Tits Alternative for Maximal Subgroups of $GL_n(D)$ *

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Abstract

Let D be a noncommutative division algebra of finite dimension over its centre F. Given a maximal subgroup M of $GL_n(D)$ with $n \geq 1$, it is proved that either M contains a noncyclic free subgroup or there exists a finite family $\{K_i\}_1^r$ of fields properly containing F with $K_i^* \subset M$ for all $1 \leq i \leq r$ such that M/A is finite if $CharF = 0$ and M/A is locally finite if $CharF = p > 0$, where $A = K_1^* \times \cdots \times K_r^*$.

1 Introduction

Let D be a division algebra of finite dimension over its centre F. Denote by $M_n(D)$ the $n \times n$ matrix ring over D and $SL_n(D)$ the commutator subgroup of the multiplicative group $GL_n(D) = M_n(D)^*$. Given a subgroup G of $GL_n(D)$, we shall say that G is *maximal* in $GL_n(D)$ if for any subgroup H of $GL_n(D)$

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with $G \subset H$, one concludes that $H = GL_n(D)$. We know, by Corollary 1 of [11], that $G(A) := A^*/RN(A^*)A'$, where $A := M_n(D)$, $A' = SL_n(D)$, and $RN(A^*)$ is the image of A^* under the reduced norm of A to F, is an abelian torsion group of a bounded exponent dividing the degree of A over F. This group is not trivial in general. For example, if A is the algebra of real quaternions, then $G(A)$ is trivial whereas for rational quaternions $G(A)$ is isomorphic to a direct product of copies of Z_2, as it is easily checked. Assume that $G(A)$ is not trivial, then by Prũfer-Baer Theorem (cf. [13, p. 105]), we conclude that $G(A)$ is isomorphic to a direct product of Z_{r_i}, where r_i divides the index of A over F. In this way one may obtain normal maximal subgroups of finite index in $GL_n(D)$. So, if $G(A)$ is not trivial, then $GL_n(D)$ contains maximal subgroups. For some examples of non-normal maximal subgroups of $GL_n(D)$, see [12] or [3, p. 140]. It is shown in [12] that even for the case $G(A) = 1$ we may have maximal subgroups in $GL_n(D)$. But the question of whether $GL_n(D)$ has a maximal subgroup for any noncommutative division algebra D, is still open. Now, let D be a noncommutative division ring not necessarily of finite dimension over its centre F. The problem of whether $GL_n(D)$ contains a noncyclic free subgroup seems to be posed first by Lichtman in [8]. Stronger versions of this problem which essentially deal with the existence of noncyclic free subgroups in normal or subnormal subgroups of $GL_n(D)$ have been investigated in [4] and [5]. It is known so far that these problems have positive answers as long as we work with a division algebra of finite dimension over its centre. Further investigations for the infinite dimensional case are also dealt with in [4] and [5]. The study of maximal subgroups of $GL_n(D)$ begins in [2] in relation with an investigation of the structure of finitely generated normal subgroups of $GL_n(D)$, where D is of finite dimension over its centre F. In [2] and [10] we actually show that maximal subgroups arise naturally in $GL_n(D)$, $n \geq 1$, and finitely generated subnormal subgroups of $GL_n(D)$, $n \geq 1$, are central. This result is used to prove that a maximal subgroup of $GL_n(D)$ can not be finitely generated for $n \geq 1$. Therefore, we are not able to apply directly Tits' result, that any finitely generated linear group either is soluble-by-finite or contains a noncyclic free group (cf. [19]), to a maximal subgroup M of $GL_n(D)$ to explore the structure of M. In [2], it is also shown that there is a similarity between the behaviour of normal or subnormal subgroups of $GL_n(D)$ and the maximal ones. So, it is natural to ask if there exists a noncyclic free group in a maximal subgroup of $GL_n(D)$. In this direction, we observe that not every subgroup of $GL_n(D)$ satisfies the Tits' Theorem though any normal subgroup does so (cf. [16, p. 154]). We also mention that the soluble subgroups of the multiplicative group of a finite dimensional division algebra were studied in 1962 by Suprunenko [17] and the soluble subgroups of the multiplicative group of a finite dimensional simple algebra were considered by Zalesskii [22]. Inspired by Suprunenko's results, it is shown in [12] that given a noncommutative maximal subgroup M of $GL_1(D)$, then either M contains a noncyclic free subgroup or there exists a maximal subfield K of D which is Galois over F such that K^* is normal in M and $M/K^* \cong Gal(K/F)$. Using some results of algebraic groups and skew linear groups, in the present note our aim is to extend this result for $n > 1$, i.e., to prove a variation of Tits' Theorem for maximal subgroups of $GL_n(D)$. To be more precise, let D be a noncommutative division algebra of finite dimension over its centre F. Given a maximal subgroup M of $GL_n(D)$, it is proved that either M contains a noncyclic free subgroup or there exists a finite family $\{K_i\}_1^r$ of fields properly containing F with $K_i^* \subset M$ for all $1 \leq i \leq r$ such that M/A is finite if $CharF = 0$ and M/A is locally finite if $CharF = p > 0$, where $A = K_1^* \times \ldots \times K_r^*$.

2 Notations and conventions

Let D be a division ring with centre F. Given a subgroup G of $GL_n(D)$, we denote by $F[G]$ the F-algebra generated by elements of G over F. We shall say that G is *absolutely irreducible* if $M_n(D) = F[G]$. For any group G we denote its centre by $Z(G)$. Given a subgroup H of G, $N_G(H)$ means the *normalizer* of H in G, $[G : H]$ denotes the *index* of H in G, and $< H, K >$ the group generated by H and K, where K is a subgroup of G. We shall say that H is *soluble-by-finite* if there is a soluble normal subgroup K of H such that H/K

is finite. Let S be a subset of $M_n(D)$, then the *centralizer* of S in $M_n(D)$ is denoted by $C_{M_n(D)}(S)$. We shall identify the centre FI of $M_n(D)$ with F. Some notations and conventions for linear groups and skew linear groups from [15], [16] and [18] are frequently used throughout.

3 Free subgroups in maximal subgroups

Given a division ring D with centre F, let M be a maximal subgroup of $GL_n(D)$. This section essentially deals with maximal subgroups of $GL_n(D)$ and how they sit in $GL_n(D)$ with respect to F^* and $SL_n(D)$. We then present some commutativity theorems that enable us to prove our main result. To be more precise, let D be a noncommutative division ring not necessarily of finite dimension over its centre F. It is shown that there exists no maximal subgroup M of $GL_n(D)$, $n \geq 1$, containing F^* such that $[M : F^*] < \infty$. It is then proved that M is nilpotent if and only if M is the multiplicative group of a maximal subfield of $M_n(D)$. We then use these results to show that given a maximal subgroup M of $GL_n(D)$, $n \geq 1$, if M is soluble, then there exists a finite family $\{K_i\}_1^r$ of fields properly containing F with $K_i^* \subset M$, $1 \leq i \leq r$, such that M/A is finite, where $A = K_1^* \times \cdots \times K_r^*$, and so M is abelian-by- finite. Using this fact, we then obtain the same conclusion when $M \cap SL_n(D)$ is commutative or $M/M \cap F^*$ is torsion for any maximal subgroup of $GL_n(D)$. Finally, we apply these results to prove our main theorem that if M is a maximal subgroup of $GL_n(D)$, $n \geq 1$, then either M contains a noncyclic free subgroup or there exists a finite family $\{K_i\}_1^r$ of fields properly containing F with $K_i^* \subset M$ for all $1 \leq i \leq r$ such that M/A is finite if $Char F = 0$ and M/A is locally finite if $Char F = p > 0$, where $A = K_1^* \times \ldots \times K_r^*$. We begin our material with

LEMMA 1. *Let D be a division ring not necessarily of finite dimension over its centre F. If either $n = 1$ and D is noncommutative or $n > 1$ and D is infinite, then there exists no maximal subgroup M of $GL_n(D)$, $n \geq 1$, containing F^* such that $[M : F^*] < \infty$.*

PROOF. Assume on the contrary that there is a maximal subgroup M such that M/F^* is finite and first suppose that $[D : F] < \infty$. Let $x_1, \dots, x_t$ be the representatives for cosets of F^* in M, i.e., $M = F^*x_1 \cup \dots \cup F^*x_t$. Then, we have $M =< x_1, \dots, x_t > F^*$, where $< x_1, \dots, x_t >$ is the group generated by $x_1, \dots, x_t$. Take $x \in GL_n(D) \backslash M$. By maximality of M, we obtain $GL_n(D) =< x_1, \dots, x_t, x > F^*$. Put $H =< x_1, \dots, x_t, x >$. Thus, $GL_n(D) = HF^*$ and consequently we have $SL_n(D) = H' \subset H$, i.e., H is normal in $GL_n(D)$. Now, by Corollary 1 of [10], we conclude that $H \subset F^*$, i.e., $GL_n(D) = F^*$ which means that $n = 1$ and $D = F$ that is a contradiction. This takes care of the finite dimensional case.

Now consider the case $[D : F] = \infty$. As above, we may assume $M = F^*x_1 \cup \dots \cup F^*x_t$. Put $A = \{\Sigma_{i=1}^n f_i x_i; f_i \in F\}$. It is clearly seen that A is a finite dimensional F-algebra and we have $M \subset A^*$. Since A is of finite dimension over F we conclude that $A \neq M_n(D)$ and so $M = A^*$ by maximality of M in $GL_n(D)$. If $n = 1$, then it is easily seen that A is a division algebra. Thus we have $[A^* : F^*] < \infty$. If A is infinite, then, by a result of Faith (cf. [7, p. 225]), $A = F$ and so $M = F^*$ which is a contradiction. So, we may assume that A is finite. Now, Wedderburn's Theorem implies that A is a finite field. So there exists an element $a \in D^*$ such that $A^* =< a >$, i.e., $a^n = 1$ for some positive integer n. Since a is non-central in D, by Herstein's Lemma (cf. [7]), there is an element $b \in D^*$ such that $bab^{-1} = a^i \neq a$. Thus, $b \in N_{D^*}(A^*)$ and so $< M, b > \subset N_{D^*}(A^*)$. Now, by maximality of M, we conclude that $N_{D^*}(A^*) = D^*$. Therefore, by Cartan-Brauer-Hua's Theorem, we have either $A \subset F$ or $A = D$, and it is clear that none of these cases can occur. This completes the proof for the case $n = 1$ and $[D : F] = \infty$.

So, we may assume that $n \geq 2$ and $[D : F] = \infty$. We claim that A is simple. To see this, we first observe that the Jacobson radical $J(A) = J$ of A is nilpotent since A is left Artinian. It is known that a multiplicative semi-group of nilpotent matrices over a division ring can be simultaneously triangularized (cf. [6, p. 135]). Thus, we may assume that each element of J is upper triangular. Now, denote by L the subring of all elements x in $M_n(D)$ such that $xJ \subset J$. It is clear that $A \subset L$. For any $d \in D$ we see that $I + de_{1n} \in L$. We claim that M contains only a finite number of these elementary matrices. To prove this, assume that $I + d_1e_{1n}, I + d_2e_{1n} \in F^*x_i$ for some $d_1, d_2 \in D^*$. Then, we have $I + d_1e_{1n} = \lambda x_i$ and $I + d_2e_{1n} = \mu x_i$ for some $\lambda, \mu \in F^*$. This implies that $d_1 = d_2$ and $\lambda = \mu$. Thus, at most t elementary matrices of the form $I + de_{1n}$ belong to M. Since $M = A^* \subset L^*$ and D is infinite we conclude that there is an element $d \in D^*$ such that $I + de_{1n} \in L^* \setminus M$. By maximality of M, this implies that $L = M_n(D)$, i.e., J is a left ideal of $M_n(D)$. A similar argument applied to the subring R of all elements $x \in M_n(D)$ such that $Jx \subset J$, we conclude that J is also a right ideal of $M_n(D)$, i.e., J is a two sided ideal of $M_n(D)$ which means that $J = 0$. So A is semisimple. Now, we observe that the centre $Z(A)$ does not contain any nonzero zero-divisor. For otherwise, if $0 \neq u \in Z(A)$ is a zero-divisor in A, consider $l = l(u)$, the left annihilator of u in A. We have $l \neq 0$ since u is a zero-divisor. Now, since $u \in Z(A)$ we conclude that l is an ideal of A and we have $lu = 0$. Since entries of each element of l belong to a division ring we may reduce the j-th column of each element of l to zero. Now, the subring R of $M_n(D)$ consisting of all elements $x \in M_n(D)$ such that $lx \subset l$ contains A and we have $le_{ji} = 0$. Thus, we conclude that $I + de_{ji} \in R$ for all $d \in D$. Now, since F^* is of finite index in $M = A^*$, we conclude as above that there exists an element $d \in D^*$ such that $I + de_{ji} \in R^* \setminus M$. Since $M = A^* \subset R^*$ by maximality of M we obtain $R = M_n(D)$, i. e., l is a right ideal of $M_n(D)$. Since l is a two sided ideal of A, a similar argument applied to the subring L of $M_n(D)$ consisting of all elements $x \in M_n(D)$ such that $xl \subset l$, we may conclude that L is also a left ideal of $M_n(D)$, i. e., $l = 0$ which is a contradiction and so no nonzero element of $Z(A)$ is a zero-divisor. Now, since A is a semisimple Artinian ring we conclude that A is simple as claimed. Therefore, there is a positive integer n_1 such that $M \cong GL_{n_1}(D_1)$, for some division ring D_1 of finite dimension over its centre K, say. We know that F^* is of finite index in M. Thus, the image of F^* in $GL_{n_1}(D_1)$, which is a subfield of K, is also of finite index in $GL_{n_1}(D_1)$. We identify this image by F^*. Thus, D_1^*/K^* is torsion and so, by a result of Kaplansky (cf. [7, 259]), we obtain $D_1 = K$, i. e., $A \cong M_{n_1}(K)$, where K is of finite dimension over F. Now, since M/F^* is finite, we conclude that $GL_{n_1}(K)/F^*$ is finite. Thus, we have $GL_{n_1}(K) = \cup_1^t F^* u_i$ for some $u_i \in GL_{n_1}(K)$. Take an elementary matrix $I + be_{ij} \in GL_{n_1}(K)$, where $b \in K^*$. As observed above, for each b, $I + be_{ij}$ may be contained in only one coset $F^* u_i$. Therefore, there is a finite number of elements $I + be_{ij}$ which occur in $M \cong GL_{n_1}(K)$. Thus, there are at most $n^2 t$ elementary matrices in $GL_{n_1}(K)$. This means that $SL_{n_1}(K)$, which is generated by $I + be_{ij}$, is finitely generated. Now, by Corollary 1 of [10], we conclude that $SL_{n_1}(K) \subset K^*$. This in turn implies that $n_1 = 1$ and consequently $M = K^*$, i. e., K^*/F^* is finite. But it is known that this is not possible unless $M = K^*$ is finite. Now, by Theorem 4 of [1], that asserts that a normal subgroup of $GL_n(D)$ does not contain a finite maximal subgroup, we arrive at a contradiction and so the proof is completes.

In the next result we show how maximal subgroups of $GL_n(D)$ sit in $GL_n(D)$ with respect to F^* and $SL_n(D)$.

PROPOSITION 2. *Let D be a division ring not necessarily of finite dimension over its centre F. Assume that M is a maximal subgroup of $GL_n(D)$, $n \geq 1$. Then we have*
(i) *M contains F^* or $SL_n(D)$.*
(ii) *Either M is (absolutely) irreducible or M is the group of units of a proper subring of $M_n(D)$.*
(iii) *Assume that D is noncommutative and $[D : F] < \infty$. Then M is nilpotent if and only if M is the multiplicative group of a maximal subfield of $M_n(D)$.*

PROOF. (i) Assume that M does not contain F^*. Then we must have $GL_n(D) = F^* M$, and consequently $SL_n(D) = M` \subset M$.

(ii) Consider the F- algebra $F[M]$ generated by M over F. By maximality of M, we have either $GL_n(D) = F[M]^*$ or $M = F[M]^*$. In the first case we obtain $M_n(D) = F[M]$, and the second case implies that M is the group of units of a proper subring of $M_n(D)$.

(iii) One way is clear. So, assume that M is nilpotent. By (i), we have either $F^* \subset M$ or $SL_n(D) \subset M$. If the second case occurs, then $SL_n(D)$ is nilpotent. Thus, in particular, the derived group D' of D^* is nilpotent and so

D^* is soluble. Therefore, by Hua's Theorem (cf. [7, 223]), D is commutative which is a contradiction. So, we may assume that $F^* \subset M$ but $SL_n(D) \not\subset M$. The case $n = 1$ follows from Theorem 7 of [2]. Thus, we may suppose $n > 1$ and consider the F-algebra $A := F[M]$ which is left Artinian since $[D : F] < \infty$. We may also view M as a linear group since $[D : F] < \infty$. By (ii), we conclude that either M is (absolutely) irreducible or $A^* = M$. If the first case happens, i. e., $M_n(D) = A$, then it is clear that $Z(M) = F^*$ and M is an irreducible linear group (cf. [18, p. 100]). Now, it is known that if a linear group M is irreducible and nilpotent, then $[M : Z(M)] < \infty$ (cf. [17, p. 57]). But this is not possible by Lemma 1. Thus, we must have $A^* = F[M]^* = M$. If we prove that A is simple, then we obtain $M \cong GL_m(\Delta)$ for some positive integer m and division ring Δ. Consequently, by Hua's Theorem as above, we conclude that $m = 1$ and $\Delta = K$ is a field and therefore M is the multiplicative group of a field. Furthermore, we then have $C_{M_n(D)}(A) = A$ by maximality of M, and this shows that M is the multiplicative group of a maximal subfield of $M_n(D)$. So, it remains to prove that A is simple. To do this, we first observe that the Jacobson radical of A, $J(A) = J$, is nilpotent. As noted in the proof of Lemma 1, the elements of J may be assumed to be upper triangular. Now, let L be the subring of all elements $x \in M_n(D)$ such that $xJ \subset J$. It is clear that $A \subset L$. For any $d \in D^*$ we see that the matrix $D_n(d) = I + (d-1)e_{nn}$ belongs to L. If M contains all matrices $D_n(d)$ for $d \in D^*$, then M contains a copy of D^*. But this is not possible by Hua's Theorem unless D is commutative which is a contradiction. Therefore, by maximality of M, we obtain $L = M_n(D)$, i.e., J is a left ideal of $M_n(D)$. A similar argument applied to the subring of all elements x in $M_n(D)$ such that $Jx \subset J$, we conclude that J is a right ideal and thus $J = 0$. This means that J is semisimple. A similar argument as used in the proof of Lemma 1, one can show that $Z(A)$ is a field and thus A is simple and the result follows.

To prove our next result, we need the following theorems:

Theorem A.(Rosenberg, [14]) *If A is a simple ring with unit, the only subrings of $M_n(A)$, $n \geq 2$, invariant under all inner automorphisms of $M_n(A)$,* *are subrings of the centre or $M_n(A)$ itself.*

THEOREM B.(Snider, [16]) *Let G be an absolutely irreducible subgroup of $GL_n(D)$, let N be a normal subgroup of G and K a subring of $M_n(D)$ normalized by G with $F \subset K$. If G/N is locally finite, then $K[N]$ is semisimple Artinian.*

THEOREM C. *Let D be a division ring that is not a locally finite field and let $n > 1$ be an integer. If N is any non-central normal subgroup of $GL_n(D)$, then N contains a noncyclic free subgroup*(cf. [16, p.154]).

THEOREM D. *Every irreducible soluble subgroup of $GL_n(F)$ has an abelian normal subgroup of finite index*(cf. [18, p. 135]).

The next result gives a criterion for when a maximal subgroup M of $GL_n(D)$ is soluble, i. e., M is soluble if and only if there exists an abelian normal subgroup A such that M/A is finite and soluble.

THEOREM 3. *Let D be a noncommutative division algebra of finite dimension over its centre F. Suppose M is a maximal subgroup of $GL_n(D)$, $n \geq 1$. If M is soluble, then there exists a finite family $\{K_i\}_1^r$ of fields properly containing F with $K_i^* \subset M$, $1 \leq i \leq r$, such that M/A is finite, where $A = K_1^* \times \cdots \times K_r^*$, and so M is abelian-by-finite.*

PROOF. The case $n = 1$ follows from Corollary 4 of [12]. So, we may assume that $n \geq 2$. By Proposition 2, we know that either $F^* \subset M$ or $SL_n(D) \subset M$. If $SL_n(D) \subset M$, then $SL_n(D)$ is soluble. But, by Theorem C, we know that $SL_n(D)$ contains a noncyclic free subgroup which is a contradiction. So, we may assume that $F^* \subset M$ but $SL_n(D)$ is not contained in M. Now, consider the F-algebra $A := F[M]$. By Proposition 2, we have either $A = F[M] = M_n(D)$ which implies that M is (absolutely) irreducible or $A^* = M$. If the first case occurs, then since $[D : F] < \infty$ we conclude that M is an irreducible linear group (cf. [18, p. 100]). Therefore, by Theorem D, M contains an abelian normal subgroup B, say, of finite index. Now, consider $K := F[B]$. By Theorem B, we conclude that $F[B]$ is semisimple Artinian. If $F[B]^* \not\subset M$, then by maximality of M we have $GL_n(D) =< F[B]^*, M >\subset N_{GL_n(D)}(F[B]^*)$.

This implies, by Theorem A, that either $F[B] = M_n(D)$ or $F[B] \subset F$. The first case can not happen since $n > 1$ and the second case implies that $[M : F^*] < \infty$ which contradicts Lemma 1. Thus, we must have $F[B]^* \subset M$. Now, since $F[B]$ is commutative we have $K \cong K_1 \times \cdots \times K_r$ for some fields K_i. It is now clearly seen that M/K^* is finite and the result follows in this case.

Therefore, suppose $A^* = M$. Since $[D : F] < \infty$, A is left Artinian. We claim that A is semisimple. To see this, we first note that the Jacobson radical $J(A) = J$ of A is nilpotent. As observed in the proof of Lemma 1, we may assume that each element of J is upper triangular. Now, denote by L the subring of all elements $x \in M_n(D)$ such that $xJ \subset J$. It is clear that $A \subset L$. For any $d \in D^*$ we see that the dilatation matrix $D_n(d) = I + (d-1)e_{nn} \in L$. If M contains $D_n(d)$ for all $d \in D^*$, then M contains a copy of D^*. Now, by Hua's Theorem (cf.[7, p. 223]), this reduces to $D = F$ which is a contradiction. Therefore, there is an element $d \in D^*$ such that $D_n(d) \in L^* \setminus M$. By maximality of M, this implies that $L = M_n(D)$, i. e., J is a left ideal of $M_n(D)$. A similar argument applied to the subring R of all elements $x \in M_n(D)$ such that $Jx \subset J$, one concludes that J is also a right ideal of $M_n(D)$. Thus, J is two-sided and we have $J = 0$. Therefore, A is semisimple, i. e., there exist positive integers n_i such that $A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$ for some division rings D_i, $1 \leq i \leq r$. This means that $M = A^* \cong GL_{n_1}(D_1) \times \cdots \times GL_{n_r}(D_r)$. But this is not possible since M is soluble whereas $GL_{n_i}(D_i)$, by Theorem C, contains a noncyclic free subgroup unless $n_i = 1$ for all $1 \leq i \leq r$. Now, use Hua's Theorem to conclude that $D_i = K_i$ for some fields K_i. This implies that $M \cong K_1^* \times \cdots \times K_r^*$ and so the result follows.

In the following result we show that if $M \cap SL_n(D)$ is commutative, then M is abelian-by-finite. This is used later on to prove that if $M/M \cap F^*$ is torsion, then M is abelian-by-finite.

Lemma 4. *Let D be a noncommutative division algebra of finite dimension over its centre F and $n \geq 1$. Suppose M is a maximal subgroup of $GL_n(D)$. If $M \cap SL_n(D)$ is commutative, then there exists a finite family $\{K_i\}_1^r$ of fields* *properly containing F with $K_i^* \subset M$, $1 \leq i \leq r$ such that M/A is finite, where $A = K_1^* \times \cdots \times K_r^*$, and so M is abelian-by- finite.*

PROOF. Assume that $M \cap SL_n(D)$ is commutative. It is clear that $SL_n(D) \not\subset M$. We have $M' \subset M \cap SL_n(D)$ and so M is soluble. Now, by Theorem 3, the result follows.

THEOREM 5. *Let D be a noncommutative division algebra of finite dimension over its centre F. Assume that M is a maximal subgroup of $GL_n(D)$ with $n \geq 1$. If $M/M \cap F^*$ is torsion, then there exists a finite family $\{K_i\}_1^r$ of fields properly containing F with $K_i^* \subset M$, $1 \leq i \leq r$ such that M/A is finite, where $A = K_1^* \times \cdots \times K_r^*$, and so M is abelian-by-finite.*

PROOF. The case $n = 1$ follows from Theorem 6 of [12]. So, we may assume that $n \geq 2$. By Proposition 2, we know that either $F^* \subset M$ or $SL_n(D) \subset M$. If $SL_n(D) \subset M$, then $PSL_n(D) = SL_n(D)/Z(SL_n(D))$ is torsion. Thus, by Corollary 2 of [9], we obtain $D = F$ which is a contradiction. So, we may assume that $F^* \subset M$ but $SL_n(D) \not\subset M$. Consider the F-algebra $A := F[M]$. Since M is maximal in $GL_n(D)$ we conclude that either $A^* = M$ or $A^* = GL_n(D)$. We now deal with these cases separately:

CASE 1. If $A^* = GL_n(D)$, then we clearly have $A = F[M] = M_n(D)$ which means that M is an irreducible linear group since $[D : F] < \infty$ (cf. [18, p.100]). We now consider two subcases:

SUBCASE 1. Char $F = 0$. By Theorem 1 of [19], either M contains a noncyclic free subgroup or it is soluble-by-finite. The first case can not occur since M/F^* is torsion. Thus, there is a soluble normal subgroup S in M such that $[M : S] < \infty$. Now, consider the F-algebra $F[S]$. We know that $< F[S]^*, M > \subset N_{GL_n(D)}(F[S]^*)$.

If $F[S]^* \not\subset M$, then, by Theorem A, we conclude that either $F[S]^* \subset F^*$ or $F[S] = M_n(D)$. If $F[S]^* \subset F^*$, then S is central and so $[M : F^*] < \infty$. Thus, by Lemma 1, we obtain a contradiction. Now, if $F[S] = M_n(D)$, then S is an irreducible linear group (cf. [18, p. 100]). Therefore, by Theorem D, we conclude that S contains an abelian normal subgroup of finite index, consequently, M contains an abelian normal subgroup B, say, of finite index.

Put $K = F[B]$. Then, we have $F[B]^* \neq GL_n(D)$ since $n > 1$. Thus, $K^* = F[B]^* \subset M$. If $K^* \subset F^*$, then $[M : F^*] < \infty$ which contradicts Lemma 1. So, assume that $F^* \subset K^* \subset M$. By 1.2.5 of [16, p.11], we conclude that K is semiprime and since $[D : F] < \infty$ we conclude that K is semiprime Artinian. Therefore, by Theorem 10.24 of [7, p.173], K is semisimple. Thus, there exist fields K_i, $1 \leq i \leq r$, and positive integers n_i such that $K^* \cong GL_{n_1}(K_1) \times \cdots \times GL_{n_i}(K_i)$, where for each i, K_i contains a copy of F. Since $K^* \subset M$ and M/F^* is torsion we conclude that K_i is radical over F. By Kaplansky's Lemma (cf. [7]), we conclude that char $F = p > 0$ which is a contradiction.

So, we may suppose $F[S]^* \subset M$. Now, by 1.2.5 of [16, p.11] again, we conclude that $F[S]$ is semiprime and as above $F[S]$ is semisimple Artinian since $[D : F] < \infty$. Therefore, $F[S]^* \cong GL_{m_1}(D_1) \times \cdots \times GL_{m_t}(D_t)$ for some positive inegers m_i and division rings D_i. This case, via a theorem of Kaplansky (cf. [7]), also leads to a contradiction. This takes care of subcase 1.

Subcase 2. Char $F = p > 0$. Consider the group $G := SL_n(D) \cap M$ which is normal in M. If $G \subset F^*$, then by Lemma 4, the result follows. So, we may assume that G is not central. Now, take $x \in G$. We know that $x^{n(x)} = a \in F^*$. Taking the redaced norm RN of $M_n(D)$ to F from both sides of the last equation, we conclude that $1 = RN(x)^{n(x)} = a^m$, where $m = \sqrt{dim_F M_n(D)}$. This means that G is a torsion group. Since $[D : F] < \infty$, we conclude that G is a torsion linear group. Thus, by Schur's Theorem (cf.[7, p. 154]), G is locally finite. Now, consider the P-algebra $S = P[G]$, where P is the prime subfield of F. By 1.1.14 of [17, p. 9], we conclude that S is a semisimple Artinian ring. Therefore, $S = S_1 \times \cdots \times S_r$, where S_i is simple Artinian. Suppose that $r = 1$. Then S is a matrix ring, $S = M_t(K)$, say, for some locally finite field K. If $S^* \not\subset M$, then $GL_n(D) =< S^*, M >\subset N_{GL_n(D)}(S^*)$. This means, by Theorem A, that either $S^* \subset F$ or $GL_n(D) = S^*$. If $S^* \subset F$, then $G \subset F^*$ and so, by Lemma 4, the result follows. Otherwise, $S = M_t(K) \cong M_n(D)$, for some positive integer t and locally finite field K. This implies that D is algebraic over its prime subfield and so $D = F$, by a result of Jacobson (cf. [7]), which is a contradiction. Therefore, we may assume that $r > 1$ and set $T = \cap_1^r N_M(S_i)$. Consider the F-algebra $F[T] = B$. Then B is semiprime by 1.2.5 of [16, p.11], and it is also left Artinian since $[D : F] < \infty$. Thus, B is semisimple Arinian. The central idempotents of S are central in B, and so B is not simple. Therefore, we have in particular $B \neq M_n(D)$. Thus, $B \cong B_1 \times \cdots \times B_t$, where B_i is simple Artinian. Now, consider $B^* = GL_{n_1}(D_1) \times \cdots \times GL_{n_t}(D_t)$ for some positive integers n_i and division rings D_i. If $B^* \not\subset M$, then $GL_n(D) =< B^*, M > \subset N_{GL_n(D)}(B^*)$. This means, by Theorem A, that either $B = M_n(D)$ which is impossible or $B^* \subset F^*$ which is nonsense since B is not simple. Therefore, $F^* \subset B^* \subset M$. Since M/F^* is torsion, by Kaplansky's Theorem (cf. [7]), we conclude that $D_i = K_i$ is commutative for all i. If for some i, $n_i > 1$, then the matrix $D_1(k) = I + (k-1)e_{11}, k \in K_i$ is torsion modulo F^*. Thus, K_i is algebraic over the prime subfield and since K_i contains a copy of F we conclude that F is algebraic over the prime subfield, i. e., D is algebraic over P and so $D = F$ by Jacobson Theorem. This contradiction allows us to assume $n_i = 1$ for all i and $D_i = K_i$ for some fields, where K_i contains a copy of F for all i. Thus, B is commutative and this in turn implies that G is commutative. Now, by Lemma 4, the result follows. This establishes Case 1.

Case 2. Assume that $A^* = M$. Then $F[M]$ is left Artinian since $[D : F] < \infty$. We claim that $F[M]$ is simple. To prove this, we first observe that the Jacobson radical of A, $J(A) = J$, is nilpotent. As we observed in the proof of Lemma 1, the elements of J may be assumed to be upper triangular. Now, let L be the subring of all elements $x \in M_n(D)$ such that $xJ \subset J$. It is clear that $A \subset L$. For any $d \in D^*$ we see that the matrix $D_n(d)$ belongs to L. If M contains all $D_n(d)$ for $d \in D^*$, then D^*/F^* is torsion and by Kaplansky's Theorem, we conclude that $D = F$ which is a contradiction. Otherwise, by maximality of M, we obtain $L = M_n(D)$, i. e., J is a left ideal of $M_n(D)$. A similar argument applied to the subring of all elements x in $M_n(D)$ such that $Jx \subset J$, we conclude that J is a right ideal and thus $J = 0$. This means that A is semisimple. A similar argument as used in the proof of Lemma 1, one can show that $Z(A)$ is a field and thus A is simple, i. e., $A \cong M_r(\Delta)$, for some positive integer r and some division ring Δ. As in the Case 1, we conclude that Δ is commutative by Kaplansky's Theorem since M/F^* is torsion. So, $M \cong GL_r(K)$ for some field K. If $r > 1$, then $D_1(k) = I + (k-1)e_{11}$, $k \in K^*$ must be torsion modulo F^*. This implies, as before, that D is algebraic over its prime subfield and so $D = F$ which is a contradiction. Thus, we must have $r = 1$ and so $M \cong K^*$ is commutative and this completes the proof of the theorem.

We are now in a position to prove our main result as

THEOREM 6. *Let D be a noncommutative division algebra of finite dimension over its centre F. Assume that M is a maximal subgroup of $GL_n(D)$, $n \geq 1$. Then either M contains a noncyclic free subgroup or there exists a finite family $\{K_i\}_1^r$ of fields properly containing F with $K_i^* \subset M$ for all $1 \leq i \leq r$ such that M/A is finite if $Char F = 0$ and M/A is locally finite if $Char F = p > 0$, where $A = K_1^* \times \ldots \times K_r^*$.*

PROOF. If M is commutative, the result follows from (iii) of Proposition 2. Thus, we may suppose that M is noncommutative. Now, the case $n = 1$ follows from Theorem 8 of [12]. So, we may assume that $n > 1$. Suppose M is a noncommutative maximal subgroup of $GL_n(D)$. We know, by Proposition 2, that either $SL_n(D) \subset M$ or $F^* \subset M$. If $SL_n(D) \subset M$, then M contain a noncyclic free subgroup by Theorem C. Thus, we may assume that $F^* \subset M$ but $SL_n(D) \not\subset M$. Since $[D : F] < \infty$ we may view M as a linear group over F. Now, consider the F-algebra $F[M]$ generated by M over F. Since M is maximal we have either $F[M]^* = M$ or $F[M] = M_n(D)$, i.e., M is absolutely irreducible. We consider these cases separately and assume that M does not contain a noncyclic free subgroup.

CASE 1. Assume that M is absolutely irreducible. If Char $F = 0$ and M does not contain a noncyclic free subgroup we conclude, by Theorem 1 of [19], that M contains a soluble normal subgroup T of finite index, i.e., $[M : T] < \infty$. If $T \subset F^*$, then by Lemma 1 we obtain a contradiction. Now, by 1.1.7 of [16], T is completely reducible, and hence it contains an abelian normal subgroup A, say, of finite index. Set $K = F[A]$. Then, we have $K^* = F[A]^* \neq GL_n(D)$ since $n > 1$. If $K^* \subset F^*$, then $[M : F^*] < \infty$ and we arrive at a contradiction by Lemma 1. So, assume that $F^* \subset K^* \subset M$. Now, by 1.2.5 of [16, p. 11], we conclude that K is semiprime. Since we are in the finite dimensional case K is semiprime Artinian, i.e., K is semisimple. Therefore, there exist fields K_i, such that $K \cong K_1 \times K_2 \times \cdots \times K_r$ and we clearly have $[M : K^*] < \infty$ and so the result follows.

Now let Char $F = p > 0$. If M does not contain a noncyclic free subgroup, then every finitely generated subgroup of M does not contain a noncyclic free subgroup. By Tit's Theorem (cf. [19]), we conclude that every finitely generated subgroup of M contains a soluble normal subgroup of finite index. Therefore, by a result of Wehrfritz (cf. [20]), $M/R(M)$ is a torsion linear group, where $R(M)$ is the unique maximal soluble normal subgroup obtained by Zassenhaus-Maltsev Theorem (cf. [21]). Thus, by Schur's Theorem, $M/R(M)$ is locally finite. Set $S = R(M)$. If $S = F^*$, then M/F^* is torsion. Thus, by Theorem 5, the result follows. So, we may assume that $F^* \subset S \subset M$. Since S is completely reducible it has an abelian normal subgroup V of finite index. We can assume in fact that it is the unique maximal abelian normal subgroup and because of this it is normal in M. If $V \subset F^*$, as above we obtain a contradiction. Now, the quotient group M/V is locally finite. By a similar argument used above, we see that the linear envelope $F[V]$ is a direct sum of fields. This completes the proof of the case $F[M] = M_n(D)$.

Case 2. Assume that $F[M]^* = M$ and M does not contain a noncyclic free subgroup. We claim that $A := F[M]$ is simple Artinian. Since the proof is more or less similar to the proof of Theorem 5, we only give the outlines. We first observe that $J(A) = J$ is nilpotent and we may take the elements of J to be upper triangular. Denote by L the subring of all elements $x \in M_n(D)$ such that $xJ \subset J$. If for all $d \in D^*$ the matrices $D_n(d) = I + (d-1)e_{nn} \in M$, then M contains a copy of D^* and this contradicts the assumption that M does not contain a noncyclic free subgroup by Goncalves' Theorem as above. Thus, there is an element $d \in D^*$ such that $D_n(d) \in L^* \setminus M$. By maximality of M, this implies that J is a left ideal of A. A similar argument, as used above, shows that J is also a right ideal and so $J = 0$. Therefore, A is semisimple Artinian since $[D : F] < \infty$. One may easily check as in the proof of Lemma 1 that $Z(A)$ is a field and so $A \cong M_r(\Delta)$ for some division ring Δ and positive integer r. Since, by our assumption $M = A^* \cong GL_r(\Delta)$ does not contain a noncyclic free subgroup we obtain $r = 1$ and $\Delta = K$ for some field K. Thus, $M = K^*$ which contradicts our assumption that M is noncommutative and so the proof is complete.

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