PURITY FOR SIMILARITY FACTORS

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Abstract. Let R be a regular local ring, K its field of fractions and A_1, A_2 two Azumaya algebras with involutions over R. We show that if $A_1 \otimes_R K$ and $A_1 \otimes_R K$ are isomorphic over K, then A_1 and A_2 are isomorphic over R. In particular, if two quadratic spaces over the ring R become similar over K then these two spaces are similar already over R. The results are consequences of a purity theorem for similarity factors.

Introduction

Let R be a regular local ring, K its field of fractions. Let (A_1, σ_1) and (A_2, σ_2) be two Azumaya algebras with involutions over R (see right below for a precise definition). Assume that $(A_1, \sigma_1) \otimes_R K$ and $(A_2, \sigma_2) \otimes_R K$ are isomorphic. Are (A_1, σ_1) and (A_2, σ_2) isomorphic too? We show that this is true if R is a regular local ring containing a field of characteristic different from 2. If A_1 and A_2 are both the $n \times n$ matrix algebra over R and the involutions are symmetric then σ_1 and σ_2 define two quadratic spaces q_1 and q_2 over R up to similarity factors. In this particular case the result looks as follows: if $q_1 \otimes_R K$ and $q_1 \otimes_R K$ are similar then q_1 and q_2 are similar too.

Grothendieck [G] conjectured that, for any reductive group scheme G over R, rationally trivial G-homogeneous spaces are trivial. Our result corresponds to the case when G is the projective unitary group $\mathrm{PU}_{A,\sigma}$ for an Azumaya algebra with involution over R. If R is an essentially smooth local k-algebra and G is defined over k (we say that G *is constant*) Grothendieck's conjecture has been proved in most cases: by Colliot-Thélène and Ojanguren [**C-TO**] for a perfect infinite field k and then by Raghunathan [**R**] for any infinite k. One notable open case is that

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of a finite base field. For a non-constant group G only few cases have been proved: when G is a torus, by Colliot-Thélène and Sansuc [**C-TS**], when G is the group $\mathrm{SL}_1(D)$ of norm one elements of an Azumaya R-algebra D, by Panin and Suslin [**PS**], when G is the unitary group $\mathrm{U}_{A,\sigma}$, by Panin and Ojanguren [**Oj-P1**], when G is the special unitary group $\mathrm{SU}_{A,\sigma}$, by Zainoulline [**Z**]. Recall as well that for semi-simple group schemes G over a discrete valuation ring the conjecture has been proved by Nisnevich in [**Ni**].

The paper is organized as follows. Section 1 contains a reduction of the main theorem (Th. 1.1) to a purity theorem for similarity factors (Th. 1.3). Section 2 is devoted to a theorem of Nisnevich and its Corollaries. The rest of the text is devoted to the proof of Theorem 1.3. The proof is given in §8. It is based on the Specialization Lemma (stated in §3 and proved in §4), the Equating Lemma (§5) and the Unramifiedness Lemma (§5).

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§1. Rationally isomorphic Azumaya algebras with involutions are locally isomorphic

Let R be a regular local ring containing a field k ($\mathrm{char}(k) \neq 2$) and let K be its quotient field. By an R- Azumaya algebra with involution (A, σ) we mean (see [**Oj-P1**]) an R-algebra A which is an Azumaya algebra over its center $Z(A)$ equipped with an involution $\sigma : A \to A^{op}$, such that $Z(A)$ is either R itself or an étale quadratic extension of R such that $Z(A)^\sigma = R$.

1.1. Theorem (Main). *Let (A_1, σ_1) and (A_2, σ_2) be two Azumaya algebras with involutions over the ring R. If the Azumaya algebras with involutions $(A_{1,K}, \sigma_{1,K})$ and $(A_{2,K}, \sigma_{2,K})$ are isomorphic, then (A_1, σ_1) and (A_2, σ_2) are already isomorphic.*

Reduction to a Purity Theorem. Since $(A_{1,K}, \sigma_{1,K}) \simeq (A_{2,K}, \sigma_{2,K})$, one concludes that the two Azumaya algebras $A_{1,K}$ and $A_{2,K}$ over Z_K are isomorphic. Thus $A_1 \simeq A_2$ (Z is regular semilocal as an étale quadratic extension of R). Therefore one may assume that $A_1 = A_2$, $Z_1 = Z_2$ and we have two involutions σ_1 and σ_2 on the same algebra (A over Z and Z is a quadratic étale extension of R (or Z just coincides with R).

Now consider the composite $A \xrightarrow{\sigma_2} A^{op} \xrightarrow{\sigma_1^{-1}} A$. It is an Azumaya algebra isomorphism. Thus it is of the form $\mathrm{Int}(\alpha)$ for an element $\alpha \in A^*$. Thus $\sigma_1 \circ \mathrm{Int}(\alpha) = \sigma_2$ and α is symmetric with respect to σ_1. Therefore we have two hermitian spaces over (A, σ_1), namely $(A, 1)$ and (A, α). Set

$$h_1 = (A, 1) \text{ and } h_2 = (A, \alpha).$$

Since $(A_K, \sigma_{1,K})$ is isomorphic to $(A_K, \sigma_{2,K})$, $h_{1,K}$ is similar to $h_{2,K}$ i.e. there exist an element $a \in K^*$ and an isometry $a \cdot h_{1,K} \simeq h_{2,K}$.

We will prove (this suffices to prove the theorem) that there exists an element $b \in R^*$ such that $b \cdot h_1 \simeq h_2$ over (A, σ_1). To find the desired element $b \in R^*$, it suffices to find a similarity factor $b_\mathfrak{m} \in K^*$ of the space $h_{1,K}$ and a unit $a_\mathfrak{m} \in R^*$ such that $a = a_\mathfrak{m} \cdot b_\mathfrak{m}$. In fact, if $b_\mathfrak{m} \in K^*$, $a_\mathfrak{m} \in R^*$ are the mentioned elements then one has a chain of relations $(h_2 \perp -a_\mathfrak{m} \cdot h_1)_K \simeq h_{2,K} \perp -a_\mathfrak{m} \cdot b_\mathfrak{m} \cdot h_{1,K} = h_{2,K} \perp -a \cdot h_{1,K} \simeq h_{2,K} \perp -h_{2,K}$. Thus $(h_2 \perp -a_\mathfrak{m} \cdot h_1)_K$ is hyperbolic and by the main theorem of [**Oj-P1**] the space $h_2 \perp -a_\mathfrak{m} \cdot h_1$ is hyperbolic, whence $h_2 \simeq a_\mathfrak{m} \cdot h_1$. Therefore putting $b = a_\mathfrak{m}$ we get $h_2 \simeq b \cdot h_1$ over (A, σ_1). It remains to find a similarity factor $b_\mathfrak{m}$ of $h_{1,K}$ and a unit $a_\mathfrak{m} \in R^*$ such that $a = a_\mathfrak{m} \cdot b_\mathfrak{m}$. By the corollary of a theorem of Nisnevich below (§3, Cor. 3.2), for a height one prime ideal $\mathfrak{p}$ in R there exist elements $b_\mathfrak{p} \in K^*$ and $a_\mathfrak{p} \in R^*$ such that

(1) $b_\mathfrak{p}$ is a similarity factor of the space $h_{1,K}$ and
(2) $a = a_\mathfrak{p} \cdot b_\mathfrak{p}$.

Thus by the Purity Theorem (Theorem 1.2) there exist a similarity factor $b_\mathfrak{m}$ of $h_{1,K}$ and a unit $a_\mathfrak{m} \in R^*$ with $a = a_\mathfrak{m} \cdot b_\mathfrak{m}$. So we have reduced Theorem 1.1 to the Purity Theorem. $\square$

1.2. Theorem (Purity Theorem). *Let R, K be as in Theorem 1.1. Let (A, σ) be an Azumaya algebra with involution over R and let h be the hermitian space $(A, 1)$ over (A, σ). Let $a \in K^*$. Suppose that for each prime ideal of height 1 $\mathfrak{p}$ in R there exist $a_\mathfrak{p} \in R^*_\mathfrak{p}$, $b_\mathfrak{p} \in K^*$ with $a = a_\mathfrak{p} \cdot b_\mathfrak{p}$ and $h_K \simeq b_\mathfrak{p} \cdot h_K$. Then there exist $b_\mathfrak{m} \in K^*$, $a_\mathfrak{m} \in R^*$ such that*

(1) *$b_\mathfrak{m}$ is a similarity factor of the space h_K,*
(2) *$a = a_\mathfrak{m} \cdot b_\mathfrak{m}$.*

It is convenient for the proof to restate Theorem 1.2 it in a slightly more technical form. For that consider the similitude group scheme $G = \mathrm{Sim}_{A,\sigma}$ of the Azumaya algebra with involution (A, σ). Recall that for an R-algebra S the S-points of G are those $\alpha \in (A \otimes_R S)^*$ for which $\alpha^\sigma \cdot \alpha \in S^*$. Further consider a group scheme morphism $\mu : G \to \mathbb{G}_m$ which takes a similitude $\alpha \in G(S)$ to its similarity factor $\mu(\alpha) = \alpha^\sigma \cdot \alpha \in S^*$. Finally for an R-algebra S consider the group $\mathcal{F}(S) = S^*/\mu(G(S))$. For an element $a \in S^*$ we will often write $\bar{a}$ for its class in $\mathcal{F}(S)$.

1.3. Theorem. *Let R, K and (A, σ) be as in Theorem 1.2. Let $a \in K^*$. If for each height 1 prime $\mathfrak{p}$ in R the class $\bar{a} \in \mathcal{F}(K)$ can be lifted in $\mathcal{F}(R_\mathfrak{p})$, then $\bar{a}$ can be lifted in $\mathcal{F}(R)$.*

Remark. Theorems 1.2 and 1.3 are equivalent. In fact, the group $\mu(G(R))$ coincides with the group $G_R(h)$ of similarity factors of the hermitian space $h = (A, 1)$.

Remark. It is quite plausible that the method of [**Z1**] could be adapted to prove Theorem 1.3.

§2. A theorem of Nisnevich

Let R be a discrete valuation ring containing a field and let K be its quotient field. Let (A, σ) be an Azumaya algebra with involution over R. The following theorem is a consequence of a theorem of Nisnevich on principal G-bundles. ([**Ni**], Theorem ??).

2.1. Theorem (Nisnevich). *Let h_1 and h_2 be two hermitian spaces over (A, σ). Suppose $h_{1,K}$ is similar to $h_{2,K}$, then h_1 is similar to h_2.*

This Theorem is a particular case of the theorem of Nisnevich just mentioned, namely the case when G is the projective unitary group scheme PU_{h_1} over R.

2.2. Corollary. *Let h_1, h_2 be two hermitian spaces over (A, σ). Let $a \in K^*$ be such that $h_{2,K} \simeq a \cdot h_{1,K}$. Then there exist an element $b' \in K^*$ and a unit $a' \in R^*$ such that*

(1) *b' is a similarity factor of the space $h_{1,K}$,*
(2) *$a = a' \cdot b'$.*

Proof. By the theorem there exists a unit $a' \in R^*$ such that $a' \cdot h_2 \simeq h_1$. Thus one has a chain of relations

$$a \cdot (a')^{-1} \cdot h_{1,K} \simeq a \cdot h_{2,K} \simeq a^2 \cdot h_{1,K} \simeq h_{1,K} \ .$$

Therefore $b' = a \cdot (a')^{-1}$ is a similarity factor of the space $h_{1,K}$ and $a = a' \cdot b'$. $\square$

2.3. Corollary. *The kernel of the map $H^1(R, Sim_{A,\sigma}) \to H^1(K, Sim_{A,\sigma})$ is trivial.*

Proof. The group scheme $\mathrm{Sim}_{A,\sigma}$ fits in an exact sequence of algebraic groups

$$0 \to R_{Z/R}(\mathbb{G}_{m,Z}) \to \mathrm{Sim}_{A,\sigma} \to \mathrm{PU}_{A,\sigma} \to 0$$

where $R_{Z/R}(\mathbb{G}_{m,Z})$ is the Weil restriction of the multiplicative group $\mathbb{G}_{m,Z}$. By Hilbert's Theorem 90 $H^1(R, R_{Z/R}(\mathbb{G}_{m,Z})) = H^1(Z, \mathbb{G}_{m,Z})) = 0$. Thus the kernel of the map $H^1(R, \mathrm{Sim}_{A,\sigma}) \to H^1(R, \mathrm{PU}_{A,\sigma})$ is trivial. On the other hand the kernel of the map $H^1(R, \mathrm{PU}_{A,\sigma}) \to H^1(K, \mathrm{PU}_{A,\sigma})$ is trivial by Theorem 3.1. Thus the kernel of the map $H^1(R, \mathrm{Sim}_{A,\sigma}) \to H^1(K, \mathrm{Sim}_{A,\sigma})$ is trivial as well, whence the Corollary.

§3. A Specialization Lemma

In this section we state a theorem which is one of the main ingredient in the proof of purity. The theorem itself will be proved in §5 below.

Let k be a field ($\mathrm{char}(k) \neq 2$) and let (A, σ) be an Azumaya algebra with involution over k (see Section 1 for the definition). Let $G = \mathrm{Sim}_{A,\sigma}$ be the similitude group of (A, σ) (see the end of Section 1 for the definition), and let $\mu : \mathrm{Sim}_{A,\sigma} \longrightarrow \mathbb{G}_m$ be a group morphism which takes a similitude α to its similarity factor $\mu(\alpha) = \alpha^\sigma \cdot \alpha$. The group G coincides with the similitude group of the hermitian space $(A, 1) = h$.

3.1. Notation. For a commutative k-algebra S, set $\mathcal{F}(S) = S^*/\mu(G(S))$. For an element $u \in S^*$ we shall write in this section $\overline{u}$ for the image of u in $\mathcal{F}(S)$. Observe that $\mu(G(S)) = G_S(h \otimes_k S)$ is the group of similarity factors of the hermitian space $h \otimes_k S$.

Let S be a k-algebra which is a Dedekind domain and let L be the quotient field of S. Let $\mathfrak{p} \subseteq S$ be a non-zero prime ideal in S and let $S_\mathfrak{p}$ be the corresponding local ring.

3.2. Definition. Let $a \in L^*$. The element $\overline{a} \in \mathcal{F}(L)$ is said to be *unramified* at a prime $\mathfrak{p}$ if $\overline{a}$ belongs to the image of the group $\mathcal{F}(S_\mathfrak{p})$ in $\mathcal{F}(L)$. In other terms, the element $\overline{a}$ is *unramified* at $\mathfrak{p}$ if $a = a_\mathfrak{p} \cdot b_\mathfrak{p}$ for certain elements $a_\mathfrak{p} \in S_\mathfrak{p}^*$ and $b_\mathfrak{p} \in \mu(G(L))$. We denote by $\mathcal{F}_{un}(S)$ the subgroup in $\mathcal{F}(L)$ consisting of all those elements in $\mathcal{F}(L)$ which are unramified at each non-zero prime $\mathfrak{p}$ in S. Elements of $\mathcal{F}(S)$ are called *S-unramified* elements.

Let $S \supseteq k[s]$ be a finite extension of the polynomial ring in one variable. Suppose S is a Dedekind domain. Let $S_1 = S/(s-1)S$ and $S_0 = S/J$, and let $\epsilon : S \longrightarrow k$ be an augmentation such that $S/tS = S/\mathrm{Ker}(\epsilon) \times S/J = k \times S/J$ for certain ideal J in S. For an element $v \in S$ we will write v_1 and v_0 for its images in S_1 and S_0 respectively. If furthermore $g \in S$ be an element coprime as to $(s-1)$ so to (s), then the canonical map $S \longrightarrow S_i$ is factorized as the composite $S \longrightarrow S_g \longrightarrow S_i$. In this case for an element $v \in S_g$ we will write v_1 and v_0 for its images in S_1 and S_0 respectively. We will denote $N_{S_i/k} : S_i^* \longrightarrow k^*$ the norm map.

3.3. Theorem (Specialization Lemma). *Let $S \supseteq k[s]$ be an integral extension of the polynomial ring in one variable $k[s]$ and suppose S is a Dedekind domain and L its quotient field. Let $\mathfrak{f} \in S$ be an element coprime to s and $(s-1)$. Let $u \in S_\mathfrak{f}^*$ be a unit. Suppose the element $\overline{u} \in \mathcal{F}(L)$ is S-unramified, i.e. $\overline{u}$ belongs to the subgroup $\mathcal{F}_{un}(S)$. Then the following relation holds in the group $\mathcal{F}(k)$*

$$(*) \qquad \overline{\epsilon(u)} = \overline{N_{S_1/k}(u_1)} \cdot \overline{N_{S_0/k}(u_0)^{-1}} \quad .$$

3.4. Remark. This theorem is proved in §4 below. Now observe only that if $u \in S^*$, then $N_{S/k[s]}(u) \in k[s]^* = k^*$ and already $\epsilon(u) = N_{S_1/k}(u_1) \cdot N_{S_0/k}(u_0)^{-1}$. So there is nothing to prove in this case. The trouble is that we do not assume $u \in S^*$.

§4. Proof of Specialization Lemma

Let k be a field of characteristic different of 2 and let (A, σ) be an Azumaya algebra with involution over k (see Section 1 for the definition). Let h be the hermitian space $(A, 1)$. We preserve in this section notation of §3.

Let K be a function field of an irreducible curve over k and let $L \supseteq K$ be a finite field extension (separable). We will consider in this section discrete valuations of K and L which are trivial on k and they will be called valuations. For valuations $x : K^* \to \mathbb{Z}$ and $y : L^* \to \mathbb{Z}$, we write y/x if y extends x. We will need completions to avoid dealing with semi-local Dedekind domains.

4.1. Notation. Let y be a valuation of L. Denote by $\hat{L}_y$ the completion of L with respect to y. Denote by $\mathcal{O}_y$ the ring of integers associated with y, i.e. $\mathcal{O}_y = \{a \in L \mid y(a) \geqslant 0\}$. And denote by $\hat{\mathcal{O}}_y$ the ring of y-integers in $\hat{L}_y$, i.e. $\hat{\mathcal{O}}_y = \{a \in \hat{L}_y \mid y(a) \geqslant 0\}$. We shall write $k(y)$ for the residue field of y, i.e. $k(y) = \mathcal{O}_y/\mathfrak{m}_y = \hat{\mathcal{O}}_y/\hat{\mathfrak{m}}_y$.

If x and y are valuations of K and L respectively and y extends x, then $\mathcal{O}_y \supseteq \mathcal{O}_x$ and $\hat{\mathcal{O}}_y \supseteq \hat{\mathcal{O}}_x$ and the ring extension $\hat{\mathcal{O}}_y \supseteq \hat{\mathcal{O}}_x$ is integral. Thus one has norm mappings $N_{\mathcal{O}_y/\hat{\mathcal{O}}_x} : \hat{\mathcal{O}}_y^* \to \hat{\mathcal{O}}_x^*$ and $N_{L_y/\hat{K}_x} : \hat{L}_y^* \to \hat{K}_x^*$ (we will use below a short notation $N_{y/x}$ for both of these maps). There is the norm map $N_{k(y)/k(x)} : k(y)^* \to k(x)^*$ and two diagrams commute

$$\begin{tikzcd} \hat{\mathcal{O}}_y^* \arrow[r] \arrow[d, "N_{y/x}"'] & \hat{L}_y^* \arrow[d, "N_{y/x}"] \\ \hat{\mathcal{O}}_x^* \arrow[r] & \hat{K}_x^*, \end{tikzcd} \qquad \begin{tikzcd} \hat{\mathcal{O}}_y^* \arrow[r] \arrow[d, "N_{y/x}"'] & k(y)^* \arrow[d, "N_{k(y)/k(x)}^{i(y/x)}"] \\ \hat{\mathcal{O}}_x^* \arrow[r] & k(x)^*, \end{tikzcd}$$

where $i(y/x) =$ the length of $\mathcal{O}_y/\mathfrak{m}_x\mathcal{O}_y$ is the ramification index of y over x.

4.2. Remark. Let $U_{A,\sigma}$ be the unitary group of the form h. It is an algebraic group over k such that for any k-algebra R the group of its R-points is the group $\{\alpha \in (A \otimes_k R)^* \mid \alpha^\sigma \cdot \alpha = 1\}$. With the notation of §3 the group $U_{A,\sigma}$ fits in an exact sequence of algebraic groups $1 \to U_{A,\sigma} \to \mathrm{Sim}_{A,\sigma} \xrightarrow{\mu} \mathbb{G}_m \to 1$. This sequence of algebraic groups induces exact sequences of pointed sets (R is a domain, K is its

quotient field)

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{F}(R) & \xrightarrow{\partial} & H^1_{et}(R, U_{A,\sigma}) & \longrightarrow & H^1_{et}(R, \mathrm{Sim}_{A,\sigma}) \\ & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_3} \\ 1 & \longrightarrow & \mathcal{F}(K) & \xrightarrow{\partial} & H^1_{et}(K, U_{A,\sigma}) & \longrightarrow & H^1_{et}(K, \mathrm{Sim}_{A,\sigma}). \end{array}$$

In the case of a Dedekind local ring R and its quotient field K the maps θ_2 and θ_3 have trivial kernels as well. This holds for θ_2 by Corollary 2.3 and for θ_3 by [**Oj**]. In particular, in this case the map $\theta_1 : \mathcal{F}(R) \to \mathcal{F}(K)$ has the trivial kernel and thus it is injective. Observe as well that for a field K the map $\mathcal{F}(K) \xrightarrow{\partial} H^1_{et}(K, U_{A,\sigma})$ is injective, i.e. $(\partial(a) = \partial(b)) \implies a = b$.

4.3. Notation. Let y be a valuation of L and let $i : L \to \hat{L}_y$ be the inclusion. Then by Remark 4.3 the map $\mathcal{F}(\hat{\mathcal{O}}_y) \xrightarrow{i_*} \mathcal{F}(\hat{L}_y)$ is injective and we will identify $\mathcal{F}(\hat{\mathcal{O}}_y)$ with its image under this map. Set

$$\mathcal{F}_y(L) = i_*^{-1}(\mathcal{F}(\hat{\mathcal{O}}_y)).$$

The inclusions $\mathcal{O}_y \hookrightarrow L$ and $\mathcal{O}_y \hookrightarrow \hat{\mathcal{O}}_y$ induce a map $\mathcal{F}(\mathcal{O}_y) \to \mathcal{F}_y(L)$ which is injective by Remark 4.3. Both groups are subgroups of $\mathcal{F}(L)$. The following lemma shows that $\mathcal{F}_y(L)$ coincides with the subgroup of $\mathcal{F}(L)$ consisting of all elements *unramified* at y.

4.4. Lemma. $\mathcal{F}(\mathcal{O}_y) = \mathcal{F}_y(L)$.

Proof. We only have to check the inclusion $\mathcal{F}_y(L) \subseteq \mathcal{F}(\mathcal{O}_y)$. Let $a_y \in \mathcal{F}_y(L)$ be an element. It determines the elements $a \in \mathcal{F}(L)$ and $\hat{a} \in \mathcal{F}(\hat{\mathcal{O}}_y)$ which coincide when regarded as elements of $\mathcal{F}(\hat{L}_y)$. We denote this common element in $\mathcal{F}(\hat{L}_y)$ by $\hat{a}_y$. Let $\xi = \partial(a) \in H^1_{et}(L, U_{A,\sigma})$, $\hat{\xi} = \partial(\hat{a}) \in H^1_{et}(\hat{\mathcal{O}}_y, U_{A,\sigma})$ and $\hat{\xi}_y = \partial(\hat{a}_y) \in H^1_{et}(\hat{L}_y, U_{A,\sigma})$. Clearly, $\hat{\xi}$ and ξ both coincide with $\hat{\xi}_y$ when regarded as elements of $H^1_{et}(\hat{L}_y, U_{A,\sigma})$. Thus one can glue ξ and $\hat{\xi}$ to get a $\xi_y \in H^1_{et}(\mathcal{O}_y, U_{A,\sigma})$ which maps to ξ under the map induced by the inclusion $\mathcal{O}_y \hookrightarrow L$ and maps to $\hat{\xi}$ under the map induced by the inclusion $\mathcal{O}_y \hookrightarrow \hat{\mathcal{O}}_y$.

We now show that ξ_y has the form $\partial(a'_y)$ for a certain $a'_y \in \mathcal{F}(\mathcal{O}_y)$. In fact, observe that the image ζ of ξ in $H^1_{et}(L, \mathrm{Sim}_{A,\sigma})$ is trivia. As mentioned in Remark 4.3 the map $H^1_{et}(\mathcal{O}_y, \mathrm{Sim}_{A,\sigma}) \to H^1_{et}(L, \mathrm{Sim}_{A,\sigma})$ has the trivial kernel. Therefore the image ζ_y of ξ_y in $H^1_{et}(\mathcal{O}_y, \mathrm{Sim}_{A,\sigma})$ is trivial as well. Thus there exists an element $a'_y \in \mathcal{F}(\mathcal{O}_y)$ with $\partial(a'_y) = \xi_y \in H^1_{et}(\mathcal{O}_y, U)$.

We now prove that a'_y coincides with a_y in $\mathcal{F}_y(L)$. Since $\mathcal{F}(\mathcal{O}_y)$ and $\mathcal{F}_y(L)$ are both subgroups of $\mathcal{F}(L)$, it suffices to show that a'_y coincides with the element a in $\mathcal{F}(L)$. By Remark 4.3 the map $\mathcal{F}(L) \xrightarrow{\partial} H^1_{et}(L, U_{A,\sigma})$ is <u>injective</u>. Thus it suffices to check that $\partial(a'_y) = \partial(a)$ in $H^1_{et}(L, U_{A,\sigma})$. This is indeed the case because $\partial(a'_y) = \xi_y$ and $\partial(a) = \xi$, and ξ_y coincides with ξ when regarded over L. We have proved that $a'_y \in \mathcal{F}(\mathcal{O}_y)$ coincides with a_y in $\mathcal{F}_y(L)$. Thus the inclusion $\mathcal{F}_y(L) \subseteq \mathcal{F}(\mathcal{O}_y)$ is proved, whence the lemma. $\square$

4.5. Definition. Let y be a valuation of L. Define a specialization map

$$s(y) : \mathcal{F}_y(L) \longrightarrow \mathcal{F}(k(y))$$

as the composite $\mathcal{F}_y(L) \longrightarrow \mathcal{F}(\hat{\mathcal{O}}_y) \xrightarrow{\mathrm{res}_y} \mathcal{F}(k(y))$ of the map $\mathcal{F}_y(L) \longrightarrow \mathcal{F}(\hat{\mathcal{O}}_y)$ induced by the map i_* (see 4.4) and the map $\mathcal{F}(\hat{\mathcal{O}}_y) \longrightarrow \mathcal{F}(k(y))$ induced by the residue map $\hat{\mathcal{O}}_y \longrightarrow k(y)$. (If we identify $\mathcal{F}_y(L)$ with $\mathcal{F}(\mathcal{O}_y)$ by Lemma 4.5, then the map $s(y) : \mathcal{F}(\mathcal{O}_y) \longrightarrow \mathcal{F}(k(y))$ coincides with the map induced by the map $\mathcal{O}_y \xrightarrow{\mathrm{res}_y} k(y)$).

4.6. Lemma-Definition. *Let K be a field containing the field k and let $K \subseteq L$ be a finite field extension. Then the norm map $N_{L/K} : L^* \longrightarrow K^*$ takes the group $G_L(h)$ into the group $G_K(h)$. Therefore the norm map $N_{L/K}$ induces a map which we still denote by*

$$N_{L/K} : \mathcal{F}(L) \longrightarrow \mathcal{F}(K).$$

Proof. The Scharlau norm principle [**KMRT**, loc. cit.] states that there is a natural inclusion $N_{L/K}(G_L(h)) \subseteq G_K(h)$, whence the lemma. $\square$

4.7. Lemma. *Let x be a valuation of K and let y be a valuation of L extending x. Then the map $N_{\hat{L}_y/\hat{K}_x} : \mathcal{F}(\hat{L}_y) \longrightarrow \mathcal{F}(\hat{K}_x)$ takes $\mathcal{F}(\hat{\mathcal{O}}_y)$ into $\mathcal{F}(\hat{\mathcal{O}}_x)$.*

Proof. The desired inclusion follows from the commutativity of the diagram

$$\begin{tikzcd}
\hat{\mathcal{O}}_y^* \arrow[rrr] \arrow[dr, two heads] \arrow[ddd, "N_{y/x}"'] & & & \hat{L}_y^* \arrow[dl, hook] \arrow[ddd, "N_{y/x}"] \\
& \mathcal{F}(\hat{\mathcal{O}}_y) \arrow[r, hook] & \mathcal{F}(\hat{L}_y) \arrow[d, "N"] & \\
& \mathcal{F}(\hat{\mathcal{O}}_x) \arrow[r, hook] & \mathcal{F}(\hat{K}_x) & \\
\hat{\mathcal{O}}_x^* \arrow[rrr] \arrow[ur] & & & \hat{K}_x^* \arrow[ul]
\end{tikzcd},$$

the surjectivity of the map $\hat{\mathcal{O}}_y^* \to \mathcal{F}(\hat{\mathcal{O}}_y)$ and the injectivity of the map $\mathcal{F}(\hat{\mathcal{O}}_x) \to \mathcal{F}(\hat{K}_x)$ (see Remark 4.3). $\square$

4.8. Notation. The map $\mathcal{F}(\hat{\mathcal{O}}_y) \to \mathcal{F}(\hat{\mathcal{O}}_x)$ will be still denoted by $N_{y/x}$.

4.9. Notation. Let x be a valuation of K. Set $\mathcal{F}_x(L) = \bigcap\limits_{y/x} \mathcal{F}_y(L)$.

4.10. Lemma. *Let x be a valuation of K. Then $N_{L/K}(\mathcal{F}_x(L)) \subseteq \mathcal{F}_x(K)$.*

Proof. The desired inclusion follows from Lemma 4.8 and the commutativity of the diagram

$$\begin{tikzcd}
\prod\limits_{y/x} \mathcal{F}(\hat{\mathcal{O}}_y) \arrow[rrr] \arrow[ddd, "\Pi N_{y/x}"'] & & & \prod\limits_{y/x} \mathcal{F}(\hat{L}_y) \arrow[ddd, "\Pi N_{y/x}"] \\
& \mathcal{F}_x(L) \arrow[ul] \arrow[r, hook] \arrow[d, "N_{L/K}"] & \mathcal{F}(L) \arrow[ur] & \\
& \mathcal{F}_x(K) \arrow[dl] \arrow[r, hook] & \mathcal{F}(K) \arrow[dr] & \\
\mathcal{F}(\hat{\mathcal{O}}_x) \arrow[rrr] & & & \mathcal{F}(\hat{K}_x) \ ,
\end{tikzcd}$$

and the definition of $\mathcal{F}_x(K)$ (see 4.4). $\square$

4.11. Lemma. *Let x be a valuation of K. Then the diagram commutes.*

$$\begin{tikzcd}
\mathcal{F}_x(L) \arrow[r, "\Pi s_y"] \arrow[d, "N_{L/K}"'] & \prod\limits_{y/x} \mathcal{F}(k(y)) \arrow[d, "\Pi N_{k(y)/k(x)}^{i(y/x)}"] \\
\mathcal{F}_x(K) \arrow[r, "s_x"] & \mathcal{F}(k(x)),
\end{tikzcd}$$

where $N_{k(y)/k(x)} : \mathcal{F}(k(y)) \to \mathcal{F}(k(x))$ is the norm map for the field extension $k(y)/k(x)$ and $N_{k(y)/k(x)}^{i(y/x)}$ is its $i(y/x)$-th power, where $i(y/x)$ is the ramification index of y over x, i.e. $i(y/x) =$ the length of $\mathcal{O}_y/\mathfrak{M}_x\mathcal{O}_y$.

Proof. Consider the diagram

$$\begin{tikzcd}
\mathcal{F}_x(L) \arrow[r] \arrow[d, "N_{L/K}"'] & \prod\limits_{y/x} \mathcal{F}(\hat{\mathcal{O}}_y) \arrow[r, "\Pi \mathrm{res}_y"] \arrow[d, "\Pi N_{y/x}"] & \prod\limits_{y/x} \mathcal{F}(k(y)) \arrow[d, "\Pi N_{k(y)/k(x)}^{i(y/x)}"] \\
\mathcal{F}_x(K) \arrow[r, "i_*"] & \mathcal{F}(\hat{\mathcal{O}}_x) \arrow[r, "\mathrm{res}_x"] & \mathcal{F}(k(x))
\end{tikzcd}$$

and observe that the left square commutes. It remains to check that the right hand square commutes. To do this it clearly suffices to check the commutativity of

$$\begin{tikzcd}
\mathcal{F}(\hat{\mathcal{O}}_y) \arrow[r, "\mathrm{res}_y"] \arrow[d, "N_{y/x}"'] & \mathcal{F}(k(y)) \arrow[d, "N^{i(y/x)}_{k(y)/k(x)}"] \\
\mathcal{F}(\hat{\mathcal{O}}_y) \arrow[r, "\mathrm{res}_x"] & \mathcal{F}(k(x)).
\end{tikzcd}$$

To see this we include it in a bigger one:

$$\begin{tikzcd}
\hat{\mathcal{O}}_y^* \arrow[rrr, "\mathrm{res}_y"] \arrow[dr, two heads, "\rho"] \arrow[ddd, "N_{y/x}"'] & & & k(y)^* \arrow[dl] \arrow[ddd, "N^{i(y/x)}_{k(y)/k(x)}"] \\
& \mathcal{F}(\hat{\mathcal{O}}_y) \arrow[r, "\mathrm{res}_y"] \arrow[d, "N_{y/x}"'] \arrow[dr, phantom, "\mathrm{I}"] \arrow[dr, phantom, "\mathrm{V}"'] & \mathcal{F}(k(y)) \arrow[d, "N^{i(y/x)}_{k(y)/k(x)}"] \arrow[dr, phantom, "\mathrm{IV}"] & \\
& \mathcal{F}(\hat{\mathcal{O}}_y) \arrow[r, "\mathrm{res}_x"] \arrow[dr, phantom, "\mathrm{III}"] & \mathcal{F}(k(x)) & \\
\hat{\mathcal{O}}_x^* \arrow[rrr, "\mathrm{res}_x"] \arrow[ur] & & & k(x)^*.
\end{tikzcd}$$

The large square in this diagram commutes and squares I to IV commute as well and the map ρ is surjective. Thus square V commutes as well and the lemma is proved. $\square$

4.12. Proposition. *Let $K = k(t)$ be the rational function field in one variable and $\mathcal{F}_{un}(k(t)) = \bigcap_{x \in \mathbb{A}^1_k} \mathcal{F}_x(k(t))$. Then the canonical map*

$$\mathcal{F}(k) \longrightarrow \mathcal{F}_{un}(k(t))$$

is an isomorphism.

Proof. Injectivity is clear, because the composite $\mathcal{F}(k) \longrightarrow \mathcal{F}_{un}(k(t)) \xrightarrow{s_0} \mathcal{F}(k)$ coincides with the identity (here s_0 is the specialization map at the point zero defined in 4.6).

It remains to check the surjectivity. Let $a \in \mathcal{F}_{un}(k(t))$. Then by Lemma 4.5 the element $\partial(a) \in H^1_{et}(k(t), U_{A,\sigma})$ is a class which for every $x \in \mathbb{A}^1_k$ belongs to the image of $H^1_{et}(\mathcal{O}_x, U_{A,\sigma})$. Thus by a lemma of Harder [H], $\partial(a)$ can be represented by an element $\xi \in H^1_{et}(k[t], U_{A,\sigma})$, where $k[t]$ is the polynomial ring. By Harder's

theorem [H], the map $H^1_{et}(k, U_{A,\sigma}) \longrightarrow H^1_{et}(k[t], U_{A,\sigma})$ is an isomorphism. Then $\xi = \rho(\xi_0)$ for an element $\xi_0 \in H^1_{et}(k, U_{A,\sigma})$. Consider the diagram

$$\begin{tikzcd}
 & a \arrow[r, mapsto] & \xi \arrow[rr, mapsto] & & * & \\
1 \arrow[r] & \mathcal{F}(k(t)) \arrow[r, "\partial"] & H^1_{et}(k(t), U_{A,\sigma}) \arrow[r] & H^1_{et}(k(t), \mathrm{Sim}_{A,\sigma}) \arrow[r] & 1 \\
1 \arrow[r] & \mathcal{F}(k) \arrow[r, "\partial"] \arrow[u, "\epsilon"] & H^1_{et}(k, U_{A,\sigma}) \arrow[r] \arrow[u, "\rho"] & H^1_{et}(k, \mathrm{Sim}_{A,\sigma}) \arrow[r] \arrow[u, "\eta"] & 1 \\
 & a_0 \arrow[r, mapsto] & \xi_0 \ , & &
\end{tikzcd}$$

where all the mapping are canonical and all the vertical arrows have trivial kernels. Since ξ goes to the trivial element in $H^1_{et}(k(t), \mathrm{Sim}_{A,\sigma})$, one concludes that ξ_0 goes to the trivial element in $H^1_{et}(k, \mathrm{Sim}_{A,\sigma})$. Thus there exists an element $a_0 \in \mathcal{F}(k)$ such that $\partial(a_0) = \xi_0$. Clearly, one has $\epsilon(a_0) = a$ (use the injectivity of the map $\mathcal{F}(k(t)) \longrightarrow H^1_{et}(k(t), U_{A,\sigma})$ mentioned in Remark 4.3). $\square$

4.13. Theorem. *Let $L \supseteq K = k(t)$ be a finite separable field extension and let $\mathcal{F}_{un}(L) = \bigcap_{y/x,\ x \in \mathbb{A}^1_k} \mathcal{F}_y(L)$. Then for an element $a \in \mathcal{F}_{un}(L)$ the following relation holds:*

$$(*) \qquad \prod_{y/0} N_{k(y)/k}(s_y(a)^{i(y/0)}) = \prod_{y/1} N_{k(y)/k}(s_y(a)^{i(y/1)}).$$

Proof. By lemma 4.11, the element $N_{L/K}(a)$ is in the group $\mathcal{F}_{un}(K)$. Now by Lemma 4.12, the left hand side of the relation (*) coincides with $s_0(N_{L/K}(a))$, where $s_0 : \mathcal{F}_{un}(K) \longrightarrow \mathcal{F}(k)$ is the specialization map (see Definition 4.6) at the point zero. The right hand side of (*) coincides with $s_1(N_{L/K}(a))$, where s_1 is the specialization map at 1. By Proposition 4.13, there exists an element $a_0 \in \mathcal{F}(k)$ whose image in $\mathcal{F}(k(t))$ is equal to $N_{L/K}(a) \in \mathcal{F}(k(t))$. Thus

$$s_0(N_{L/K}(a)) = s_0(a_0) = a_0 = s_1(a_0) = s_1(N_{L/K}(a)).$$

The theorem is proved. $\square$

4.14. Corollary. *The Specialization Lemma (Theorem 3.3) holds.*

Proof. We use notation of §3. Let $S \supseteq k[s]$ be the integral extension of the polynomial ring in one variable and suppose (as in the hypothesis of the Specialization Lemma) that S is an integral Dedekind domain. Let L be the quotient field of S,

$K = k(s)$, and $u \in S_{\mathsf{f}}^*$ for the element f from the hypotheses of the Specialization Lemma.

The element $\overline{u} \in \mathcal{F}(L)$ is S-unramified, i.e. $\overline{u} \in \mathcal{F}_{un}(S)$. Thus $\overline{u} \in \mathcal{F}_{un}(L)$. Theorem 5.14 shows that the relation

$$(**) \qquad \prod_{y/1} N_{k(y)/k}(s_y(\overline{u})^{i(y/1)}) = \prod_{y/0} N_{k(y)/k}(s_y(\overline{u})^{i(y/0)})$$

holds in $\mathcal{F}(k)$. It remains to check that the left hand side of the relation (**) coincides with the element $\overline{N_{S_1/k}(u_1)}$ in $\mathcal{F}(k)$ and the right hand side of the relation (**) coincides with the element $\overline{N_{S_0/k}(u_0)} \cdot \overline{\epsilon(u)}$ in $\mathcal{F}(k)$.

Let $S_{1,y}$ be the localization at y of the Artinian ring $S_1 = S/(s-1)S$. Clearly, the diagram

$$\begin{array}{ccc} S_{1,y} & \xrightarrow{p_y} & k(y) \\ \uparrow & & \uparrow \\ S_y & \longrightarrow & \hat{S}_y, \end{array}$$

(where all the mappings are the canonical ones) commutes. For an element $v \in S_1$ let v_y be its image in $S_{1,y}$. Now Lemma 4.5 and Definition 4.6 show that the element $p_y((u_1)_y)$ coincides with the element $s_y(u)$ in $\mathcal{F}(k(y))$. Observe as well that $S_1 = \prod_{y/1} S_{1,y}$ and that the diagrams

$$\begin{array}{ccc} S_1^* & \xrightarrow{\sim} & \prod_{y/1} S_{1,y}^* \\ {\scriptstyle N_{S_1/k}}\downarrow & & \downarrow{\scriptstyle \Pi N_{S_{1,y}/k}} \\ k^* & \xrightarrow{\mathrm{id}} & k^*, \end{array} \qquad \begin{array}{ccc} S_{1,y}^* & \xrightarrow{p_y} & k(y)^* \\ {\scriptstyle N_{S_{1,y}/k}}\downarrow & & \downarrow{\scriptstyle N_{k(y)/k}^{i(y/1)}} \\ k^* & \xrightarrow{\mathrm{id}} & k^*. \end{array}$$

commute. This proves the relation $\prod_{y/1} N_{k(y)/k}(s_y(\overline{u})^{i(y/1)}) = \prod_{y/1} N_{S_{1,y}/k}((u_1)_y) = N_{S_1/k}(u_1)$ in $\mathcal{F}(k)$. The relation $\prod_{y/0} N_{k(y)/k}(s_y(\overline{u})^{i(y/0)}) = N_{S_0/k}(u_0) \cdot \epsilon(u)$ in $\mathcal{F}(k)$ is proved similarly (use that $S/sS = k \times S_0$ and the map $S \longrightarrow k$ is the augmentation $\epsilon : S \longrightarrow k$). The Corollary is proved. $\square$

§6. Two lemmas

Let k be an infinite field and $\mathcal{O}$ an essentially smooth local k-algebra.

5.1. Definition. A *perfect triple* $(\mathcal{R} \underset{i}{\overset{\epsilon}{\rightleftarrows}} \mathcal{O}, f)$ over $\mathcal{O}$ consists of a commutative $\mathcal{O}$-algebra $i : \mathcal{O} \longrightarrow \mathcal{R}$, an augmentation map $\epsilon : \mathcal{R} \longrightarrow \mathcal{O}$ and an element $f \in \mathcal{R}$ which are subjected to the following conditions:

(1) $\epsilon \circ i = id_{\mathcal{O}}$,

(2) the $\mathcal{O}$-algebra $\mathcal{R}$ is smooth at each prime $\mathfrak{p}$ containing $\mathrm{Ker}(\epsilon)$,
(3) $\mathcal{R}$ is essentially k-smooth and $\mathcal{R}$ is domain,
(4) $\mathcal{R}/f\mathcal{R}$ is a finitely generated $\mathcal{O}$-module,
(5) there exists an element $t \in \mathcal{R}$ such that $\mathcal{O}[t]$ is the polynomial ring in one variable and $\mathcal{R}$ is a finitely generated $\mathcal{O}[t]$-module.

5.2. Remark. The condition (5) shows that Spec $\mathcal{R}$ is a relative curve over Spec $\mathcal{O}$. The condition (3) shows that $\mathcal{R}$ is a regular ring. Since $\mathcal{O}[t]$ is a regular ring as well, a theorem of Grothendieck [**Eis**, Corollary 18.17] together with (5) show that $\mathcal{R}$ is flat over $\mathcal{O}[t]$ and thus it is a finitely generated projective $\mathcal{O}[t]$-module.

Let $(\mathcal{R}\underset{i}{\overset{\epsilon}{\rightleftarrows}}\mathcal{O}, f)$ a perfect triple. Let (A, σ) be an Azumaya algebra with involution over $\mathcal{R}$ and let $(A_0, \sigma_0) = \mathcal{O} \otimes_{\mathcal{R}} (\mathcal{A}, \sigma)$, where $\mathcal{O}$ is considered as an $\mathcal{R}$-algebra by means of the augmentation ϵ.

5.3. Lemma (Equating Lemma). *There exist a quasi-finite étale extension $\widetilde{j} : \mathcal{R} \hookrightarrow \widetilde{\mathcal{R}}$ and a lifting $\widetilde{\epsilon} : \widetilde{\mathcal{R}} \to \mathcal{O}$ of the augmentation ϵ (i.e. $\widetilde{\epsilon} \circ \widetilde{j} = \epsilon$) and an isomorphism $\Phi : \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} (\mathcal{A}, \sigma) \to \widetilde{\mathcal{R}} \otimes_{\mathcal{O}} (A_0, \sigma_0)$ of Azumaya algebras with involutions over $\widetilde{\mathcal{R}}$ such that the triple $(\widetilde{\mathcal{R}}\underset{\widetilde{i}}{\overset{\widetilde{\epsilon}}{\rightleftarrows}}\mathcal{O}, \widetilde{f})$ with $\widetilde{i} = \widetilde{j} \circ i$ and $\widetilde{f} = \widetilde{j}(f)$ is still perfect and the map $\mathcal{O} \otimes_{\widetilde{\mathcal{R}}} \Phi : (A_0, \sigma_0) \to (A_0, \sigma_0)$ is the identity.*

Proof. The required quasi-finite étale extension $\widetilde{j} : \mathcal{R} \hookrightarrow \widetilde{\mathcal{R}}$ and lifting $\widetilde{\epsilon} : \widetilde{\mathcal{R}} \to \mathcal{O}$ of the augmentation ϵ and the isomorphism $\Phi : \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} (\mathcal{A}, \sigma) \to \widetilde{\mathcal{R}} \otimes_{\mathcal{O}} (A_0, \sigma_0)$ of Azumaya algebras with involutions are constucted using geometric terminology in [**Oj-P1**, Proof of 8.1].

To see this set $\mathcal{X} = \mathrm{Spec}(\mathcal{R})$, $U = \mathrm{Spec}(\mathcal{O})$ and consider the morphisms $p : \mathcal{X} \to U$ and $\Delta : U \to \mathcal{X}$ induced by the ring homomorphisms i and ϵ. Let $q : \mathcal{X} \to U \times \mathbb{A}^1$ be the finite surjective U-morphism corresponding to the integral extension $\mathcal{O}[t] \subset \mathcal{R}$. Let $\mathcal{Z} \subset \mathcal{X}$ be the vanishing locus of f.

Now consider certain scheme morphisms from [**Oj-P1**, Proof of 8.1]. Namely, consider the quasi-finite étale morphism $\tilde{\mathcal{X}} \to \mathcal{X}$ which is the composition of the finite surjective etale morphism $\pi : \tilde{\mathcal{X}} \to \mathcal{W}$ and the open inclusion $\mathcal{W} \subset \mathcal{X}$. Consider the section $\widetilde{\Delta} : U \to \tilde{\mathcal{X}}$ and the isomorphism of Azumaya algebras with involutions Φ from [**Oj-P1**, Proof of 8.1]. Recall that $\widetilde{\Delta}^*(\Phi)$ is the identity, $\Delta = \pi \circ \widetilde{\Delta}$, $\Delta(U) \subset \mathcal{W}$, $\mathcal{Z} \subset \mathcal{W}$, and that there is a finite surjective U-morphism $r : \mathcal{W} \to U \times \mathbb{A}^1$.

Let $\widetilde{j} : \mathcal{R} \hookrightarrow \widetilde{\mathcal{R}}$ be the inclusion induced by $\tilde{\mathcal{X}} \to \mathcal{X}$ and $\widetilde{\epsilon} : \widetilde{\mathcal{R}} \to \mathcal{O}$ the $\mathcal{O}$-augmentation induced by $\widetilde{\Delta} : U \to \tilde{\mathcal{X}}$. We claim that $\widetilde{j}$, $\widetilde{\epsilon}$ and Φ satisfy the Lemma.

In fact, $\mathcal{O} \otimes_{\widetilde{\mathcal{R}}} \Phi = \widetilde{\Delta}^*(\Phi)$ is the identity. The relation $\widetilde{\epsilon} \circ \widetilde{j} = \epsilon$ follows from

the equality $\Delta = \pi \circ \widetilde{\Delta}$ mentioned just above. It remains to check that the triple $(\widetilde{\mathcal{R}} \underset{\widetilde{i}}{\overset{\widetilde{\epsilon}}{\rightleftarrows}} \mathcal{O}, \widetilde{f})$ is perfect. To check this note that $\widetilde{\epsilon} \circ \widetilde{i} = \widetilde{\epsilon} \circ \widetilde{j} \circ i = \epsilon \circ i = id_{\mathcal{O}}$. The $\mathcal{O}$-algebra $\widetilde{\mathcal{R}}$ is smooth at each prime containing $\mathrm{Ker}(\widetilde{\Delta})$ because $\Delta = \pi \circ \widetilde{\Delta}$ (with π an étale morphism) and $p : \mathcal{X} \to U$ is smooth along $\Delta(U)$. The k-algebra $\widetilde{\mathcal{R}}$ is essentially smooth because the k-algebra $\mathcal{R}$ is essentially smooth and $\widetilde{j} : \mathcal{R} \to \widetilde{\mathcal{R}}$ is etale. The vanishing locus $\widetilde{\mathcal{Z}} \subset \widetilde{\mathcal{X}}$ of $\widetilde{f}$ is finite over U because $\mathcal{Z} \subset \mathcal{W}$ and $\pi : \tilde{\mathcal{X}} \to \mathcal{W}$ is finite. Since $\widetilde{\mathcal{Z}}$ is finite over U the $\mathcal{O}$-module $\mathcal{R}/f\mathcal{R}$ is finitely generated. It remains to check that there is a finite surjective U-morphism $\widetilde{\mathcal{X}} \to U \times \mathbb{A}^1$. For that consider the finite surjective morphism $r : \mathcal{W} \to U \times \mathbb{A}^1$ and take the composition $r \circ \pi : \widetilde{\mathcal{X}} \to U \times \mathbb{A}^1$. $\square$

Let R be a commutative k-algebra and let (A, σ) be an Azumaya algebra with involution over R. Let $G = \mathrm{Sim}_{A,\sigma}$ be the similitude group of (A, σ) and let $\mu : G \longrightarrow \mathbb{G}_m$ be a group homomorphism which takes a similitude α to its similarity factor $\mu(\alpha) = \alpha^\sigma \cdot \alpha$. Observe that $\mu(G(S)) = G_S(h)$ from 3.1. $\square$

5.4. Notation. For every commutative R-algebra S denote by $\mathcal{F}(S)$ the group $S^*/\mu(G(S))$. An R-algebra homomorphism $S \xrightarrow{\alpha} T$ clearly induces a group map $\mathcal{F}(S) \xrightarrow{\alpha_*} \mathcal{F}(T)$. For an element $u \in S^*$ we shall write $\overline{u}$ for its image in $\mathcal{F}(S)$. The homomorphism α_* takes $\overline{u}$ to $\overline{\alpha(u)}$.

5.5. Definition. Let S be an $\mathcal{R}$-algebra which is a domain with the quotient field K and let $\mathfrak{p}$ be a height 1 prime ideal in S. An element $\nu \in \mathcal{F}(K)$ is called unramified at $\mathfrak{p}$ iff ν belongs to the image of $\mathcal{F}(S_\mathfrak{p})$ in $\mathcal{F}(K)$. An element $\nu \in \mathcal{F}(K)$ is called S-unramified if it is unramified at each height 1 prime $\mathfrak{p}$ in S.

5.6. Lemma (Unramifiedness Lemma). *Let R and S be domains with quotient fields K and L respectively. Let $R \xrightarrow{\alpha} S$ be an injective flat homomorphism of finite type and let $\beta : K \longrightarrow L$ be the induced inclusion of the quotient fields. Then for each localization $T \supset S$ of S the map $\beta_* : \mathcal{F}(K) \longrightarrow \mathcal{F}(L)$ takes S-unramified elements to T-unramified elements.*

Proof. Let $\nu \in K^*$ and let $\mathfrak{r}$ be height 1 primes of T. Then $\mathfrak{q} = S \cap \mathfrak{r}$ is a height 1 prime of S. Let $\mathfrak{p} = R \cap \mathfrak{q}$. Since the R-algebra S is flat of finite type one has $\mathrm{ht}(\mathfrak{q}) \geq \mathrm{ht}(\mathfrak{p})$. Thus $\mathrm{ht}(\mathfrak{p})$ is 1 or 0. The commutative diagram

$$\begin{array}{ccc} \mathcal{F}(K) & \longrightarrow & \mathcal{F}(L) \\ \uparrow & & \uparrow \\ \mathcal{F}(R_\mathfrak{p}) & \longrightarrow & \mathcal{F}(T_\mathfrak{r}) \end{array}$$

shows that the class $\overline{\beta(\nu)}$ is in the image of $\mathcal{F}(T_\mathfrak{r})$. Whence the class $\overline{\beta(\nu)} \in \mathcal{F}(L)$ is T-unramified. The Lemma follows. $\square$

§6. Relative Specialization Lemma

Let $\mathcal{O}$ be a regular local ring containing an infinite field k and which is an essentially smooth k-algebra. Let K be the quotient field of $\mathcal{O}$. Let $(\mathcal{R}\underset{i}{\overset{\epsilon}{\rightleftarrows}}\mathcal{O}, f)$ be a perfect triple. Denote by $\epsilon_K : \mathcal{R}_K = \mathcal{R}\otimes_{\mathcal{O}} K \to K$ the homomorphism $\epsilon\otimes_{\mathcal{O}} K$. We will consider $\mathcal{O}$ and K as $\mathcal{R}$-algebras via ϵ and ϵ_K respectively. So for an Azumaya algebra with involution $(\mathcal{A}, \sigma)$ over $\mathcal{R}$ it makes sense to speak about the groups $\mathcal{F}(\mathcal{O})$ and $\mathcal{F}(K)$ (see Definition 5.5).

6.1. Lemma (Relative Specialization Lemma). *Let $(\mathcal{R}\underset{i}{\overset{\epsilon}{\rightleftarrows}}\mathcal{O}, f)$ be a perfect triple and $(\mathcal{A}, \sigma)$ an Azumaya algebra with involution over $\mathcal{R}$. Let $\mathcal{K}$ be the quotient field of $\mathcal{R}$ and let $u \in \mathcal{R}^*_f$ be a unit such that the class $\bar{u} \in \mathcal{F}(\mathcal{K})$ is $\mathcal{R}$-unramified. If $\epsilon(f) \neq 0$ then the class $\overline{\epsilon_K(u \otimes 1)} \in \mathcal{F}(K)$ can be lifted to $\mathcal{F}(\mathcal{O})$.*

Proof. Set $(A_0, \sigma_0) = (\mathcal{O}\otimes_{\mathcal{R}}\mathcal{A}, \mathcal{O}\otimes_{\mathcal{R}}\sigma)$, where $\mathcal{O}$ is an $\mathcal{R}$-algebra by means of ϵ. Set $(\mathcal{A}_0, \sigma_0) = (\mathcal{R}\otimes_{\mathcal{O}} A_0, \mathcal{R}\otimes_{\mathcal{O}}\sigma_0)$, where $\mathcal{R}$ is regarded as an $\mathcal{O}$-algebra by means of the map i. There are two Azumaya algebras with involutions $(\mathcal{A}, \sigma)$ and $(\mathcal{A}_0, \sigma_0)$ over $\mathcal{R}$. Their scalar extensions $(\mathcal{A}, \sigma)\otimes_{\mathcal{R}}\mathcal{O}$ and $(\mathcal{A}_0, \sigma_0)\otimes_{\mathcal{R}}\mathcal{O}$ tautologically coincides because the composite map $\mathcal{O} \xrightarrow{i} \mathcal{R} \xrightarrow{\epsilon} \mathcal{O}$ is the identity. Thus by the Equating Lemma 5.3 one can find a quasi-finite étale extension $j : \mathcal{R} \hookrightarrow \widetilde{\mathcal{R}}$ and a lifting $\widetilde{\epsilon} : \widetilde{\mathcal{R}} \to \mathcal{O}$ of the augmentation ϵ and an isomorphism $\Phi : (\widetilde{\mathcal{A}}, \widetilde{\sigma}) \to (\widetilde{\mathcal{A}_0}, \widetilde{\sigma_0})$ of Azumaya algebras with involutions over $\widetilde{\mathcal{R}}$ such that $(\widetilde{\mathcal{R}}\underset{\widetilde{i}}{\overset{\widetilde{\epsilon}}{\rightleftarrows}}\mathcal{O}, j(f))$ is still a perfect triple and the isomorphism $\mathcal{O}\otimes_{\widetilde{\mathcal{R}}}\Phi$ is the identity. Here $(\widetilde{\mathcal{A}}, \widetilde{\sigma}) = \widetilde{\mathcal{R}}\otimes_{\mathcal{R}}(\mathcal{A}, \sigma)$, $(\widetilde{\mathcal{A}_0}, \widetilde{\sigma_0}) = \widetilde{\mathcal{R}}\otimes_{\mathcal{O}}(A_0, \sigma_0)$ and $\mathcal{O}$ is regarded as an $\widetilde{\mathcal{R}}$-algebra by means of $\widetilde{\epsilon} : \widetilde{\mathcal{R}} \to \mathcal{O}$.

Denote by $\tilde{\epsilon}_K : \tilde{\mathcal{R}}_K = \mathcal{R}\otimes_{\mathcal{O}} K \to K$ the augmentation $\tilde{\epsilon}\otimes_{\mathcal{O}} K$. Set $\tilde{f} = j(f)$ and $\tilde{u} = j(u) \in \tilde{\mathcal{R}}^*_{\tilde{f}}$. Since $\tilde{\epsilon}_K(\tilde{u}\otimes 1) = \epsilon_K(u\otimes 1)$ it suffices to check that the class $\overline{\tilde{\epsilon}_K(\tilde{u}\otimes 1)}$ can be lifted to $\mathcal{F}(\mathcal{O})$.

Let $\tilde{\mathcal{K}}$ be the quotient field of $\tilde{\mathcal{R}}$. By the Unramifiedness Lemma the class $\overline{\tilde{u}} \in \mathcal{F}(\tilde{\mathcal{K}})$ is $\tilde{\mathcal{R}}$-unramified. So replacing $(\mathcal{R}\underset{i}{\overset{\epsilon}{\rightleftarrows}}\mathcal{O}, f)$, $(\mathcal{A}, \sigma)$ and u by $(\widetilde{\mathcal{R}}\underset{\widetilde{i}}{\overset{\widetilde{\epsilon}}{\rightleftarrows}}\mathcal{O}, j(f))$, $(\tilde{\mathcal{A}}, \tilde{\sigma})$ and $\tilde{u}$ we may assume that $(\mathcal{R}\underset{i}{\overset{\epsilon}{\rightleftarrows}}\mathcal{O}, f)$, is a perfect triple, $(\mathcal{A}, \sigma) = \mathcal{R}\otimes_{\mathcal{O}}(A_0, \sigma_0)$ for an Azumaya algebra with involution (A_0, σ_0) over $\mathcal{O}$, and $u \in \mathcal{R}^*_f$ is such that the class $\bar{u} \in \mathcal{F}(\mathcal{K})$ is $\mathcal{R}$-unramified. We must check that the class $\overline{\epsilon_K(u_K)} \in \mathcal{F}(K)$ can be lifted in $\mathcal{F}(\mathcal{O})$.

Since the triple $(\mathcal{R}\underset{i}{\overset{\epsilon}{\rightleftarrows}}\mathcal{O}, f)$ is perfect, the geometric presentation lemma [**Oj-P**, Lemma 5.2] shows that one can choose an element $s \in \mathcal{R}$ such that the extension

$\mathcal{R} \supseteq \mathcal{O}[s]$ is finite, the ring $\mathcal{O}[s]$ is the polynomial ring in one variable over $\mathcal{O}$ and the following holds:

(1) $(1-s)\mathcal{R} + f\mathcal{R} = \mathcal{R}$,
(2) $s\mathcal{R} = \mathrm{Ker}(\epsilon) \cap J$ for a certain ideal J and
(3) $J + f\mathcal{R} = \mathcal{R}$ and
(4) the map $\mathcal{R}/s\mathcal{R} \longrightarrow \mathcal{R}/\mathrm{Ker}(\epsilon) \times \mathcal{R}/J = \mathcal{O} \times \mathcal{R}/J$ is an isomorphism.

Since $\mathcal{R}$ and $\mathcal{O}[s]$ are both essentially smooth k-algebras (and thus regular rings) and since the extesion $\mathcal{R}$ over $\mathcal{O}[s]$ is finite, a theorem of Grothendieck [**Eis**, Corollary 18.17] shows that $\mathcal{R}$ is a flat $\mathcal{O}[s]$-module. Therefore $\mathcal{R}$ is a finitely generated projective $\mathcal{O}$-module. Thus $\mathcal{R}_1 = \mathcal{R}/(1-s)\mathcal{R}$ and $\mathcal{R}_0 = \mathcal{R}/J$ are finitely generated projective $\mathcal{O}$-modules.

Consider the elements $u_1 = u \bmod (1-s)\mathcal{R}_f$ in $\mathcal{R}^*_{1,f}$ and $u_0 = u \bmod J_f$ in $\mathcal{R}^*_{0,f}$. By (1) and (3) one has $\mathcal{R}_i = \mathcal{R}_{i,f}$ and thus $u_i \in \mathcal{R}^*_i$ $(i = 0, 1)$. Since $\mathcal{R}_1$ and $\mathcal{R}_0$ are finitely generated projective $\mathcal{O}$-modules, there are the norm mappings $N_{\mathcal{R}_i/\mathcal{O}} : \mathcal{R}^*_i \longrightarrow \mathcal{O}^*$ $(i = 0, 1)$ given by $(v \mapsto \det(\text{mult. by } v))$. Set

$$\phi(u) = N_{\mathcal{R}_1/\mathcal{O}}(u_1) \cdot N_{\mathcal{R}_0/\mathcal{O}}(u_0^{-1}) \in \mathcal{O}^* \subseteq K^* \ .$$

Claim. $\overline{\phi(u)} = \overline{\epsilon_K(u_K)}$ *in the group* $K^*/\mu(G(K)) = \mathcal{F}(K)$.

Since $\phi(u) \in \mathcal{O}^*$, the Claim clearly completes the proof of purity. The rest of the section is devoted to the proof of the Claim.

Set $\mathcal{R}_K = K \otimes_{\mathcal{O}} \mathcal{R}$ and $u_K = 1 \otimes u \in \mathcal{R}_K$. Set $\mathcal{R}_{i,K} = K \otimes_{\mathcal{O}} \mathcal{R}_i$ and $u_{i,K} = 1 \otimes u_i \in \mathcal{R}_K \in \mathcal{R}^*_{i,K}$. Finally set $\mathcal{R}_{f,K} = \mathcal{R}_{K,1\otimes f}$. Clearly it suffices to prove the relation

$$(\dagger) \qquad \overline{\epsilon_K(1 \otimes u)} = \overline{N_{\mathcal{R}_{1,K}/K}(1 \otimes u_1)} \cdot \overline{N_{\mathcal{R}_{0,K}/K}(1 \otimes u_0)^{-1}}$$

in the group $\mathcal{F}(K)$. The relation $(\dagger)$ will be checked below in this proof applying the Specialization Lemma (Theorem 3.3) to the integral extension $\mathcal{R}_K \supseteq K \otimes_{\mathcal{O}} \mathcal{O}[s] = K[s]$ and the Azumaya algebra with involution $(A_0, \sigma_0) \otimes_{\mathcal{O}} K$ over K.

Check the hypotheses of the Specialization Lemma. Since $\mathcal{R}$ is regular domain and $\mathcal{R}_K$ is its localization $\mathcal{R}_K$ is a regular domain as well. Since $\mathcal{R}_K$ is an integral extension of the polynomial ring $K[s]$, the dimension of $\mathcal{R}$ is one. A regular domain of dimension 1 is a Dedekind domain. <u>Thus $\mathcal{R}$ is a Dedekind domain.</u>

The class $\bar{u} \in \mathcal{F}(\mathcal{K})$ of the element $u \in \mathcal{R}^*_f$ is $\mathcal{R}$-unramified. Thus the class $\bar{u}_K \in \mathcal{F}(\mathcal{K})$ of the element $u_K \in \mathcal{R}^*_{f,K}$ is $\mathcal{R}_K$-unramified.

Now check that the element $1 \otimes f \in \mathcal{R}_K$ is coprime with both s and $s-1$ in $\mathcal{R}_K$. Recall the conditions (1) to (4) mentioned above in this proof. The element $1 \otimes f$ is coprime with $(s-1)$ by condition (1). The element $\epsilon_K(1 \otimes f) = \epsilon(f)_K$ is non-zero in K by the very assumption on f. The element $1 \otimes f$ is coprime with the ideal

J_K by condition (3). Thus $1 \otimes f$ is coprime with s by condition (4). We already checked that the class $\bar{u}$ is $\mathcal{R}_K$-unramified. Thus by Theorem 3.3 the relation (†) holds in $\mathcal{F}(K)$. The Claim is proved. The Relative Specialization Lemma follows.

§7. Geometric case of the Purity Theorem

Under the notation of 5.4 and 5.5 the following theorem holds.

7.1. Theorem. *Let $\mathcal{O}$ be a local, essentially smooth algebra over a field k and let K be its quotient field. Let (A, σ) be an Azumaya algebra with involution over $\mathcal{O}$ and let $\nu \in K^*$ be such that the class $\bar{\nu} \in \mathcal{F}(K)$ is $\mathcal{O}$-unramified. Then $\bar{\nu}$ can be lifted in $\mathcal{F}(\mathcal{O})$.*

Proof. We begin with the case of an infinite field k. By assumption there exist a smooth d-dimensional k-algebra $R = k[t_1, \dots, t_n]$ and a prime ideal $\mathfrak{p}$ of R such that $A = R_{\mathfrak{p}}$. We first reduce the proof to the case in which $\mathfrak{p}$ is maximal. To do this we choose a maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$. Since k is infinite, by a standart general position argument we can find d algebraically independent elements $X_1, X_2, \dots, X_d$ such that R is finite over $k[X_1, \dots, X_d]$ and étale at $\mathfrak{m}$. After a linear change of coordinates we may assume that $R/\mathfrak{p}$ is finite over $B = k[X_1, \dots, X_m]$, where m is the dimension of $R/\mathfrak{p}$. Clearly R is smooth over B at $\mathfrak{m}$ and thus, for some $h \in R - \mathfrak{m}$, the localization R_h is smooth over B. Let S be the set of nonzero elements of B, $k' = S^{-1}B$ the field of fractions of B and $R' = S^{-1}R_h$. The prime ideal $\mathfrak{p}' = S^{-1}\mathfrak{p}_h$ is maximal in R', the k'-algebra R' is smooth and $A = R'_{\mathfrak{p}'}$.

From now on and till the end of the proof of Theorem 8.1 we assume that $\mathcal{O} = \mathcal{O}_{X,x}$ is the local ring of a closed point x of a smooth d-dimensional irreducible affine variety X over k.

Replacing X by a sufficiently small affine neighbourhood of x we may assume that

(1) the algebra with involution (A, σ) is defined over $k[X]$ and is an Azumaya algebra with involution already over $k[X]$,
(2) the element ν is a unit in $k[X]_g$ for certain nonzero element $g \in k[X]$,
(3) the class $\bar{\nu} \in \mathcal{F}(K)$ is $k[X]$-unramified.

We must prove that $\bar{\nu}$ can be lifted in $\mathcal{F}(\mathcal{O})$.

By Quillen's trick there exists a polinomial subalgebra $k[t_1, t_2, \dots, t_n]$ in $k[X]$ such that the algebra $R = k[X]$ is finite over $k[t_1, t_2, \dots, t_n]$, the algebra R is smooth over $k[t_1, t_2, \dots, t_{n-1}]$ at the maximal ideal $\mathfrak{m}$ and the $k[t_1, t_2, \dots, t_{n-1}]$-module R/fR is finite. Set $P = k[t_1, t_2, \dots, t_{n-1}]$, $\mathcal{R} = \mathcal{O} \otimes_P R$, consider ring homomorphisms $j : R \to \mathcal{R}$, $i : \mathcal{O} \to \mathcal{R}$ and $\epsilon : \mathcal{R} \to \mathcal{O}$ given by $j(a) = 1 \otimes a$, $i(b) = b \otimes 1$ and $\epsilon(a \otimes b) = ab$ respectively.

We claim that $(\mathcal{R} \underset{i}{\overset{\epsilon}{\rightleftarrows}} \mathcal{O}, f)$ with $f = j(\mathrm{f})$ is a perfect triple (see 6.1 for definition).

This is checked in [Oj-P] using geometric terminology. This perfect triple fits in the diagram

$$\begin{tikzcd} \mathcal{R} \arrow[d, shift right, "\epsilon"'] & R \arrow[l, "j"'] \arrow[dl, "can"] \\ \mathcal{O} \arrow[u, shift right, "i"'] & \end{tikzcd}$$

with the localization map can. Clearly $can = \epsilon \circ j$.

Set $(\mathcal{A}, \sigma) = \mathcal{R} \otimes_{k[X]} (A, \sigma)$ and $u = j(\nu) \in \mathcal{R}_f^*$. Let $\mathcal{K}$ be the quotient field of $\mathcal{R}$. By the Unramifiedness Lemma the class $\bar{u} \in \mathcal{F}(\mathcal{K})$ is $\mathcal{R}$-unramified. Since $\epsilon(f) = \epsilon(j(f)) = f$ is nonzero element of $\mathcal{O}$ we are under the hypotheses of the Relative Specialization Lemma. Thus the class $\bar{\epsilon}_K(u) \in \mathcal{F}(K)$ can be lifted in $\mathcal{F}(\mathcal{O})$. It remain to note that

$$\epsilon_K(u) = \epsilon_K(j(\nu)) = \nu \in K.$$

Thus the class $\bar{\nu} \in \mathcal{F}(K)$ can be lifted in $\mathcal{F}(\mathcal{O})$.

Now suppose that k is finite. So $\mathcal{O}$ is a local essentially smooth k-algebra with maximal ideal $\mathfrak{m}$. Let $\nu \in K^*$ be such that the class $\bar{\nu} \in \mathcal{F}(K)$ is $\mathcal{O}$-unramified. Let p^m be the cardinality of the algebraic closure of k in $A/\mathfrak{m}$ and s be an odd integer greater than 2 and prime to m. For any i let l_i be the field (in some fixed algebraic closure of k) of degree s^i over k. Let l be the union of all l_i. Since $l \otimes_k (\mathcal{O}/\mathfrak{m})$ is still a field, $R = l \otimes_k \mathcal{O}$ is a local essentially smooth algebra over the infinite field l. Let $L = l \otimes_k K$ be its field of fractions. The image $\bar{\nu}_L$ of $\bar{\nu}$ in $\mathcal{F}(L)$ is R-unramified. In fact, let $\mathfrak{q}$ be a hight-one prime of R and $\mathfrak{p} = \mathcal{O} \cap \mathfrak{q}$. By assumption $\bar{\nu}$ is in the image of $\mathcal{F}(\mathcal{O}_\mathfrak{p})$ and since $\mathcal{O}_\mathfrak{p} \to L$ factors through $R_\mathfrak{q}$ the class $\bar{\nu}_L$ is in $\mathcal{F}(R_\mathfrak{q})$ for every $\mathfrak{q}$. We can now find a finite subfield l' of l, and for $\mathcal{O}' = l' \otimes_k \mathcal{O}$, a $\nu' \in \mathcal{O}'$ which maps to $\bar{\nu}_L$. Let K' be the field of fractions of $\mathcal{O}'$. Further enlarging l' we may assume that the images $\bar{\nu}$ and $\bar{\nu}'$ in $\mathcal{F}(K')$ coincide. Consider the diagram

$$\begin{tikzcd} \mathcal{O}^* \arrow[r] \arrow[d] & (O')^* \arrow[r, "N"] \arrow[d] & \mathcal{O}^* \arrow[d, "\alpha"] \\ \mathcal{F}(K) \arrow[r] & \mathcal{F}(K') \arrow[r, "\bar{N}"] & \mathcal{F}(K). \end{tikzcd}$$

where $\bar{N}$ is the norm map (it is well-defined by the Scharlau norm principle). Since the composition of the horizontal maps is the identity, we have $\alpha \circ N(\nu') = \bar{\nu}$ in $\mathcal{F}(K)$. Thus $\bar{\nu}$ is indeed in the image of $\mathcal{O}^*$. Theorem 7.1 is proved.

§8. Proof of the Purity Theorem

Proof of Theorem 1.3. Let k be the prime subfield of the ring R. By Popescu's theorem [P], [Sw] $R = \varinjlim R_\alpha$ (a filtered direct limit), where R_α's are smooth k-algebras. We first observe that we may replace the direct system of the R_α's by a system of essentially smooth local k-algebras. In fact, if $\mathfrak{m}$ is the maximal ideal of R, we can replace each R_α by $(R_\alpha)_{\mathfrak{p}_\alpha}$ where $\mathfrak{p}_\alpha = \mathfrak{m} \cap R_\alpha$. Note that in this case the canonical morphisms $\phi_\alpha : R_\alpha \to R$ are local and that every R_α is a regular local ring thus in particular a factorial ring.

Now let K be the field of fractions of R and, for each α, let K_α be the field of fractions of R_α. The ideal $\mathfrak{r}_\alpha = \ker(\phi_\alpha)$ is prime. Set $S_\alpha = (R_\alpha)_{\mathfrak{r}_\alpha}$. Note that the S_α's form a direct system of regular local rings with $K = \varinjlim S_\alpha$ (a filtered direct limit).

We may assume that there exists an index α and an Azumaya algebra with involution $(A_\alpha, \sigma_\alpha)$ over R_α such that $(A, \sigma) = (A_\alpha, \sigma_\alpha) \otimes_{R_\alpha} R$. Replacing the direct system of indeces α's by the subsystem of indeces β satisfying $\beta \geq \alpha$ we may assume that we are given with a direct system of Azumaya algebras with involutions $(A_\alpha, \sigma_\alpha)$ over the R_α's such that $(A, \sigma) = \varinjlim (A_\alpha, \sigma_\alpha)$.

Let $G_\alpha = \mathrm{Sim}_{A_\alpha, \sigma_\alpha}$. Then one has $G(R) = \varinjlim G_\alpha(R_\alpha)$ and $G(K) = \varinjlim G_\alpha(S_\alpha)$. Let $\mu_\alpha : G_\alpha \to \mathbb{G}_m$ be the group morphism which takes a similitude to its similarity factor (see the Introduction).

Let $\bar{a} \in \mathcal{F}(K)$ be an R-unramified class. We may represent $\bar{a}$ by a unit $a \in R_f^*$, where $0 \neq f \in R$. Let $f = p_1 p_2 \dots p_n$ be a prime decomposition of f in R. Since $\bar{a}$ is R-unramified for every index $i = 1, 2, \dots, n$ there exist elements $h_i \in R - p_i R$ and $a_i \in R_{h_i}^*$ and $g_i \in G(K)$ such that $a = a_i \mu(g_i)$.

We can now choose an index α, elements $p_{\alpha,i}$ and $h_{\alpha,i}$ with $\phi_\alpha(p_{\alpha,i}) = p_i$ and $\phi_\alpha(h_{\alpha,i}) = h_i$. Set $f_\alpha = p_{\alpha,1} p_{\alpha,2} \dots p_{\alpha,n}$. Since $\phi_\alpha(f_\alpha) = f \neq 0$ and $\phi_\alpha(h_{\alpha,i}) = h_i \neq 0$ one has the inclusions $R_{\alpha,f_\alpha} \subset S_\alpha$ and $R_{\alpha,h_{\alpha,i}} \subset S_\alpha$. Futher enlarging the index α we can choose $a_\alpha \in R_{\alpha,f_\alpha}^*$ and elements $a_{\alpha,i} \in R_{\alpha,h_{\alpha,i}}^*$ and $g_{\alpha,i} \in G_\alpha(S_\alpha)$ which are preimages of the a and the a_i's and the g_i's respectively. Having choosen these preimages consider the relations

$$a_\alpha = a_{\alpha,i} \mu_\alpha(g_{\alpha,i})$$

in S_α. Since they hold over K, we may assume, after replacing α by some larger index, that they hold over S_α. We claim that the class $\bar{a}_\alpha \in \mathcal{F}(K_\alpha)$ is R_α-unramified.

To prove this note first that each $p_{\alpha,i}$ is prime. In fact, $\phi_\alpha(p_{\alpha,i}) = p_i$, the element p_i is prime and ϕ_α is a local homomorphism of local factorial rings. Thus $p_{\alpha,i}$ is indeed prime. Since $a_\alpha \in R_{\alpha,f_\alpha}^*$ and $f_\alpha = p_{\alpha,1} p_{\alpha,2} \dots p_{\alpha,n}$ the class $\bar{a}_\alpha$ can be ramified at most at one of the $p_{\alpha,i}$'s. However the relations $a_\alpha = a_{\alpha,i} \mu_\alpha(g_{\alpha,i})$ with

$a_{\alpha,i} \in R^*_{\alpha,h_{\alpha,i}}$ and the fact that $p_{\alpha,i}$ does not divide $h_{\alpha,i}$ prove that the class $\bar{a}_\alpha$ is unramified at each the $p_{\alpha,i}$. Thus the class $\bar{a}_\alpha$ is indeed R_α-unramified.

By purity for R_α there exists an $a'_\alpha \in R^*_\alpha$ such that $\bar{a}'_\alpha = \bar{a}_\alpha$ in $\mathcal{F}(K_\alpha)$. The exact sequence $1 \to U_{A,\sigma} \to \mathrm{Sim}_{A,\sigma} \xrightarrow{\mu} \mathbb{G}_m \to 1$ of algebraic group schemes over S_α shows that the kernel of the boundary map $\partial : \mathcal{F}(S_\alpha) \to H^1(S_\alpha, U_{A,\sigma})$ is trivial. The Main Theorem of [Oj-P1] states that the kernel

$$\ker[H^1(S_\alpha, U_{A,\sigma}) \to H^1(K_\alpha, U_{A,\sigma})]$$

is trivial. Thus $\mathcal{F}(S_\alpha)$ injects into $\mathcal{F}(K_\alpha)$ and $\bar{a}'_\alpha = \bar{a}_\alpha$ already in $\mathcal{F}(S_\alpha)$. The commutative diagram

$$\begin{array}{ccc} R_\alpha & \xrightarrow{\phi_\alpha} & R \\ \downarrow & & \downarrow \\ S_\alpha & \longrightarrow & K\ . \end{array}$$

shows that $\overline{\phi_\alpha(\alpha')} = \bar{a}$ in $\mathcal{F}(K)$. This completes the proof of Theorem 1.2.

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