$ and the resulting functor is obviously exact in $P$.

To prove the second statement note that $H^p(J^*(B,C^m)^{B_{(r)}}) = H^p(B_{(r)},C^m)$. Moreover

$$C^m = \bigoplus_{i_0,\dots,i_m} k[U_{i_0} \cap \dots \cap U_{i_m}] \otimes_{k[G^{(r)}]} I$$

Since the action of $B^{(r)}$ on $I$ is $k[G^{(r)}]$-linear we conclude from Lemma 1.5 that

$$H^p(B_{(r)},C^m) = \bigoplus_{i_0,\dots,i_m} k[U_{i_0} \cap \dots \cap U_{i_m}] \otimes_{k[G^{(r)}]} H^p(B_{(r)},I)$$

Thus the complex $H^p(B_{(r)},C^*)$ coincides with the Čech complex defined by the covering $\mathcal{U}$ and a $k[G^{(r)}]$-module $H^p(B_{(r)},I)$ and hence is a resolution of $H^p(B_{(r)},I)$ according to Lemma 1.7.

Call a monomorphism of complexes $X^* \xrightarrow{i} Y^*$ admissible provided it induces injective maps on all cohomology groups. Call a short exact sequence of complexes

$$0 \to X^* \xrightarrow{i} Y^* \xrightarrow{p} Z^* \to 0$$

admissible in case $i$ is an admissible monomorphism. Finally call a long exact sequence of complexes admissible in case all the corresponding short exact sequences are admissible. The following result is (apparently) well-known (and easy to verify), we leave it as an exercise to the reader.

**Lemma 2.4.2.** *a) An exact sequence of complexes*

$$0 \to X \to X^0 \to X^1 \to \ \ldots$$

*is admissible iff for all $p$ the corresponding cohomology sequence $0 \to H^p(X) \to H^p(X^0) \to H^p(X^1) \to \ \ldots$ is exact.*
*b) Cartan-Eilenberg resolution of a complex $X$ is nothing but its relative (with respect to the class of admissible monomorphisms ) injective resolution in the abelian category $\mathcal{A}$.*

Lemmas 2.4.1 and 2.4.2 show in particular that $J^*(B, C^*)^{B_{(r)}}$ is a relative resolution of the complex $J^*(B, I)^{B_{(r)}}$, whereas $J^{*,*}$ is its relative injective resolution. The standard comparison theorem for resolutions implies that there exists (a unique up to homotopy) homomorphism of reolutions $J^*(B, C^*)^{B_{(r)}} \xrightarrow{F} J^{*,*}$. Taking $B^{(r)}$-invariants we get a homomorphism of bicomplexes

$$J^*(B, C^*)^B \xrightarrow{F^{B^{(r)}}} (J^{*,*})^{B^{(r)}}$$

which gives the induced homomorphisms on the corresponding spectral sequences. The spectral sequence corresponding to the bicomplex on the left is exactly the hypercohomology spectral sequence introduced in § 1, whereas the spectral sequence corresponding to the bicomplex on the right coincides with the Hochschild-Serre spectral sequence. To prove that the induced map of the spectral sequences is an isomorphism we need one more Lemma.

**Lemma 2.4.3.** *a) For all $p$ and all $m$ the $B^{(r)}$-module $H^p(J^*(B, (C^m))^{B_{(r)}}) = H^p(B_{(r)}, C^m)$ is acyclic.*
*b) Let $X^* \in \mathcal{A}$ be a bounded below complex such that all $X^p$ and all $H^p(X^*)$ are acyclic $B^{(r)}$-modules. Then all cycle and boundary modules $Z^p(X^*), B^p(X^*)$ are acyclic as well. Moreover for such a complex we have canonical identifications $H^p(X^{*B^{(r)}}) = H^p(X^*)^{B^{(r)}}$.*

*Proof.* As we saw in the proof of Lemma 2.4.1

$$H^p(B_{(r)}, C^m) = \bigoplus_{i_0,\ldots,i_m} k[U_{i_0} \cap \ldots \cap U_{i_m}] \otimes_{k[G^{(r)}]} H^p(B_{(r)}, I)$$

Our first statement follows from this identification and Lemma 2.2.

The second statement is proved by an immediate induction on $p$.

**Corollary 2.4.4.** *The homomorphism of spectral sequences induced by the homomorphism of bycomplexes $F^{B^{(r)}}$ is an isomorphism.*

*Proof.* According to Lemmas 2.4.1 and 2.4.3 for all $p$ the homomorphism $F^{B^{(r)}}$ of bicomplexes defines a homomorphism of resolutions of a rational $B^{(r)}$-module $H^p(B_{(r)}, I)$:

$$\begin{array}{ccccc} H^p(I^*(C^0)^{B_{(r)}}) & \longrightarrow & H^p(I^*(C^1)^{B_{(r)}}) & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ H^p(J^{*,0}) & \longrightarrow & H^p(J^{*,1}) & \longrightarrow & \ldots \end{array}$$

Moreover the top resolution consists of acyclic $B^{(r)}$-modules, whereas the bottom one is the injective resolution. The standard comparison for resolutions implies readily that the induced map on complexes of $B^{(r)}$-invariants is a quasiisomorphism.

Finally we describe the situation in which the spectral sequence of § 1 will be used for the proof of the detection theorem. In conditions and notations introduced at the beginning of this section assume further that $\pi$ is a finite subgroup in $B(k)$. Let finally $M$ be a rational $\pi \ltimes G_{(r)}$-module. If we want to compute the cohomology groups $H^*(\pi \ltimes G_{(r)}, M)$ we may note that the quotient scheme $\pi \ltimes G/\pi \ltimes G_{(r)} = G/G_{(r)} = G^{(r)}$ is affine and hence, using once again the Theorem of Clein, Parshall, Scott and van der Kallen we get the following identifications:

$$H^*(\pi \ltimes G_{(r)}, M) = H^*(\pi \ltimes G, I) = H^*(\pi \ltimes B, I)$$

where $I = Ind_{\pi \ltimes G_{(r)}}^{\pi \ltimes G} M$ is the corresponding induced module. Note further that the same as before $I$ is a module over the commutative ring $k[\pi \ltimes G]^{\pi \ltimes G_{(r)}} = k[G^{(r)}]$ and the action of $\pi \ltimes G$ on $I$ is compatible with this module structure. Finally one checks easily that the action of $\pi \ltimes G$ on $k[G^{(r)}]$ which comes from the identification $k[\pi \ltimes G]^{\pi \ltimes G_{(r)}} = k[G^{(r)}]$ looks as follows: the group $G$ acts on $k[G^{(r)}]$ via the composition of the Frobenius map and the right regular representation whereas the group $\pi$ acts by conjugation). We take $X$ to be the quotient scheme of $G^{(r)}$ with respect to the action of $\pi \ltimes B^{(r)}$ (where as before $B^{(r)}$ acts on $G^{(r)}$ via the right regular representation and $\pi$ acts by conjugation. Note that the quotient exists and moreover we have a natural identification $X = G^{(r)}/(\pi \ltimes B^{(r)}) = \pi \backslash (G/B)^{(r)}$, where the action of $\pi$ on $(G/B)^{(r)} = G^{(r)}/B^{(r)}$ by conjugation may be also identified with the left regular action.This shows that the scheme $X$ may be also identified with the variety of double cosets: $X = \pi \backslash G^{(r)}/B^{(r)}$. The projection morphism $p : G^{(r)} \longrightarrow X$ may be written as a composition $G^{(r)} \to G^{(r)}/B^{(r)} = (G/B)^{(r)} \to \pi \backslash (G/B)^{(r)}$ and hence is affine. According to Theorem 1.8 we get canonical quasicoherent sheaves $\mathcal{H}^q$ on the scheme

$$X = G^{(r)}/\pi \ltimes B^{(r)} = \pi \setminus (G/B)^{(r)}$$

and a spectral sequence

$$E_2^{pq} = H^p(X, \mathcal{H}^q) \Longrightarrow H^{p+q}(\pi \ltimes B, I) = H^{p+q}(\pi \ltimes G_{(r)}, M) \tag{2.5.0}$$

We are interested in the multiplicative structure of $H^*(\pi \ltimes G_{(r)}, M)$ so we assume that $M = \Lambda$ is a rational $\pi \ltimes G_{(r)}$-algebra. In this case $I = Ind_{\pi\ltimes G_{(r)}}^{\pi\ltimes G}\Lambda$ is a rational $\pi \ltimes B$-algebra and hence (according to the results of § 1) the spectral sequence (2.5.0) has a canonical multiplicative structure. Since the cohomology $H^p(X, \mathcal{H}^q)$ are trivial for $p > dimX$ we conclude in the usual way (cf. [S-F-B], § 4) that the kernel of the edge homomorphism

$$H^*(\pi \ltimes G_{(r)}, \Lambda) \xrightarrow{\rho} H^0(Y, \mathcal{H}^*)$$

is a nilpotent ideal in $H^*(\pi \ltimes G_{(r)}, \Lambda)$. Thus we have established the following result.

**Theorem 2.5.** *A cohomology class $z \in H^*(\pi \ltimes G_{(r)}, \Lambda)$ is nilpotent if and only if its image in $H^0(Y, \mathcal{H}^*)$ is.*

**Corollary 2.5.1.** *Assume that the cohomology class $z \in H^*(\pi\ltimes G_{(r)}, \Lambda) = H^*(\pi\ltimes B, I)$ has the property that for every affine open subset $V \subset X$ with pull-back $U = p^{-1}(V) \subset G^{(r)}$ the image of $z$ under the canonical homomorphism*

$$H^*(\pi \ltimes B, I) \xrightarrow{\rho_V} H^*(\pi \ltimes B, k[U] \otimes_{k[G^{(r)}]} I)$$

*induced by the homomorphism of the rational $\pi \ltimes B$-algebras $I \longrightarrow k[U] \otimes_{k[G^{(r)}]} I$ is nilpotent. Then $z$ is itself nilpotent.*

*Proof.* It suffices to note the restriction of the global section $\rho(z) \in H^0(X, \mathcal{H}^*)$ to the affine open subset $V$ is an element in $\Gamma(V, \mathcal{H}^*) = H^*(\pi \ltimes B, k[U] \otimes_{k[G^{(r)}]} I)$, which according to the construction of the spectral sequence coincides with $\rho_V(z)$.

**Remark 2.5.2.** *The reason why Corollary 2.5.1 is useful is that cohomology $H^*(\pi \ltimes B, k[U] \otimes_{k[G^{(r)}]} I)$ is in many ways easier to understand than $H^*(\pi \ltimes B, I)$ since $k[U] \otimes_{k[G^{(r)}]} I$ has a large ring of operators with respect to which the action of $\pi\ltimes B$ is linear – namely $k[V] = k[U]^{\pi\ltimes B}$ whereas for $I$ itself the corresponding ring of operators is trivial (coincides with $k$) since the variety $X = \pi\backslash G/B$ is projective.*

## § 3. The Detection Theorem for $\pi \ltimes G_{(r)}$.

In this section we are going to prove the Detection Theorem for the finite group scheme $\pi \ltimes G_{(r)}$, where $\pi$, $B$ and $G$ are as in § 2.

**Theorem 3.1.** *Let $G$ be a connected smooth affine algebraic group over a field $k$ of characteristic $p$, let $B \subset G$ be a Borel subgroup in $G$ and let $\pi$ a finite $p$-subgroup in $B(k)$. Let finally $\Lambda$ be an associative unital rational $\pi \ltimes G_{(r)}$-algebra and let $z \in H^n(\pi \ltimes G_{(r)}, \Lambda)$ be a cohomology class which has the property that for any field extension $K/k$ and any elementary abelian subgroup scheme $i : \pi' \times \mathbb{G}_{a(r),K} \hookrightarrow \pi \ltimes G_{(r),K}$ the restriction $i^*(z_K)$ of $z_K$ to $\pi' \times \mathbb{G}_{a(r),K}$ is nilpotent. Then $z$ is itself nilpotent.*

*Proof.* We start with the following Lemma.

**Lemma 3.1.0.** *In the above notations and assumptions the detection theorem holds for the group scheme $\pi \ltimes B_{(r)}$*

*Proof.* Denote by $U$ the unipotent radical of $B$ (this is a local notation – in the main part of the proof $U$ will denote something entirely different). Since $\pi$ is a $p$-subgroup in $B(k)$ and $T(k)$ has no $p$-torsion (where $T = B/U$ is the corresponding torus) we immediately conclude that $\pi \subset U(k)$ and hence $\pi \ltimes U_{(r)}$ is a normal subgroup in $\pi \ltimes B_{(r)}$ with the corresponding qoutient group being equal to $T_{(r)}$. Since the group scheme $T_{(r)}$ has no cohomology in positive degrees (see [J], ch. 1, 4.3.) we conclude that the corresponding Hochschild-Serre spectral sequence degenerates providing isomorphisms

$$H^*(\pi \ltimes B_{(r)}, \Lambda) = H^*(\pi \ltimes U_{(r)}, \Lambda)^{T_{(r)}} \subset H^*(\pi \ltimes U_{(r)}, \Lambda)$$

Thus the restriction map $H^*(\pi \ltimes B_{(r)}, \Lambda) \to H^*(\pi \ltimes U_{(r)}, \Lambda)$ is injective and to finish the proof it suffices to note that for $\pi \ltimes U_{(r)}$ the detection theorem holds according to the theorem of Chris Bendel [Be].

Corollary 2.5.1 shows that to prove the nilpotence of $z$ it suffices to show that for any open affine $V \subset \pi \setminus G^{(r)}/B^{(r)}$ the image of $z \in H^n(\pi \ltimes G_{(r)}, \Lambda) = H^n(\pi \ltimes B, Ind\ \Lambda)$ under the canonical homomorphism

$$H^n(\pi \ltimes B, Ind\ \Lambda) \xrightarrow{\rho_V} H^n(\pi \ltimes B, k[U] \otimes_{k[G^{(r)}]} Ind\ \Lambda)$$

(where $U = p^{-1}(V) \subset G^{(r)}$) is nilpotent.

To make the forthcoming computations more transparent we need the following elementary facts about Frobenius twist.

**Lemma 3.2.** *Let $Y$ be any scheme of finite type over the field $k$ of characteristic $p$. Consider the Frobenius morphism $Y \xrightarrow{F^r} Y^{(r)}$. For an open subscheme $W \subset Y^{(r)}$ denote by $W^{(-r)}$ the open subscheme $W^{(-r)} = (F^r)^{-1}(W) \subset Y$. Then we get a canonical isomorphism $(W^{(-r)})^{(r)} = W$ and the the restriction $(F^r_Y)_{|W^{(-r)}} : W^{(-r)} \to W$ coincides with $F^r_{W^{(-r)}}$.*

*Proof.* We may obviously assume that $r = 1$ and the scheme $Y = Spec\ A$ is affine. Note that for any $Y$ we have (according to definitions) a cartesian square

$$\begin{array}{ccc} Y^{(1)} & \longrightarrow & Spec\ k \\ {\scriptstyle \Phi_Y}\downarrow & & \downarrow{\scriptstyle Specf} \\ Y & \longrightarrow & Spec\ k \end{array}$$

where $f : k \to k$ is the Frobenius embedding $f(\lambda) = \lambda^p$. From this we readily conclude that for any morphism $s : W \to Y$ of schemes over $k$ the following square is cartesian

$$\begin{array}{ccc} W^{(1)} & \xrightarrow{s^{(1)}} & Y^{(1)} \\ {\scriptstyle \Phi_W}\downarrow & & \downarrow{\scriptstyle \Phi_Y} \\ W & \xrightarrow{s} & Y \end{array}$$

The previous remark implies in particular that for an open subscheme $W \subset Y$ the scheme $W^{(1)}$ coincides with the open subscheme $(\Phi_Y)^{-1}(W)$ of $Y^{(1)}$. Thus we get two operations $\Phi_Y^{-1}$ and $F_Y^{-1}$ relating open subschemees in $Y$ and $Y^{(1)}$. We have to show that these two operations are mutually inverse bijections. Since both operations commute with taking finite unions and intersections it suffices to consider their action on principal open subschemes. Start with an open subscheme $W = Y_a \subset Y$ $(a \in A)$. Then $F_Y^{-1}(\Phi_Y^{-1}(W)) = Y_{a^p} = Y_a = W$. Start with an open subscheme $U = (Y^{(1)})_b \subset Y^{(1)}$ $(b \in A \otimes_f k)$. Write $b = \sum a_i \otimes_f \lambda_i$ for appropriate $a_i \in A, \lambda_i \in k$. In this case $\Phi_Y^{-1}(F_Y^{-1}(U))$ is the principal open defined by the element $\sum \lambda_i a_i^p \otimes_f 1 = b^p$ and the statement follows.

**Lemma 3.2.1.** *In conditions and notations of Lemma 3.2 an open subscheme $W \subset Y^{(r)}$ is affine iff $W^{(-r)}$ is affine.*

*Proof.* If $W^{(-r)}$ is affine then $W = (W^{(-r)})^{(r)}$ is obviously also affine. On the other hand if $W$ is affine then $W^{(-r)}$ is affine as well since the morphism $F^r : Y \to Y^{(r)}$ is finite.

**Lemma 3.2.2.** *Let $\pi$ be a finite group acting (on the left) on a quaiprojective scheme $Y$. Then we get an induced action of $\pi$ on $Y^{(r)}$ and furthermore $\pi \setminus Y^{(r)} = (\pi \setminus Y)^{(r)}$.*

*Proof.* The first statement is obvious. To prove the second one it suffices obviously to treat the case when $Y = Spec\ A$ is an affine scheme. In this case our statement is equivalent to the relation $(A \otimes_f k)^\pi = A^\pi \otimes_f k$ which follows from the fact that $- \otimes_f k$ is an exact functor.

Denote by $I^\bullet$ the standard $\pi \ltimes G_{(r)}$-injective resolution of $\Lambda$. Thus $I^n = \Lambda \otimes k[\pi \ltimes G_{(r)}]^{\otimes(n+1)}$ with the right regular action of $\pi \ltimes G_{(r)}$ on the last tensor factor. The differential in $I^\bullet$ is given by the formula (in which we identify $I^n$ with $Mor((\pi \ltimes G_{(r)})^{\times(n+1)}, \Lambda_a)$ -cf. [J] ch. 1, 3.3).

$$df(g_0, ..., g_{n+1}) = {}^{g_0}f(g_1, ..., g_{n+1}) + \sum_{i=1}^{n+1} (-1)^i f(g_0, ..., g_{i-1}g_i, ..., g_{n+1})$$

Here $A$ is an arbitrary commutative $k$-algebra, $g_i \in (\pi \ltimes G_{(r)})(A)$ and we utilize left exponential notation for the action of $\pi \ltimes G_{(r)}$ on $\Lambda$. Note that $I^\bullet$ is a special case of the complex ( which we denote $C^\bullet(\Lambda, \pi \ltimes G_{(r)}, Z)) = C_k^\bullet(\Lambda, \pi \ltimes G_{(r)}, Z))$ defined for every affine scheme $Z$ provided with the (left) action of $\pi \ltimes G_{(r)}$. The terms of this complex have the form

$$C^n(\Lambda, \pi \ltimes G_{(r)}, Z) = \Lambda \otimes k[\pi \ltimes G_{(r)}]^{\otimes(n)} \otimes k[Z]$$

and the differential is given by essentially the same formula as above but where we use the left action of $\pi \ltimes G_{(r)}$ on $Z$ in the last summand. Note further that for any scheme $Z$ provided with an action of $\pi \ltimes G_{(r)}$ the complex $C^\bullet(\Lambda, \pi \ltimes G_{(r)}, Z)$ is a differential graded $k$-algebra with respect to the product operation given by the

formula (in which we use the same identification $C^n(\Lambda, \pi \ltimes G_{(r)}, Z) = Mor((\pi \ltimes G_{(r)})^{\times n} \times Z, \Lambda_a)$ as above)
(3.3.0)
$(f \cup f')(g_1, ..., g_{n+m}, z) = f(g_1, ..., g_n, g_{n+1} \cdot ... \cdot g_{n+m} \cdot z) \cdot {}^{g_1 \cdot ... \cdot g_n} f'(g_{n+1}, ..., g_{n+m}, z)$

Here $A$ is an arbitrary commutative $k$-algebra, $g_i \in (\pi \ltimes G_{(r)})(A), z \in Z(A)$ and we utilize left exponential notation for the action of $\pi \ltimes G_{(r)}$ on $\Lambda$. Since the induction functor $Ind = Ind_{\pi \ltimes G_{(r)}}^{\pi \ltimes G}$ is exact and takes injectives to injectives we conclude that $Ind\ I^\bullet$ is an injective resolution of the rational $\pi \ltimes G$-module $Ind\ \Lambda$. Moreover since $Ind\ k[\pi \ltimes G_{(r)}] = k[\pi \ltimes G]$ we conclude that the terms of this resolution have the form $Ind\ I^n = \Lambda \otimes (k[\pi \ltimes G_{(r)}])^{\otimes n} \otimes k[\pi \ltimes G]$. The presence for any two rational $\pi \ltimes G_{(r)}$-modules $M, M'$ of the natural homomorphism $Ind\ M \otimes Ind\ M' \to Ind\ (M \otimes M')$ implies readily that $Ind\ I^\bullet$ inherits a structure of the differential graded algebra. Moreover a straightforward verification shows that this DGA coincides with $C^\bullet(\Lambda, \pi \ltimes G_{(r)}, \pi \ltimes G)$. We note also that $Ind\ I^\bullet$ is not just a differential graded algebra over $k$ but actually a differential graded algebra over $k[\pi \ltimes G]^{\pi \ltimes G_{(r)}} = k[G]^{G_{(r)}} = k[G^{(r)}]$.

Let $V$ be an open affine in $X = \pi \setminus G^{(r)}/B^{(r)}$ and let $W$ and $U$ denote its inverse images to $G^{(r)}/B^{(r)} = (G/B)^{(r)}$ and $G^{(r)}$ respectively. Denote further by $V^{(-r)} \subset \pi \setminus G/B, W^{(-r)} \subset G/B$ and $U^{(-r)} \subset G$ the open affines corresponding to $V, W$ and $U$ according to Lemma 3.2.

According to what was said above we have to compute the image of $z$ in $H^n(\pi \ltimes B, k[U] \otimes_{k[G^{(r)}]} Ind\ \Lambda)$. To compute the latter cohomology group we note that $k[U] \otimes_{k[G^{(r)}]} Ind\ I^\bullet$ is a resolution of the rational $\pi \ltimes B$-module $k[U] \otimes_{k[G^{(r)}]} Ind\ \Lambda$. Moreover Lemma 3.4 below shows that this resolution consists of acyclic $\pi \ltimes B$-modules.

**Lemma 3.4.** *Let $M$ be an injective $\pi \ltimes G_{(r)}$-module. Then $k[U] \otimes_{k[G^{(r)}]} Ind\ M$ is an acyclic $\pi \ltimes B$-module.*

*Proof.* It clearly suffices to treat the special case $M = k[\pi \ltimes G_{(r)}]$. In this case $Ind\ M = k[\pi \ltimes G] = k[G]^{\times \pi}$. The action of the group $G$ here is componentwise and the action of $\pi$ looks as follows:

$$(\tau \cdot f)_\sigma = f^\tau_{\sigma\tau}$$

where for a function $f \in k[G]$ we denote by $f^\tau$ the conjugated function $f^\tau(g) = f(\tau^{-1} g \tau)$. The identification of $k[G^{(r)}] = k[G]^{G_{(r)}}$ with $k[\pi \ltimes G]^{\pi \ltimes G_{(r)}}$ associates to a $G_{(r)}$-invariant function $f \in k[G]$ a $\pi \ltimes G_{(r)}$-invariant function on $\pi \ltimes G$ with $\sigma$-component equal to $f^{\sigma^{-1}}$. Thus the action of the ring $k[G^{(r)}]$ on $k[G]^{\times \pi}$ is componentwise and the action on the $\sigma$-component coincides with multiplication by $f^{\sigma^{-1}}$. Since the action of $k[G^{(r)}]$ on $k[G]^{\times \pi}$ is componentwise we conclude that

$$k[U] \otimes_{k[G^{(r)}]} k[\pi \ltimes G] = \bigoplus_{\sigma \in \pi} k[U]^\sigma \otimes_{k[G^{(r)}]} k[G]$$

where $k[U]^\sigma$ coincides with $k[U]$, but has a new $k[G^{(r)}]$-algebra structure given by the homomorphism $k[G^{(r)}] \xrightarrow{f\mapsto f^{\sigma^{-1}}} k[G^{(r)}] \to k[U]$. The action of the group $B_{(r)}$ on $k[\pi \ltimes G]$ is $k[G^{(r)}]$-linear and hence

$$H^*(B_{(r)}, k[U] \otimes_{k[G^{(r)}]} k[\pi \ltimes G]) = k[U] \otimes_{k[G^{(r)}]} H^*(B_{(r)}, k[\pi \ltimes G]) = 0 \text{ for } * > 0$$

Where the vanishing of $H^*(B_{(r)}, k[\pi \ltimes G])$ follows from [J], ch.1, 4.12 and 5.13. Furthermore the $B^{(r)}$-module $H^0(B_{(r)}, k[U] \otimes_{k[G^{(r)}]} k[\pi \ltimes G])$ is acyclic according to Corollary 2.3 and hence $k[U] \otimes_{k[G^{(r)}]} k[\pi \ltimes G]$ is an acyclic $B$-module. Finally

$$H^0(B, k[U] \otimes_{k[G^{(r)}]} k[\pi \ltimes G]) = \bigoplus_{\sigma\in\pi} H^0(B, k[U]^\sigma \otimes_{k[G^{(r)}]} k[G]))$$

This is an induced and hence acyclic $\pi$-module. Note for future use that we have also established the following formula:

$$\begin{aligned} H^0(\pi \ltimes B, k[U] \otimes_{k[G^{(r)}]} k[\pi \ltimes G]) = H^0(B, k[U] \otimes_{k[G^{(r)}]} k[G]) &= H^0(B, k[U^{(-r)}]) \\ &= k[W^{(-r)}] \end{aligned} \tag{3.4.0}$$

Here at the last stage of computation we used the following result

**Lemma 3.4.1.** *Let $W \subset G/B$ be an open affine subset. Denote by $U$ its inverse image in $G$. Then $U$ is also affine and the canonical map $k[W] \to k[U]^B$ is an isomorphism.*

*Proof.* The fact that $U$ is affine is proved in [J], ch. 1, 5.7. To prove the second statement we note that since $k[U]$ is a faithfully flat $k[W]$-algebra (see[J], ch. 1, 5.7) it suffices to show that the induced homomorphism $k[U] \to k[U] \otimes_{k[W]} H^0(B, k[U]) = H^0(B, k[U]_{\text{triv}} \otimes_{k[W]} k[U])$ is an isomorphism. The Cartesian diagram

$$\begin{array}{ccc} U \times B & \xrightarrow{\mu} & U \\ {\scriptstyle pr_1}\downarrow & & \downarrow \\ U & \longrightarrow & W \end{array}$$

shows that $B$-module $k[U]_{\text{triv}} \otimes_{k[W]} k[U])$ identifies canonically with $k[U]_{\text{triv}} \otimes k[B]$ and hence $H^0(B, k[U]_{\text{triv}} \otimes_{k[W]} k[U]) = H^0(B, k[U]_{\text{triv}} \otimes k[B]) = k[U]$.

Lemma 3.4 implies that $H^*(\pi \ltimes B, k[U] \otimes_{k[G^{(r)}]} Ind\ \Lambda)$ coincides with the cohomology of the differential graded $k[V]$-algebra $J^\bullet = (k[U] \otimes_{k[G^{(r)}]} Ind\ I^\bullet)^{\pi\ltimes B}$. According to [S-F-B] Proposition 4.2 to verify that the cohomology class $t = \rho_V(z) \in H^n(J^\bullet)$ is nilpotent it suffices to check that for every point $v \in V$ the image $t(v)$ of $t$ in $H^n(k(v) \otimes_{k[V]} J^\bullet)$ is nilpotent. Note further that $J^n = \Lambda \otimes k[\pi \ltimes G_{(r)}]^{\otimes n} \otimes (k[U] \otimes_{k[G^{(r)}]} k[\pi \ltimes G])^{\pi\ltimes B}$. Applying (3.4.0) we get the following formula for $J^n$:

$$J^n = \Lambda \otimes k[\pi \ltimes G_{(r)}]^{\otimes n} \otimes k[W^{(-r)}]$$

which readily implies that the DGA $J^\bullet$ coincides with $C^\bullet(\Lambda, \pi \ltimes G_{(r)}, W^{(-r)})$, where we use implicitely the obvious fact that $W^{(-r)}$ is stable with respect to the left action of $\pi \ltimes G_{(r)}$ on $G/B$.

Fix a point $v \in V$ and tensor $J^\bullet$ with $k(v)$ over $k[V]$. In this way we get a differential graded $k(v)$-algebra $J^\bullet(v) = k(v) \otimes_{k[V]} J^\bullet$. The terms of this DGA look as follows:

$$J^n(v) = \Lambda \otimes k[\pi \ltimes G_{(r)}]^{\otimes n} \otimes k(v)[W_v^{(-r)}] = \Lambda_{k(v)} \otimes_{k(v)} k(v)[\pi \ltimes \mathbb{G}_{(r)}]^{\otimes n} \otimes_{k(v)} \\ \otimes_{k(v)} k(v)[W_v^{(-r)}]$$

where $W_v^{(-r)}$ denotes the fiber of the finite morphism $W^{(-r)} \xrightarrow{F^r} W \to \pi \setminus W = V$ over the point $v \in V$.

Fix an algebraic closure $K$ of $k(v)$ and denote by $\overline{v}$ the corresponding geometric point $Spec\ K \to Spec\ k(v) \to V$. Finally extend scalars in the differential graded algebra $J^\bullet(v)$ from $k(v)$ to $K$, thus getting a DGA $J^\bullet(\overline{v}) = K \otimes_{k(v)} J^\bullet(v)$ over $K$. The terms of this DGA are of the form

$$J^n(\overline{v}) = \Lambda_K \otimes_K K[\pi \ltimes G_{(r)}]^{\otimes n} \otimes_K K[W_{\overline{v}}^{(-r)}]$$

where $W_{\overline{v}}^{(-r)}$ denotes the fiber of the finite morphism $W^{(-r)} \xrightarrow{F^r} W \xrightarrow{q} V = \pi \setminus W$ over the geometric point $\overline{v}$. An easy computation shows that the DGA $J^\bullet(\overline{v})$ coincides with $C_K^\bullet(\Lambda_K, \pi \ltimes G_{(r)}, W_{\overline{v}}^{(-r)})$. In particular the products in $J^\bullet(\overline{v})$ are given by the formula

(3.3.1)
$$(f \cup f')(g_1, ..., g_{n+m}, w) = f(g_1, ..., g_n, g_{n+1} \cdot ... \cdot g_{n+m} \cdot w) \cdot {}^{g_1 \cdot ... \cdot g_n} f'(g_{n+1}, ..., g_{n+m}, w)$$

where this time $A$ is an arbitrary commutative $K$-algebra, $g_i \in (\pi \ltimes G_{(r)})(A), w \in W_{\overline{v}}^{(-r)}(A)$.

Note that all terms of $J^\bullet(\overline{v})$ are modules over $K[W_{\overline{v}}]$ (even over $K[W_{\overline{v}}^{(-r)}]$), however $J^\bullet(\overline{v})$ is not a DGA over $K[W_{\overline{v}}]$ ($K[W_v]$ is not in the center of $J^\bullet(\overline{v})$ and does not consist of cocycles). Denote by $w_1, ..., w_n$ the (closed) points of $W_{\overline{v}}$. Since the action of $\pi$ on the set of closed points of $W_v$ is transitive, for every point $w \in W_v$ the extension field $k(w)$ is normal over $k(v)$ and canonical homomorphism $Stab_\pi(w) \to Gal(k(w)/k(v)$ is surjective (see [Bou],?) one concludes easily (see [SGA], ex. 5, sec. 2 or [S-V] Lemma 5.1) that the action of $\pi$ on the set $\{w_1, ..., w_n\}$ is still transitive. Denote finally by $I$ the ideal of $K[W_{\overline{v}}]$, consisting of functions vanishing at all points $w_i$. Thus $I$ is a nilpotent ideal in $K[W_{\overline{v}}]$ and $K[W_{\overline{v}}]/I = \prod_i K(w_i) = \prod_i K$.

**Lemma 3.5.** *The left ideal $J^\bullet(\overline{v}) \cdot I$ is actually a nilpotent two-sided Differential Graded Ideal (DGI).*

*Proof.* To prove the statement we have to rewrite the formula defining the product in the DGA $J^\bullet(\overline{v})$ in a more algebraic way. To do so we define the following operations on cochains. Given a pair of integers $n, m$ and cochains $f \in J^n(\overline{v}) = Mor((\pi \ltimes G_{(r)})^{\times n} \times W_{\overline{v}}, \Lambda_{K,a}), f' \in J^m(\overline{v}) = Mor((\pi \ltimes G_{(r)})^{\times m} \times W_{\overline{v}}, \Lambda_{K,a})$ we define two $(n+m)$-dimensional cochains $P_{n,m}(f), Q_{n,m}(f')$ via the formula

$$P_{n,m}(f)(g_1, ..., g_{n+m}, w) = f(g_1, ..., g_n, g_{n+1} \cdot ... \cdot g_{n+m} \cdot w)$$
$$Q_{n,m}(f')(g_1, ..., g_{n+m}, w) = {}^{g_1 \cdot ... \cdot g_n} f'(g_{n+1}, ..., g_{n+m}, w)$$

It's clear from the above definition that the homomorphism $P_{n,m} : \Lambda_K \otimes K[\pi \ltimes G_{(r)}]^{\otimes n} \otimes K[W_{\overline{v}}] \longrightarrow \Lambda_K \otimes K[\pi \ltimes G_{(r)}]^{\otimes (n+m)} \otimes K[W_{\overline{v}}]$ is induced by the homomorphism of commutative $K$-algebras $K[W_{\overline{v}}] \longrightarrow K[(\pi \ltimes G_{(r)})^{\times m} \times W_{\overline{v}}] = K[(\pi \ltimes G_{(r)})]^{\otimes m} \otimes K[W_{\overline{v}}]$, corresponding to the multiplication morphism $(\pi \ltimes G_{(r)})^{\times m} \times W_{\overline{v}} \longrightarrow W_{\overline{v}}$, whereas the homomorphism $Q_{n,m} : \Lambda_K \otimes K[\pi \ltimes G_{(r)}]^{\otimes m} \otimes K[W_v] \longrightarrow \Lambda_K \otimes K[\pi \ltimes G_{(r)}]^{\otimes (n+m)} \otimes K[W_{\overline{v}}]$ is induced by the iterated diagonal morphism $\Delta : \Lambda \longrightarrow \Lambda \otimes k[\pi \ltimes G_{(r)}]^{\otimes n}$ corresponding to the rational $\pi \ltimes G_{(r)}$-module $\Lambda$. Using these operations $P$ and $Q$ we may rewrite the formula for $f \cup f'$ in the following form

$$f \cup f' = P_{n,m}(f) * Q_{n,m}(f')$$

where $*$ on the right denotes the usual product operation on the algebra $\Lambda_K \otimes K[\pi \ltimes G_{(r)}]^{\otimes (n+m)} \otimes K[W_{\overline{v}}]$. An immediate verification shows $J^m(\overline{v}) \cdot I^k = \Lambda_K \otimes K[\pi \ltimes G_{(r)}]^{\otimes n} \otimes I^k$. Moreover since the ideals $I^k$ are stable with respect to the action of $\pi \ltimes G_{(r)}$ (note that $G_{(r)}$ acts trivially on $W$ and hence on $W_{\overline{v}}$ and $I$ is obviously stable with respect to the action of the discrete group $\pi$) we conclude easily from the above formulae for $P$ and $Q$ that $P(J^n(\overline{v}) \cdot I^k) \subset J^{n+m}(\overline{v}) \cdot I^k, Q(J^m(\overline{v}) \cdot I^k) \subset J^{n+m}(\overline{v}) \cdot I^k$. The obvious formula $(J^{n+m}(\overline{v}) \cdot I^k) * (J^{n+m}(\overline{v}) \cdot I^l) = J^{n+m}(\overline{v}) \cdot I^{k+l}$ and the nilpotence of $I$ show that $J^\bullet(\overline{v}) \cdot I$ is a nilpotent two-sided ideal. To show that this ideal is a DGI it suffices to check that $d(f) \in J^1(\overline{v}) \cdot I$ for any function $f \in I$, which is straightforward from the above computations and definitions.

**Remark 3.5.1.** *If we try to replace $I$ by the corresponding ideal $I' \subset K[W_{\overline{v}}^{(-r)}]$ the proof won't go since the ideal $I'$ is not stable with respect to the action of $G_{(r)}$.*

Lemma 3.5 shows that to verify nilpotence of $t(v) \in H^*(J^\bullet(\overline{v}))$ we may reduce $J^\bullet(\overline{v})$ modulo the ideal $J^\bullet(\overline{v}) \cdot I$. Reducing $J^\bullet(\overline{v})$ modulo $J^\bullet(\overline{v}) \cdot I$ amounts to replacing the tensor factor $K[W_{\overline{v}}^{(-r)}]$ appearing in $J^n(\overline{v})$ by $K[W_{\overline{v}}^{(-r)}]/I = \prod_i K[W_{w_i}^{(-r)}]$, where $W_{w_i}^{(-r)}$ denotes the fiber of the finite morphism $W^{(-r)} \xrightarrow{F^r} W$ over $w_i$. We need to know the structure of this rational $\pi \ltimes G_{(r)}$-module. To do so we need to figure out the fibers of the Frobenius morphism $F^r : G/B \longrightarrow (G/B)^{(r)} = G^{(r)}/B^{(r)}$.

**Lemma 3.6.** *Let $g \in G(k)$ be a rational point and let $g^{(r)}$ be its image in $G^{(r)}$. Then the fiber of the Frobenius morphism $F^r : G/B \longrightarrow G^{(r)}/B^{(r)}$ over the rational point $g^{(r)} \cdot B^{(r)} \in G^{(r)}/B^{(r)}$ identifies canonically with $G_{(r)}/gB_{(r)}g^{-1}$.*

*Proof.* Note that the scheme $G^{(r)}/B^{(r)} = (G/G_{(r)})/(B/B_{(r)})$ may be identified with $G/B \cdot G_{(r)}$. This identification implies readily that the following diagram of schemes is Cartesian

$$\begin{array}{ccc} G/B_{(r)} & \longrightarrow & G/G_{(r)} = G^{(r)} \\ \downarrow & & \downarrow \\ G/B & \longrightarrow & G/B \cdot G_{(r)} = G^{(r)}/B^{(r)} \end{array}$$

Hence the fiber of the bottom arrow over $g^{(r)} \cdot B^{(r)}$ coincides with the fiber of the top arrow over $g^{(r)}$. Finally the fiber of the top arrow over $g^{(r)}$ coincides with $G_{(r)}/gB_{(r)}g^{-1}$ as one sees from the Cartesian diagram (see [S-F-B] (4.1.1)):

$$\begin{array}{ccc} G_{(r)}/gB_{(r)}g^{-1} & \longrightarrow & Spec\ k \\ {\scriptstyle x\mapsto xg}\downarrow & & {\scriptstyle g^{(r)}}\downarrow \\ G/B_{(r)} & \longrightarrow & G/G_{(r)} = G^{(r)} \end{array}$$

**Corollary 3.6.1.** *In conditions and notations of Lemma 3.6 let $h \in G(k)$ be another rational point. Then we have a commutative diagram relating the fibers of $F^r$ over the rational points $g^{(r)}, (hg)^{(r)} \in G^{(r)}$*

$$\begin{array}{ccc} G_{(r)}/gB_{(r)}g^{-1} & \xrightarrow{\sim} & (G/B)_{g^{(r)}} \\ {\scriptstyle Ad(h)}\downarrow & & {\scriptstyle h}\downarrow \\ G_{(r)}/hgB_{(r)}(hg)^{-1} & \xrightarrow{\sim} & (G/B)_{(hg)^{(r)}} \end{array}$$

*Here the horisontal arrows are the identifications of Lemma 3.6, the left vertical arrow is conjugation by $h$ and right vertical arrow is left multiplication by $h$.*

*Proof.* This follows immediately from the explicit formulae for the maps involved in the above diagram.

**Proposition 3.7.** *The rational $(\pi \ltimes G_{(r)})_K$-module $K[W_{\overline{v}}^{(-r)}]/I = \prod_i K[W_{w_i}^{(-r)}]$ identifies with $K[\pi \ltimes G_{(r)}]^{\pi_1 \ltimes gB_{(r),K}g^{-1}}$, where $\pi_1$ is the stabilizer of the point $w_1$ and $g \in G(K)$ is an appropriate element.*

*Proof.* Since the field $K$ is algebraically closed and the morphism $F^r : G/B \to (G/B)^{(r)}$ is purely inseparable, the point $w_1$ lifts canonically to a point $w_1^{(-r)} \in G/B$. Using once again the fact that $K$ is algebraically closed we conclude that $w_1^{(-r)}$ admits a lifting $g \in G(K)$, i.e. $w_1^{(-r)} = g \cdot B$, $w_1 = g^{(r)} \cdot B^{(r)}$. According to Lemma 3.6 for any $\sigma \in \pi$ the fiber $(G/B)_{\sigma w_1}$ identifies with $G_{(r),K}/\sigma g B_{(r),K}(\sigma g)^{-1}$. So $\coprod_i W_{w_i}^{(-r)} = \coprod_{\sigma \in \pi/\pi_1} G_{(r),K}/\sigma g B_{(r),K}(\sigma g)^{-1} = (\pi \ltimes G_{(r),K})/(\pi_1 \ltimes gB_{(r),K}g^{-1})$. Finally Corollary 3.6.1 implies easily that this isomorphism is compatible with the left action of $\pi \ltimes G_{(r),K}$ on schemes involved.

**Corollary 3.7.1.** *The DGA* $J^\bullet(\overline{v})/J^\bullet(\overline{v})\cdot I$ *identifies with* $(I^\bullet \otimes_k K)^{\pi_1 \ltimes gB_{(r),K}g^{-1}}$ *and hence its cohomology identifies with* $H^*(\pi_1 \ltimes gB_{(r),K}g^{-1}, \Lambda_K)$*.*

**End of the Proof of the Theorem 3.1** We can sum up all the previous computations by saying that to verify the nilpotence of the cohomology class $z \in H^n(\pi \ltimes G_{(r)}, \Lambda) = H^n(\pi \ltimes B, Ind\ \Lambda)$ it suffices to check that for any open affine $V \subset \pi \setminus G^{(r)}/B^{(r)}$ and any geometric point $\overline{v} : Spec\ K \to V$ the image of $z$ under the natural homomorphism

$$\begin{aligned} H^n(\pi \ltimes G_{(r)}, \Lambda) &= H^n(\pi \ltimes B, Ind\ \Lambda) \to H^n(\pi \ltimes B, k[U] \otimes_{k[G^{(r)}]} Ind\ \Lambda) = \\ &= H^n(C^*(\Lambda, \pi \ltimes G_{(r)}, W^{(-r)})) \to H^n(C^*_K(\Lambda_K, \pi \ltimes G_{(r),K}, W^{(-r)}_{\overline{v}}) \to \\ &\to H^n(C^*_K(\Lambda_K, \pi \ltimes G_{(r),K}, \coprod_i W^{(-r)}_{w_i}) = H^n(\pi_1 \ltimes gB_{(r),K}g^{-1}, \Lambda_K) \end{aligned}$$

is nilpotent. An easy computation shows that the above composition takes $z$ to $(z_K)_{|\pi_1 \ltimes gB_{(r)}g^{-1}}$. Furthermore the resulting cohomology class of $\pi_1 \ltimes gB_{(r),K}g^{-1}$ still have the same defining property : it restricts nilpotently to all elementary abelian subgroups and hence is nilpotent according to Lemma 3.1.0.

## §4. The General Case of the Detection Theorem

**Theorem 4.1.** *Let $G/k$ be a finite group scheme over a field $k$ of positive characteristic $p$. Let further $\Lambda$ be an associative unital rational $G$-algebra and let $z \in H^n(G, \Lambda)$ be a cohomology class. Assume that for any field extension $K/k$ and any elementary abelian subgroup scheme $\nu : \pi_0 \times \mathbb{G}_{a(r),K} \hookrightarrow G_K$ the pull-back $\nu^*(z_K) \in H^n(\pi_0 \times \mathbb{G}_{a(r),K}, \Lambda)$ is nilpotent. Then $z$ is nilpotent itself.*

*Proof.* Extending scalars we may obviously assume that the base field $k$ is algebraically closed. In this case the group scheme $G$ identifies canonically with the semidirect product $G = \pi \ltimes G_0$ where $\pi = G(k)$ is a finite discrete group and $G_0$ is the connected component of $G$. The group scheme $G_0$ is obviously infinitesimal and we denote by $r$ its height. Denote by $\pi'$ the Sylow $p$-subgroup in $\pi$. The usual transfer argument (see [Be], Lemma 5.4) shows that the restriction homomorphism $H^*(\pi \ltimes G_0, \Lambda) \xrightarrow{res} H^*(\pi' \ltimes G_0, \Lambda)$ is injective. Thus in what follows we may assume that $G = \pi \ltimes G_0$ where $\pi$ is a finite $p$-group and $G_0$ is an infinitesimal group scheme of height $r$. Embed $G$ into $GL_n = GL_{n,k}$ with $n$ large enough.

**Lemma 4.2.** *There exists a Borel subgroup $B \subset GL_n$ such that $\pi \subset B(k)$.*

*Proof.* The embedding $G \hookrightarrow GL_n$ makes $k^n$ into a rational $G$-module and in particular into a $\pi$-module. Since $\pi$ is a $p$-group and $char\ k = p$ the only simple $\pi$-module is the trivial $\pi$-module $k$. This implies readily the existence of a flag of $\pi$-invariant subspaces

$$0 = W_0 \subset W_1 \subset\ \ldots\ \subset W_n = k^n \qquad dim_k W_i = i$$

such that the action of $\pi$ on all subsequent factors $W_i/W_{i-1}$ is trivial. Now it suffices to take $B$ to be the Borel subgroup in $GL_n$ determined by this flag.

Since $G_0$ was assumed to be of height $r$ we conclude that $G_0 \subset GL_{n(r)}$ and hence $G \subset G' = \pi \ltimes GL_{n(r)}$. Set $\Lambda' = Ind_G^{G'} \Lambda$. Then $\Lambda'$ is an associative unital $G'$-algebra. Moreover $H^*(G, \Lambda) = H^*(G', \Lambda')$ as graded algebras. Denote by $z' \in H^*(G', \Lambda')$ the cohomology class corresponding to $z$ under the above identification. In view of Theorem 3.1 to show that $z'$ is nilpotent it suffices to show that for any field extension $K/k$ and any elementary abelian subgroup scheme $\nu' : \pi_0 \times \mathbb{G}_{a(s),K} \hookrightarrow G'_K$ the pull-back $(\nu')^*(z'_K)$ is nilpotent. This fact is proved in exactly the same way as the corresponding statement was proved for infinitesimal groups in [S-F-B] , Theorem 4.3. For the sake of completeness we remind the proof.

Extending scalars and replacing $GL_{n,k}$ by $GL_{n,K}$ it suffices to show that for any elementary abelian subgroup scheme $\nu' : \pi_0 \times \mathbb{G}_{a(s)} \hookrightarrow \pi \ltimes GL_{n(r)}$ the restriction of $z'$ to $\pi_0 \times \mathbb{G}_{a(s)}$ is nilpotent. Denote $\pi_0 \times \mathbb{G}_{a(s)}$ by $H'$ and set $H = H' \times_{G'} G$, so that we have a cartesian diagram of finite group schemes

$$\begin{tikzcd} H \arrow[r, "\nu"] \arrow[d] & G \arrow[d] \\ H' \arrow[r, "\nu'"] & G' \end{tikzcd}$$

Note that $H$ is a closed subgroup scheme in an elementary abelian group scheme $H'$ and hence is elementary abelian as well. Consider the homomorphism of rational $H'$-modules

$$\theta : \Lambda' = Ind_G^{G'}(\Lambda) \to Ind_H^{H'}(\Lambda)$$

determined by adjointness of $Ind$ and $Res$. This homomorphism is surjective. To prove this consider the commutative diagram of schemes:

$$\begin{tikzcd} H' \arrow[r, "\nu'"] \arrow[d, "p_H"'] & G' \arrow[d, "p_G"'] \\ H'/H \arrow[r, "\overline{\nu}'"] & G'/G \end{tikzcd}$$

Note that there are canonical quasicoherent sheaves $\text{Ł}_G(\Lambda)$ on $G'/G$ and $\text{Ł}_H(\Lambda)$ on $H'/H$ with the property that $\Gamma(G'/G, \text{Ł}_G(\Lambda)) = Ind_G^{G'}(\Lambda), \Gamma(H'/H, \text{Ł}_H(\Lambda)) = Ind_H^{H'}(\Lambda)$ and $\Gamma(G', p_G^*(\text{Ł}_G)) = k[G'] \otimes \Lambda, \Gamma(H', p_H^*(\text{Ł}_H)) = k[H'] \otimes \Lambda$ - see [J], ch.1, 5.8 or [S-F-B], §3. Thus $p_H^*(\overline{\nu}')^*(\text{Ł}_G) = (\nu')^* p_G^*(\text{Ł}_G)$ is a quasicoherent $\mathcal{O}_{H'}$-module corresponding to the $k[H']$-module $k[H'] \otimes \Lambda$ and the same is true for $p_H^*(\text{Ł}_H)$. This shows that the canonical homomorphism of quasicoherent $\mathcal{O}_{H'/H}$-modules $(\overline{\nu}')^*(\text{Ł}_G) \to \text{Ł}_H$ becomes an isomorphism when we apply $p_H^*$ to it. Since the morphism $p_H$ is faithfully flat we conclude that $\text{Ł}_H = (\overline{\nu}')^*(\text{Ł}_G)$. Taking global sections of both sheaves we see that $Ind_H^{H'}(\Lambda) = k[H'/H] \otimes_{k[G'/G]} Ind_G^{G'}(\Lambda)$. Finally it's not hard to see that $H'/H \xrightarrow{\overline{\mu}'} G'/G$ is a closed embedding (see Theorem 4.3 below) and hence the homomorphism $k[G'/G] \to k[H'/H]$ is surjective

As was seen above the homomorphism $\theta$ is actually a homomorphism of $k[G'/G]$-algebras which may be identified with the composition

$$Ind_G^{G'}(\Lambda) \to k[H'/H] \otimes_{k[G'/G]} Ind_G^{G'}(\Lambda) = Ind_H^{H'}(\Lambda)$$

Since $k[G'/G] = k[GL_{n(r)}/G_0]$ is a local artinian $k$-algebra we conclude that the ideal $J = \mathrm{Ker}(k[G'/G] \to k[H'/H]$ is nilpotent and hence the ideal $I = \mathrm{Ker}\,\theta = Ind_G^{G'}(\Lambda) \cdot J$ is nilpotent as well. Now consider the following portion of the exact cohomology sequence:

$$H^n(H', I) \to H^n(H', \Lambda') \to H^n(H', Ind_H^{H'}\Lambda) = H^n(H, \Lambda)$$

To prove that $(\nu')^*(z') \in H^n(H', \Lambda')$ is nilpotent we first observe that its image in $H^n(H, \Lambda)$ is nilpotent by our original hypothesis on $z$ for this image equals $\nu^*(z)$. Thus, replacing $z$ by its appropriate power, we may assume that ${\nu'}^*(z')$ is in the image of $H^n(H', I)$. Since $I$ is nilpotent the ring without unit $H^*(H', I)$ is also nilpotent and hence ${\nu'}^*(z')$ is nilpotent.

**Theorem 4.3.** *Let $\nu' : H' \to G'$ be a homomorphism of affine group schemes over $k$. Let further $G \subset G'$ be a subgroup scheme and let $H$ denote its inverse image in $H$. The canonical morphism of quotient schemes $\overline{\nu}' : H'/H \to G'/G$ is always a locally closed embedding. If all group schemes involved are finite then $\overline{\nu}'$ is a closed embedding.*

This statement is of too general character not to be well-known. Unfortunately I was not able to find a reference so I will provide a detailed proof of this Theorem in the next section. This proof (worked out jointly with Eric Friedlander) follows closely the proof presented in [Wa] of a theorem that says that any homomorphism of affine group schemes with trivial kernel is a closed embedding. Right now I would note only that the second statement of the above theorem is an obvious corollary of the first one since for finite group schemes all morphisms appearing above are obviously finite.

## §5. Proof of the Theorem 4.3.

In this section we are going to provide the proof of the theorem 4.3. We start with few preliminary definitions and remarks.

**Definition 5.0.** *We'll be saying that the morphism $f : Y \to X$ is a weak epimorphism iff the closed image of $Y$ in $X$ coincides with $X$, i.e. iff the associated map of sheaves of rings $f^* : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$ is injective.*

Note that a weak epimorphism need not be surjective even in the set theoretical sence. The best we can say is that $f(Y)$ contains an open dense subset of $X$.

**Theorem 5.1.** *Let $G$ be an affine group scheme over an algebraically closed field $k$. Let further $X/k$ be a scheme provided with an action of $G$ and let $x \in X(k)$ be a $k$-rational point of $X$. Assume that the translation map $m_x : G \to X \quad g \mapsto g \cdot x$ is weakly surjective then the $m_x$ is flat and, in particular its image is open in $X$.*

*Proof.* Note that the image of $m_x$ coincides with the orbit of $x$ and hence is locally closed in $X$ (see [Hu] ch. 2, 8.3) since on the other hand the closure of this image coincides with $X$ we conclude that $U = m_x(G)$ is open in $X$. Replacing $X$ by $U$ we may assume in the future that the map $m_x$ is surjective i.e. the action of $G$ on $X$ is transitive. We first consider the case when $X$ is reduced. In this case our statement follows from the following more general result.

**Lemma 5.2.** *Let $f : Y \to X$ be a $G$-equivariant morphism of schemes provided with an action of $G$. Assume that the scheme $X$ is reduced and the action of $G$ on $X$ is transitive. Then $f$ is flat.*

*Proof.* Note that every morphism to an integral scheme is flat in the neighbourhood of the generic point. Thus there exists an open dense subset $V \subset X$ such that $f$ is flat over $V$. Since the morphism $f$ is $G$-equivariant we conclude further that $f$ is flat over $g \cdot V$ for any $g \in G(k)$. Finally $\cup_{g \in G(k)} g \cdot V = X$ since the action of $G$ on $X$ is transitive and hence $f$ is everywhere flat.

Next we consider the case when $X$ has only one point (namely $x$), i.e. $k[X]$ is a finite dimensional local $k$-algebra. in this case we show that the $k[X]$-algebra $k[G]$ is actually a free $k[G]$-module. Denote by $I$ the maximal ideal of $k[X]$, which obviously is nilpotent. Set $G_x = Stab_G(x) = G \times_X Spec\, k$ (where $x : Spec\, k \to G$ is the unique point of $X$. One checks immediately that $G_x$ is a closed subgroup scheme in $G$, moreover $k[G_x] = k[G]/I \cdot k[G]$. Choose a family of functions $\{f_j \in k[G]\}_{j \in J}$ such that their restrictions $\overline{f}_j$ to $G_x$ form a basis of $k[G_x]$ over $k$.

**Lemma 5.3.** *In conditions and notations introduced above the functions $\{f_j\}_{j \in J}$ form a basis of $k[G]$ over $k[X]$.*

*Proof.* Consider the homomorphism of $k[X]$-modules

$$f : \bigoplus_{j \in J} k[X] \xrightarrow{e_j \mapsto f_j} k[G]$$

If we reduce both sides modulo $I$, then $f$ becomes an isomorphism since $\overline{f}_j$ form a basis of $k[G_x] = k[G]/I \cdot k[G]$ over $k[X]/I = k$. Thus denoting by $C$ the cokernel of $f$ we conclude that $I \cdot C = C$ and hence $C = 0$ since $I$ is nilpotent. This shows that $f$ is an epimorphism. To show that $f$ is also a monomorphism we tensor it with $k[G]$ over $k[X]$ and consider the diagram

$$\begin{tikzcd} \bigoplus_{j \in J} k[X] \arrow[r, "f"] \arrow[d] & k[G] \arrow[d] \\ \bigoplus_{j \in J} k[G] \arrow[r, "1_{k[G]} \otimes_{k[X]} f"] & k[G] \otimes_{k[X]} k[G] \end{tikzcd}$$

where the vertical arrows are canonical maps induced by the $k$-algebra homomorphism $k[X] \xrightarrow{(m_x)^*} k[G]$. Since the morphism $m_x$ was assumed to be weakly surjective we conclude that $k[X] \xrightarrow{(m_x)^*} k[G]$ is a monomorphism and hence the left vertical arrow in the above diagram is a monomorphism. To conclude that the top horizontal arrow is a monomorphism it suffices now to verify that the bottom horizontal arrow is a monomorphism. To do so we note that the bottom horizontal arrow is an epimorphism (since $f$ is an epimorphism) and moreover becomes an isomorphism if we reduce modulo a nilpotent ideal $I \cdot k[G]$. To conclude now that the bottom horizontal arrow is actually an isomorphism we observe that the $k[G]$-module $k[G] \otimes_{k[X]} k[G] = k[G \times_X G]$ is actually free and hence the bottom horizontal arrow is a split epimorphism. In fact we have a canonical isomorphism of schemes over $G$: $G \times_X G = G \times G_x$ and hence an isomorphism of $k[G]$-algebras $k[G] \otimes_{k[X]} k[G] = k[G \times_X G] = k[G] \otimes_k k[G_x]$. Since $1_G \otimes_{k[X]} f$ splits and also becomes an isomorphism being reduced modulo $I \cdot k[G]$ we conclude that its kernel $K$ satisfies the property $K = I \cdot K$ and hence $K = 0$ since $I$ is nilpotent.

We now treat the general case. Assume first that *char* $k = 0$. Since every group scheme over a field of characteristic zero is reduced we conclude first that $\mathcal{O}_G$ has no nontrivial nilpotents. According to our assumptions the homomorphism $\mathcal{O}_X \to (m_x)_*(\mathcal{O}_G)$ is a monomorphism, which implies that the sheaf $\mathcal{O}_X$ also has no nontrivial nilpotents, i.e. the scheme $X$ is reduced. However this case was settled above in Lemma 5.2.

Assume now that *char* $k = p > 0$. For any scheme $X$ we denote by $X^{[n]}$ the closed image of $X$ under the Frobenius morphism $F^n : X \to X^{(n)}$.

**Lemma 5.4.** *a) If $X$ is reduced then $X^{(n)}$ is reduced as well and $X^{[n]} = X^{(n)}$ for all $n$.*
*b) For any $X/k$ there exists $N \geq 0$ such that $X^{[n]}$ is reduced for all $n \geq N$.*

*Proof.* In both cases it clearly suffices to treat the case of affine schemes. If $X = Spec\ A$ is affine then $X^{(n)} = Spec\ A^{(n)}$, where the $k$-algebra $A^{(n)}$ may be identified with $A$ with a new $k$-algebra structure, given by the formula $\lambda * a = \lambda^{p^{-n}} a (\lambda \in k, a \in A)$. Moreover after this identification the Frobenius homomorphism $F^n : A^{(n)} \to A$ coincides with the raising to the power $p^n$. The first statement is now obvious: if $A$ is reduced $A^{(n)}$ is also reduced and the Frobenius homomorphism $F^n : A^{(n)} \to A$ is injective. To prove the second statement we note that there exists $n \geq 0$ such that $a^{p^n} = 0$ for all $a$ from the nilradical of $A$ and hence $k[X^{[n]}]$ which may be identified with the subring of $A = k[X]$ consisting of $p^n$-th powers of all elements of $A$ is reduced.

**End of the Proof of the Theorem 5.1** We assume that *char* $k = p > 0$ and the action of $G$ on $X$ is transitive (see beginning of the proof). Pick up an integer $n$ such the scheme $Y = X^{[n]}$ is reduced and denote by $y$ the image of $x$ in $Y$. The action of $G$ on $X$ determines an action of $G^{(n)}$ on $X^{(n)}$ and hence determines also an action (via the Frobenius homomorphism $F^n : G \to G^{(n)}$) of $G$ on $X^{(n)}$. Moreover the Frobenius map $F^n : X \to X^{(n)}$ is $G$-equivariant and hence the closed image $X^{[n]}$ of $X$ in $X^{(n)}$ is a $G$-invariant closed subscheme. Note next that the morphism $m_y : G \to Y$ may be written as a composition $G \xrightarrow{m_x} X \xrightarrow{F^n} Y$. Since both arrows are weak epimorphisms we conclude immediately that their composition is a weak epimorphism as'well. Finally the morphism $F^n : X \to Y$ gives a bijection on the set of $k$-points from which we conclude that the action of $G$ on $Y$ is transitive as well. Lemma 5.2 implies that both $m_y$ and $F^n : X \to Y$ are flat. Thus we get a diagram of $G$-equivariant morphisms

$$\begin{tikzcd} G \arrow[rr, "m_x"] \arrow[dr, "m_y"'] & & X \arrow[dl, "F^n"] \\ & Y & \end{tikzcd}$$

with flat diagonals. Using once again the fact that $G$ acts transitively on $X$ we see that it suffices to establish that $m_x$ is flat over $x$, and for that (since $m_y$ and $F^n$ are flat) it suffice to establish that the induced morphism on fibers over $y$ is flat – see [SGA1] ex. 4, 5.9 or [Bou] III 5.4. Denote by $G_y = Stab_G(y) = G \times_Y y$ and $X_y = X \times_Y y$ the fibers of $G$ and $X$ respectively over $y$. Note further that we have canonical identifications

$$G \times G_y \xrightarrow{\sim} G \times_Y G \qquad (g, h) \mapsto (g, gh)$$

$$G \times X_y \xrightarrow{\sim} G \times_Y X \qquad (g, z) \mapsto (g, gz)$$

Moreover under these identifications the morphism $1_G \times_Y m_x$ identifies with $1_G \times (m_x)_{|G_x}$. Since $\mathcal{O}_X \to (m_x)_*(\mathcal{O}_G)$ is a monomorphism and $\mathcal{O}_G$ is a flat $\mathcal{O}_Y$-algebra we conclude easily that $\mathcal{O}_{G\times_Y X} \to (1_G \times_Y m_x)_*(\mathcal{O}_{G\times_Y G})$ is still a monomorphism i.e. $1_G \times_Y m_x = 1_G \times (m_x)_{|G_x}$ is a weak epimorphism, which readily implies that $(m_x)_{|G_x}$ is a weak epoimorphism. Finally the scheme $X_y$ has only one point i.e. we are in the infinitesimal situation considered in Lemma 5.3 and hence may conclude that $k[G_y]$ is even a free $k[X_y]$-module. In any event the induced morphism on the fibers $G_y \to X_y$ is flat and hence $m_x : G \to X$ is flat.

**Proof of the Theorem 4.3** Recall that we start with a homomorphism $\nu' : H' \to G'$ of affine group schemes over $k$ and a closed subgroup scheme $G \subset G'$. We set $H = H' \times_H G$ and want to conclude that the induced map on the quotient schemes $H'/H \to G'/G$ is a locally closed embedding. Denote by $x$ the distinguished point of $G'/G$, i.e. $x = G/G$. The composition morphism $H' \to H'/H \to G'/G$ coincides obviously with the map $m_x$ considered above (corresponding to the action of $H'$ on $G'/G$ given by the composition of the homomorphism $\nu' : H' \to G'$ and the left regular action of $G'$ on $G'/G$). Denote by $X$ the closed image of $H'/H$ in $G'/G$ or what amounts to the same thing the closed image of $H'$ in $G'/G$. According to Theorem 5.1 the image $U$ of $m_x$ is open in $X$, i.e. locally closed in $G'/G$ and moreover the composition morphism

$$H' \xrightarrow{p} H'/H \to U$$

is faithfully flat. Since the projection $p : H' \to H'/H$ is faithfully flat as well we immediately conclude that the arrow $H'/H \to U$ is faithfully flat. We next observe that the morphism of fppf-sheaves on $Sch/k$ defined by the morphism $\overline{\nu}'$ is injective. In fact the fppf-sheaf represented by the scheme $G'/G$ is associated to the presheaf $X \mapsto G'(X)/G(X)$ - see [J], ch.1, § 5. Since for any $X$ the corresponding map $H'(X)/H(X) \to G'(X)/G(X)$ is clearly injective we conclude that the map of presheaves is injective and since the functor of associated sheaf is exact the same is true for associated sheaves. Now it suffices to use the following Lemma.

**Lemma 5.5.** *Let $f : X \to Y$ be a faithfully flat morphism. Assume that for any scheme $S \in Sch/k$ the corresponding map $Hom_{Sch_k}(S, X) \to Hom_{Sch_k}(S, Y)$ is injective. Then $f$ is an isomorphism.*

*Proof.* Take $S = X \times_Y X$. The projections $p_1, p_2 : X \times_Y X \longrightarrow X$ coincide being composed with $f$. Thus our condition implies that $p_1 = p_2$. This implies further that the morphisms $p_1 = p_2 : X \times_Y X \longrightarrow X$ and $\Delta_X : X \to X \times_Y X$ are mutually inverse isomorphisms. Finally the morphism $p_2$ is obtained from $f$ making the base change $X \to Y$. Since $f$ becomes an isomorphism after a faithfully flat base change it was an isomorphism from the starts.

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