KOSZUL COMPLEXES AND SYMMETRIC FORMS
OVER THE PUNCTURED AFFINE SPACE

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ABSTRACT. Let X be a scheme which is not of equicharacteristic 2 and let $\mathbb{U}_X^n \subset \mathbb{A}_X^n$ be the punctured affine n-space over X. If $n \equiv \pm 1$ modulo 4, we show that there exists a ± 1-symmetric bilinear space $(\mathcal{E}_X^{(n)}, \varphi_X^{(n)})$ over $\mathbb{U}_X^n$ which can not be extended to the whole affine space $\mathbb{A}_X^n$ and which is locally metabolic for $n \geq 2$. If X is regular, contains $\frac{1}{2}$ and is of finite Krull dimension, we show that the total Witt ring $\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n)$ of $\mathbb{U}_X^n$ is a free $\mathrm{W}^{\mathrm{tot}}(X)$-module with two generators: the Witt classes of $<1>$ and of the above $(\mathcal{E}_X^{(n)}, \varphi_X^{(n)})$. We describe $\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n)$ similarly when n is even.

INTRODUCTION

Let X be scheme. We are studying the *(total) graded Witt ring*

$$\mathrm{W}^{\mathrm{tot}}(X) := \bigoplus_{i \in \mathbb{Z}} \mathrm{W}^i(X)$$

where the groups W^i are the derived Witt groups of Balmer [2, 3] and where the multiplicative structure is the one of Gille-Nenashev [10]. See more in Section 2.

We fix an integer $n \geq 1$ for the entire article. Consider the following open subset $\mathbb{U}_{\mathbb{Z}}^n \subset \mathbb{A}_{\mathbb{Z}}^n$ of the affine space $\mathbb{A}_{\mathbb{Z}}^n = \mathrm{Spec}(\mathbb{Z}[T_1, \ldots, T_n])$:

$$\mathbb{U}_{\mathbb{Z}}^n := \bigcup_{j=1}^n \mathrm{Spec}\left(\mathbb{Z}[T_1, \ldots, T_n, T_j^{-1}]\right) \subset \mathbb{A}_{\mathbb{Z}}^n.$$

For any scheme X, define by base-change the open subscheme $\mathbb{U}_X^n \subset \mathbb{A}_X^n$, called *the punctured affine space over X*, i.e. define $\mathbb{U}_X^n$ by the following pull-back square:

$$
\begin{array}{ccc}
\mathbb{U}_X^n & \xrightarrow{\sigma_X} & X \\
{\scriptstyle v_X}\downarrow & & \downarrow \\
\mathbb{U}_{\mathbb{Z}}^n & \longrightarrow & \mathrm{Spec}(\mathbb{Z}).
\end{array}
\tag{1}
$$

Our main result is Theorem 7.14 below, which says in particular:

Theorem. *If the scheme X is regular, contains $\frac{1}{2}$ and has finite Krull dimension, there is a decomposition $\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n) = \mathrm{W}^{\mathrm{tot}}(X) \oplus \mathrm{W}^{\mathrm{tot}}(X) \cdot \varepsilon$ for some Witt class $\varepsilon = \varepsilon_X^{(n)}$ in $\mathrm{W}^{n-1}(\mathbb{U}_X^n)$. If n=1, we have $\varepsilon^2 = 1$. If $n \geq 2$, we have $\varepsilon^2 = 0$ and an isomorphism*

$$\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n) \cong \frac{\mathrm{W}^{\mathrm{tot}}(X)[\varepsilon]}{\varepsilon^2}$$

of graded rings, with the generator ε in degree $n-1$.

Our second goal is a "classical" description of the generator $\varepsilon_X^{(n)} \in \mathrm{W}^{n-1}(\mathbb{U}_X^n)$.

Recall a few facts. First, the derived Witt groups are 4-periodic: $\mathrm{W}^i = \mathrm{W}^{i+4}$. Secondly, W^0 and W^2 are naturally isomorphic to the usual Witt groups $\mathrm{W}_{\mathrm{us}}^+$ and $\mathrm{W}_{\mathrm{us}}^-$ of symmetric and skew-symmetric vector bundles respectively, as defined by Knebusch [12]. Thirdly, W^1 and $\mathrm{W}^3 = \mathrm{W}^{-1}$ are groups of formations, see Walter [15]. Therefore, if we want to describe in classical terms our generator $\varepsilon_X^{(n)}$ in W^{n-1}, we are bound to produce an explicit element of the above nature, *i.e.* a ± 1-symmetric form or formation, depending on the congruence of n modulo 4.

In this introduction, we focus on the case where n is odd and we write $n-1 = 2\ell$. In this case, we have to describe a $(-1)^\ell$- symmetric bundle

$$\left(\mathcal{E}_X^{(n)} , \varphi_X^{(n)} \right)$$

over $\mathbb{U}_X^n$ whose class in $\mathrm{W}_{\mathrm{us}}^\pm$ will be our generator $\varepsilon_X^{(n)}$. Let us stress that this will be a (skew-)symmetric space of classical nature, which does not involve triangulated categories. By the very naturality of the original problem, it suffices to construct this (skew-)symmetric bundle when $X = \mathrm{Spec}(\mathbb{Z})$ and then to pull it back over an arbitrary scheme X. Therefore, we start with a description of $(\mathcal{E}_\mathbb{Z}^{(n)}, \varphi_\mathbb{Z}^{(n)})$.

Let us denote by $A := \mathbb{Z}[T_1, \dots, T_n]$ the polynomial ring in n variables and by

$$K_\bullet = K_\bullet(A, \underline{T})$$

the (homological) Koszul complex over A for the A-sequence $\underline{T} := (T_1, \dots, T_n)$. There is a well-known isomorphism of complexes $\Theta_\bullet : K_\bullet \xrightarrow{\simeq} \mathrm{Hom}_A(K_\bullet, A)[n]$, see [7, Chap. 1.6] for instance. Since $K_\bullet|_{\mathbb{U}_\mathbb{Z}^n}$ is locally split, the $\mathcal{O}_{\mathbb{U}_\mathbb{Z}^n}$-module

$$\mathcal{E}_\mathbb{Z}^{(n)} := \mathrm{Coker}\left(K_{\ell+2} \xrightarrow{d_{\ell+2}} K_{\ell+1} \right)\big|_{\mathbb{U}_\mathbb{Z}^n} \simeq \mathrm{Ker}\, d_\ell|_{\mathbb{U}_\mathbb{Z}^n}$$

is locally free. From $n = 2\ell+1$, one easily sees that $\Theta_\ell \circ d_{\ell+1}$ induces an isomorphism

$$\varphi_\mathbb{Z}^{(n)} : \mathcal{E}_\mathbb{Z}^{(n)} \xrightarrow{\simeq} \mathcal{H}om_{\mathcal{O}_{\mathbb{U}_\mathbb{Z}^n}}(\mathcal{E}_\mathbb{Z}^{(n)}, \mathcal{O}_{\mathbb{U}_\mathbb{Z}^n})$$

which is $(-1)^\ell$-symmetric. For a scheme X, with the base-change morphism $\upsilon_X : \mathbb{U}_X^n \longrightarrow \mathbb{U}_\mathbb{Z}^n$ as in diagram (1), we define $(\mathcal{E}_X^{(n)}, \varphi_X^{(n)}) := \upsilon_X^*(\mathcal{E}_\mathbb{Z}^{(n)}, \varphi_\mathbb{Z}^{(n)})$.

If n is even, we can construct similarly a complex of length 1, with a suitable (skew-)symmetric form, *i.e.* a formation, whose class in the Witt group $\mathrm{W}^{n-1}(X)$ is our wanted $\varepsilon_X^{(n)}$. This complex is also obtained by chopping off some parts of the above Koszul complex.

Putting things together, if we define the integer $-1 \le r \le 2$ by the equation $n-1 \equiv r \bmod 4$, we produce "short symmetric r-spaces" $(\mathcal{F}_X^{(n)}, \phi_X^{(n)})$ (if n is odd this complex is concentrated in one degree and corresponds to $(\mathcal{E}_X^{(n)}, \varphi_X^{(n)})$ above via the natural embedding $\mathrm{VB}_X \hookrightarrow \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$), having the following properties (Theorems 7.13 and 8.2):

Theorem. *Let X be a scheme, not of equicharacteristic 2 (in particular $2 \ne 0$).*

(i) *The symmetric r-space $(\mathcal{F}_X^{(n)}, \phi_X^{(n)})$ can not be extended to $\mathbb{A}_X^n$, i.e. there does not exist a symmetric r-space (P, ϕ) over $\mathbb{A}_X^n$ whose restriction $(P, \phi)\big|_{\mathbb{U}_X^n}$ is isometric (nor Witt equivalent) to $(\mathcal{F}_X^{(n)}, \phi_X^{(n)})$. In particular, $(\mathcal{F}_X^{(n)}, \phi_X^{(n)})$ is not extended from X either. Hence, if n is odd, the same is true for the classical (skew-)symmetric space $(\mathcal{E}_X^{(n)}, \varphi_X^{(n)})$.*

(ii) *Assume moreover that 2 is invertible in X and that $n \geq 2$. Then the space $(\mathcal{F}_X^{(n)}, \phi_X^{(n)})$ is locally hyperbolic, i.e. for any $x \in \mathbb{U}_X^n$ we have $[(\mathcal{F}_X^{(n)}, \phi_X^{(n)})_x] = 0$ in $\mathrm{W}^r(\mathcal{O}_{\mathbb{U}_X^n, x})$, and moreover its square is Witt trivial, i.e. $[\mathcal{F}_X^{(n)}, \phi_X^{(n)}]^2$ is zero in $\mathrm{W}^{2r}(\mathbb{U}_X^n)$.*

This theorem says that the spaces $(\mathcal{F}_X^{(n)}, \phi_X^{(n)})$ are quite specific to $\mathbb{U}_X^n$. They can not be extended to $\mathbb{A}_X^n$, not even up to Witt equivalence. In particular, these spaces are not metabolic on $\mathbb{U}_X^n$. On the other hand, they do become metabolic as soon as we localize them to some principal open given by $T_i \neq 0$, see 7.11.

There are two appendices. In the first one, for the sake of completeness, we show that when $n \geq 3$ our locally free $\mathcal{O}_{\mathbb{U}_X^n}$-module $\mathcal{E}_X^{(n)}$ can not be extended to a locally free $\mathcal{O}_{\mathbb{A}_X^n}$-module and in particular $\mathcal{E}_X^{(n)}$ is not free. The second appendix contains the compatibility between product and 4-periodicity, a fact which we use several times in this work.

Contents

1. Conventions and notations

We collect here the notations which are kept unchanged in all sections.

First of all, recall that we have fixed an integer $n \geq 1$. We decompose it as

$$n = 4\,q + r + 1 \tag{2}$$

where $q \in \mathbb{N}$ and $r \in \{-1, 0, 1, 2\}$. Note that $n - 1 \equiv r \bmod 4$. We also baptize

$$\left[\frac{n}{2}\right] =: \ell. \tag{3}$$

Convention 1.1. Unless mentioned, a *ring* means a commutative ring with unit.

Convention 1.2. As always, when using a notation defined for schemes X in the affine case, $X = \mathrm{Spec}(R)$, we shall drop "Spec" as for instance: VB_R, $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_R)$, $\mathrm{W}^i(R)$ instead of $\mathrm{VB}_{\mathrm{Spec}(R)}$, $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_{\mathrm{Spec}(R)})$, $\mathrm{W}^i(\mathrm{Spec}(R))$, and so on. See 2.11.

Convention 1.3. We shall say that a scheme is *regular* if it is noetherian and separated and if all its local rings are regular.

Notation 1.4. Let X be a scheme. We denote by $\mathbb{A}_X^n$ the affine n-space and by $\mathbb{U}_X^n$ the punctured affine n-space over X. The obvious structure morphisms and base-change morphisms will be denoted as follows:

$$
\begin{array}{ccccc}
& & \sigma_Y & & \\
\mathbb{U}_Y^n & \xrightarrow{\ \iota_Y\ } & \mathbb{A}_Y^n & \xrightarrow{\ \pi_Y\ } & Y \\
\downarrow{\scriptstyle v_f} & & \downarrow{\scriptstyle \alpha_f} & & \downarrow{\scriptstyle f} \\
\mathbb{U}_X^n & \xrightarrow{\ \iota_X\ } & \mathbb{A}_X^n & \xrightarrow{\ \pi_X\ } & X \\
\downarrow{\scriptstyle v_X} & & \downarrow{\scriptstyle \alpha_X} & & \downarrow \\
\mathbb{U}_{\mathbb{Z}}^n & \xrightarrow{\ \iota_{\mathbb{Z}}\ } & \mathbb{A}_{\mathbb{Z}}^n & \xrightarrow{\ \pi_{\mathbb{Z}}\ } & \mathrm{Spec}(\mathbb{Z})
\end{array}
\tag{4}
$$

for any morphism of schemes $f : Y \to X$.

2. Recalling derived Witt groups

This section is a quick course on triangular Witt groups over schemes, included only for the reader's convenience. Here, X is a scheme with structure bundle $\mathcal{O}_X$.

2.1. *Categories and dualities.*

We denote by the symbol VB_X the exact category of locally free $\mathcal{O}_X$-modules of finite rank, *i.e. vector bundles*. The usual duality on VB_X is abbreviated

$$(-)^\vee := \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X).$$

$\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ stands for the bounded derived category of VB_X. We use homological notations for complexes. The translation functor $\Sigma : \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X) \longrightarrow \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$, also written $P_\bullet \longmapsto P_\bullet[1]$, is given by $(P_\bullet[1])_j = P_{j-1}$; as usual, Σ changes the sign of all differentials: $d_j^{P[1]} = -d_{j-1}^P$.

Let $P_\bullet = (P_\bullet, d_\bullet^P)$ be a complex in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$. Its *dual* $\mathcal{D}_X(P_\bullet)$ is the complex

$$
\mathcal{D}_X(P_\bullet) := \qquad \cdots \longrightarrow P_{-j}{}^\vee \xrightarrow{\ d_{-j+1}^P{}^\vee\ } P_{-(j-1)}{}^\vee \longrightarrow \cdots
$$
$$
\qquad\qquad\qquad\quad \deg j \qquad\qquad\qquad \deg (j-1)
$$

and similarly for morphisms of complexes. In other words, $\mathcal{D}_X$ is the derived functor of $(-)^\vee = \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$. This defines a duality on $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ turning it into a triangulated category with duality in the sense of [2]. The isomorphism between the identity and the double dual, $\varpi : \mathrm{id}_{\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)} \xrightarrow{\simeq} \mathcal{D}_X \mathcal{D}_X$, is given in each degree j by the canonical (evaluation) isomorphism $\mathrm{can}_{P_j} : P_j \longrightarrow P_j{}^{\vee\vee}$. We consider VB_X

as a subcategory of $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ via the natural embedding $c_0 : \mathrm{VB}_X \longrightarrow \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$, which maps a vector bundle $P \in \mathrm{VB}_X$ to the complex

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$$\deg 0$$

and does the same on morphisms. The restriction of the duality $\mathcal{D}_X$ to this subcategory is the original duality of VB_X and the restriction of ϖ is the above can .

Definition 2.2. Let $P_\bullet$ be a complex in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$. Let $i \in \mathbb{Z}$, and $\phi : P_\bullet \longrightarrow \mathcal{D}_X(P_\bullet)[i]$ be a morphism in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$. We say that ϕ is an *symmetric i-form* on the complex $P_\bullet$ if

$$\mathcal{D}_X(\phi)[i] \cdot \varpi_{P_\bullet} = (-1)^{\frac{i(i+1)}{2}} \phi .$$

We then say that $(P_\bullet, \phi)$ is an *symmetric i-pair*. If ϕ is moreover an isomorphism we say that $(P_\bullet, \phi)$ is a symmetric *i-space* over X. Two symmetric i-pairs $(P_\bullet, \phi)$ and $(Q_\bullet, \psi)$ are called *isometric* if there exists in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ an *isometry* between them, that is, an isomorphism $h : P_\bullet \xrightarrow{\cong} Q_\bullet$ such that $\phi = \mathcal{D}_X(h)[i] \cdot \psi \cdot h$.

Remark 2.3. Note that if $(P_\bullet, \phi)$ is a symmetric i-pair then $(P_\bullet[2], \phi[2])$ is a symmetric $(i+4)$-pair because $\mathcal{D}_X(P_\bullet)[1] = \mathcal{D}_X(P_\bullet[-1])$ for all $P_\bullet \in \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$.

Let $f : Y \longrightarrow X$ be a morphism of schemes. There is a natural isomorphism of functors $\eta_f : f^* \mathcal{D}_X \xrightarrow{\cong} \mathcal{D}_Y f^*$ which is induced by the natural isomorphism of locally free $\mathcal{O}_Y$-modules $f^* \mathcal{H}om_{\mathcal{O}_X}(P, \mathcal{O}_X) \xrightarrow{\cong} \mathcal{H}om_{\mathcal{O}_Y}(f^*P, \mathcal{O}_Y)$. If now $(P_\bullet, \phi)$ is a symmetric i-space over X then the isomorphism

$$f^*(P_\bullet) \xrightarrow{f^*\phi} f^*(\mathcal{D}_X(P_\bullet)[i]) = f^*(\mathcal{D}_X(P_\bullet))[i] \xrightarrow{\eta_{f,P}[i]} \mathcal{D}_Y(f^*P_\bullet)[i]$$

is a symmetric i-form and so $f^*(P_\bullet, \phi) := (f^*(P_\bullet), \ \eta_{f,P}[i] \cdot f^*(\phi))$ a symmetric i-space over Y.

2.4. *"Short" i-forms : Forms and formations.*

We present examples of symmetric i-pairs $(P_\bullet, \phi)$ in four cases $i = -1, 0, 1, 2$.

In each case, the complexes $P_\bullet$ and $P_\bullet^\vee$ are depicted horizontally and the symmetric i-form $\phi : P_\bullet \longrightarrow \mathcal{D}_X(P_\bullet)[i]$ vertically. The symmetric pairs of the left-hand column

are classical symmetric and skew-symmetric forms embedded in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ via the functor c_0 (slightly pushed to the left for $i = 2$). These symmetric i-pairs are i-*spaces* exactly when ϕ_0 and ϕ_1 is an isomorphism. The symmetric i-pairs of the right-hand column are *i-spaces* when ϕ is a quasi-isomorphism, *i.e.* when its cone is an exact complex; these are *formations*; we call them symmetric if $i = -1$ and skew-symmetric if $i = 1$. The four types of i-form presented above will be called *short*, for the obvious reasons.

2.5. *Product of symmetric spaces.*

The precise definition of this product is given in [10], where the reader will also find an explanation for the existence of *two* different products – the left and the right one – which differ by signs. To fix the ideas, *we will use here the left product*. Let $(P_\bullet, \phi)$ be a symmetric i-form and $(Q_\bullet, \psi)$ a symmetric j-form. The product

$$(P_\bullet, \phi) \star (Q_\bullet, \psi)$$

is then a symmetric $(i + j)$-form on the tensor product (of complexes) $P_\bullet \otimes_{\mathcal{O}_X} Q_\bullet$ and we denote it by $(P_\bullet \otimes_{\mathcal{O}_X} Q_\bullet , \phi \star \psi)$. Up to signs and identifications like for instance $P_\bullet \otimes_{\mathcal{O}_X} (Q_\bullet^\vee[j]) \simeq (P_\bullet \otimes_{\mathcal{O}_X} Q_\bullet^\vee)[j]$, the morphism of complexes $\phi \star \psi$ is equal to the tensor product $\phi \otimes \psi$. Via c_0, this product coincides on short 0-spaces with the usual tensor product of symmetric spaces as defined in Knebusch [12].

2.6. *Symmetric cones.*

We now recall the important *cone construction*. Let $\phi : P_\bullet \longrightarrow \mathcal{D}_X(P_\bullet)[i]$ be a symmetric i-form (maybe not an isomorphism). Let $Q_\bullet$ be the mapping cone of ϕ. Then, there exists an isomorphism ψ such that the following diagram commutes:

$$
\begin{array}{ccccccc}
P_\bullet & \xrightarrow{\ \phi\ } & \mathcal{D}_X(P_\bullet)[i] & \xrightarrow{\ u\ } & Q_\bullet & \xrightarrow{\qquad v \qquad} & P_\bullet[1] \\
\cong \downarrow {\scriptstyle (-1)^{\frac{i(i+1)}{2}} \cdot \varpi_P} & & \downarrow {\scriptstyle =} & & \psi \downarrow {\scriptstyle \simeq} & {\scriptstyle (-1)^{\frac{i(i+1)}{2}} \cdot \varpi_P[1]} \downarrow {\scriptstyle \cong} & \\
\mathcal{D}_X\mathcal{D}_X(P_\bullet) & \xrightarrow[\mathcal{D}_X(\phi)[i]]{} & \mathcal{D}_X(P_\bullet)[i] & \xrightarrow[-\mathcal{D}_X(v)[i+1]]{} & \mathcal{D}_X(Q_\bullet)[i+1] & \xrightarrow[(-1)^i \mathcal{D}_X(u)[i+1]]{} & \mathcal{D}_X\mathcal{D}_X(P_\bullet)[1]
\end{array}
$$

If the isomorphism ψ is moreover a *symmetric $(i + 1)$-form*, we call such a diagram *a cone diagram* (over ϕ) and we say that $(Q_\bullet, \psi)$ is *a symmetric cone* of the pair $(P_\bullet, \phi)$.

Note that both rows of the diagram are exact triangles in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$: The upper one by definition and the lower one is the dual of the upper row, shifted i times. Assume for a moment that 2 is invertible over our scheme X. Then we can always choose the isomorphism ψ to be a symmetric $(i + 1)$-form, see [2]. Moreover, if $(Q_\bullet', \psi')$ is another symmetric cone of ϕ, then there exists an isometry $(Q_\bullet, \psi) \simeq (Q_\bullet', \psi')$. We say then that $(Q_\bullet, \psi)$ is *the* symmetric cone of ϕ, in symbols:

$$(Q_\bullet, \psi) = \operatorname{cone} \phi = \operatorname{cone}(P_\bullet, \phi).$$

2.7. *Witt groups.*

The *usual Witt group* of symmetric (respectively skew-symmetric) spaces $\mathrm{W}_{\mathrm{us}}(X)$ (respectively $\mathrm{W}_{\mathrm{us}}^-(X)$) classifies these spaces up to isometry and modulo metabolic ones. More information about these Witt groups can be found in the fundamental paper of Knebusch [12]. The i-th *derived Witt group* $\mathrm{W}^i(X)$ classifies symmetric

i-spaces up to isometry and modulo neutral spaces, *i.e.* spaces with Lagrangian (*cf.* [2], Sect. 2). In fact, a symmetric i-space is neutral exactly if it is a symmetric cone of some symmetric $(i-1)$-form, as described above. Observe that this definition does not require 2 to be invertible in X. We denote by $[P_\bullet, \phi]$ the Witt class of $(P_\bullet, \phi)$.

The Witt groups are contravariant functors. If $f : Y \longrightarrow X$ is a morphism of schemes then the assignment $[P_\bullet, \phi] \longmapsto [f^*(P_\bullet, \phi)]$ defines a homomorphism $f^* : \mathrm{W}^i(X) \longrightarrow \mathrm{W}^i(Y)$ for all $i \in \mathbb{Z}$.

2.8. *Periodicity.*

The derived Witt groups are 4-periodic. The shift by two: $P_\bullet \longmapsto P_\bullet[2]$ induces an isomorphism $\tau : \mathrm{W}^i(X) \xrightarrow{\simeq} \mathrm{W}^{i+4}(X)$ for all $i \in \mathbb{Z}$ and all schemes X. The same periodicity applies to the Witt groups with support defined below.

2.9. *Agreement and localization* (*with* $\frac{1}{2}$).

We assume now that "X contains $\frac{1}{2}$", *i.e.* that X is a $\mathbb{Z}[1/2]$-scheme, *i.e.* 2 is invertible in the ring $\Gamma(X, \mathcal{O}_X)$. The main result of [3] is that the functor $c_0 : \mathrm{VB}_X \longrightarrow \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ induces isomorphisms:

$$\mathrm{W}(X) = \mathrm{W}_{\mathrm{us}}(X) \xrightarrow{\simeq} \mathrm{W}^0(X) \qquad [P, \phi] \longmapsto [c_0(P), c_0(\phi)]$$

and

$$\mathrm{W}^-(X) = \mathrm{W}_{\mathrm{us}}^-(X) \xrightarrow{\simeq} \mathrm{W}^2(X) \qquad [Q, \psi] \longmapsto [c_0(Q)[1], c_0(\psi)[1]].$$

Other Witt groups appearing in this work are the *Witt groups with support*. For a complex $P_\bullet \in \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ let

$$\operatorname{supp} P_\bullet := \{x \in X \mid H_j(P_\bullet)_x \neq 0 \quad \text{for at least one } j\},$$

be its (homological) support. Let Z be a closed subscheme of X with open complement U. The full triangulated subcategory of $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ which consists of complexes with support contained in Z is denoted $\mathbb{D}_Z^{\mathrm{b}}(\mathrm{VB}_X)$. The restriction of the duality $\mathcal{D}_X$ to $\mathbb{D}_Z^{\mathrm{b}}(\mathrm{VB}_X)$ is again a duality, turning $\mathbb{D}_Z^{\mathrm{b}}(\mathrm{VB}_X)$ into a triangulated category with duality. The corresponding triangular Witt groups $\mathrm{W}_Z^i(X)$ ($i \in \mathbb{Z}$) are called the derived Witt groups of X with support in Z. They appear in the localization sequence of Balmer [2]. If X is a regular scheme then there is an exact sequence

$$\cdots \longrightarrow \mathrm{W}^i(X) \longrightarrow \mathrm{W}^i(U) \xrightarrow{\partial} \mathrm{W}_Z^{i+1}(X) \longrightarrow \mathrm{W}^{i+1}(X) \longrightarrow \cdots.$$

The connecting morphism ∂ comes from the cone construction 2.6 as follows. Let $w \in \mathrm{W}^i(U)$. Then $\partial(w) = [\operatorname{cone}(P_\bullet, \phi)]$, where $(P_\bullet, \phi)$ is any symmetric i-pair over X with $[(P_\bullet, \phi)|_U] = w$ (the existence of $(P_\bullet, \phi)$ is guaranteed by regularity of X). The Witt groups with support are natural and so is the localization sequence.

2.10. *The graded Witt ring.*

The (left) product of symmetric spaces of 2.5 yields a product structure

$$\star : \mathrm{W}^i(X) \times \mathrm{W}_Z^j(X) \longrightarrow \mathrm{W}_Z^{i+j}(X) \quad ([P_\bullet, \phi], [Q_\bullet, \psi]) \longmapsto [(P_\bullet, \phi) \star (Q_\bullet, \psi)]$$

for any $i, j \in \mathbb{Z}$, any scheme X and closed subset $Z \subseteq X$. Via this pairing, $\mathrm{W}^{\mathrm{tot}}(X) := \bigoplus_{i \in \mathbb{Z}} \mathrm{W}^i(X)$ is a graded skew-commutative associative $\mathrm{W}^0(X)$-algebra, the *graded Witt ring* of X and $\mathrm{W}_Z^{\mathrm{tot}}(X) := \bigoplus_{i \in \mathbb{Z}} \mathrm{W}_Z^i(X)$ is a graded $\mathrm{W}^{\mathrm{tot}}(X)$-module.

This product is compatible with the connecting homomorphism ∂ in the localization sequence by [10, Thm. 2.11], *i.e.* we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{W}^i(X) \times \mathrm{W}^j(U) & \xrightarrow{\;\;\star\;\;} & \mathrm{W}^{i+j}(U) \\
{\scriptstyle \mathrm{id} \times \partial} \downarrow & & \downarrow {\scriptstyle \partial} \\
\mathrm{W}^i(X) \times \mathrm{W}_Z^{j+1}(X) & \xrightarrow{\;\;\star\;\;} & \mathrm{W}_Z^{i+j+1}(X) \,.
\end{array}
\tag{5}
$$

In words, the connecting homomorphism is a left $\mathrm{W}^{\mathrm{tot}}(X)$-linear map. But note that it is not right $\mathrm{W}^{\mathrm{tot}}(X)$-linear:

$$
\partial(y \star x) = (-1)^{ij}\partial(x \star y) \overset{(5)}{=} (-1)^{ij} x \star \partial(y) = (-1)^i \partial(y) \star x
\tag{6}
$$

(the first and last equation by skew-commutativity), where $x \in \mathrm{W}^i(X)$ and $y \in \mathrm{W}^j(U)$.

Remark 2.11. Of course, Convention 1.2 applies here as well. For instance, if $X = \mathrm{Spec}(R)$ is affine and $Z \subset X$ is defined by the ideal I we might say that a complex "has support in the ideal I" and we shall write $\mathrm{W}_I^i(R)$ instead of $\mathrm{W}_Z^i(X)$.

3. Basic facts about Koszul complexes

In this section, A is a ring and $\underline{T} = (T_1, \ldots, T_n)$ is any sequence in A. As before, we write the dual as $M^\vee := \mathrm{Hom}_A(M, A)$, for any A-module M.

We first recall the definition of the Koszul complex

$$
K_\bullet(A, \underline{T}) =: (K_\bullet, d_\bullet).
$$

Let $e_1, e_2, \ldots, e_n$ be a basis of the free A-module $A^n = \bigoplus_{i=1}^n A \cdot e_i$. The A-module $K_i = K_i(A, \underline{T}) := \bigwedge^i A^n$ is by definition the i-th exterior power of A^n. As is well-known, the module K_i is free with basis $\{ e_{j_1} \wedge \cdots \wedge e_{j_i} \mid 1 \leq j_1 < \ldots < j_i \leq n \}$. The differential $d_i = d_i(A, \underline{T}) : K_i \longrightarrow K_{i-1}$ is given by

$$
e_{j_1} \wedge \ldots \wedge e_{j_i} \longmapsto \sum_{k=1}^i (-1)^{k-1} T_{j_k} \cdot e_{j_1} \wedge \ldots \widehat{e_{j_k}} \ldots \wedge e_{j_i} \,,
$$

where the symbol $\widehat{e_{j_k}}$ indicates that e_{j_k} has been omitted. We consider this (homological) *Koszul complex* $K_\bullet(A, \underline{T})$:

$$
\cdots 0 \longrightarrow K_n(A, \underline{T}) \xrightarrow{d_n(A, \underline{T})} K_{n-1}(A, \underline{T}) \longrightarrow \cdots \xrightarrow{d_1(A, \underline{T})} K_0(A, \underline{T}) \longrightarrow 0 \cdots
$$

as an element of $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$ with $K_j(A, \underline{T})$ in degree j.

There is a structure of symmetric n-space on $K_\bullet(A, \underline{T})$, that we now give in an economic way; see more details in Remark 3.3. For each $i = 1, \ldots, n$, let $K_\bullet(A, T_i) \in \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$ be the short Koszul complex for the one-element sequence (T_i), *i.e.*

$$
K_\bullet(A, T_i) = \qquad \cdots \longrightarrow 0 \longrightarrow A \xrightarrow{\;\cdot T_i\;} A \longrightarrow 0 \longrightarrow \cdots
$$

$$
\qquad\qquad\qquad\qquad\qquad \deg 1 \qquad\quad \deg 0 \,.
$$

This complex can be equipped with the following symmetric 1-form (see 2.2):

$$
\begin{array}{ccccccccc}
K_\bullet(A,T_i) = & & \cdots\, 0 & \longrightarrow & A & \xrightarrow{\;\cdot T_i\;} & A & \longrightarrow & 0 \cdots \\[2pt]
\Theta(A,T_i)\Big\downarrow := & & & & \text{id}\Big\downarrow & & \Big\downarrow -\,\text{id} & & \\[2pt]
\mathcal{D}_A(K_\bullet(A,T_i))[1] = & & \cdots\, 0 & \longrightarrow & A & \xrightarrow{\;\cdot(-T_i)\;} & A & \longrightarrow & 0 \cdots \\
\end{array}
$$

$$\deg 1 \qquad\quad \deg 0\,,$$

where we identify $A = \mathrm{Hom}_A(A,A)$ as usual. This is of course a symmetric 1-space since $\Theta(A,T_i)$ is an isomorphism of complexes, hence a quasi-isomorphism. It is easily checked that the tensor product of complexes $K_\bullet(A,T_1)\otimes_A \ldots \otimes_A K_\bullet(A,T_n)$ is equal to the Koszul complex $K_\bullet(A,\underline{T})$ of the sequence $\underline{T} = (T_1,\ldots,T_n)$ and therefore we can give the following:

Definition 3.1. With the above notations, we define a symmetric n-form

$$\Theta(A,\underline{T})\,:\,K_\bullet(A,\underline{T}) \longrightarrow \mathcal{D}_A(K_\bullet(A,\underline{T}))[n]$$

as the product (see 2.5)

$$\big(K_\bullet(A,\underline{T})\,,\,\Theta(A,\underline{T})\big) \,:=\, \big(K_\bullet(A,T_1),\Theta(A,T_1)\big) \star \ldots \star \big(K_\bullet(A,T_n),\Theta(A,T_n)\big)\,.$$

This defines a symmetric n-space $\big(K_\bullet(A,\underline{T})\,,\,\Theta(A,\underline{T})\big)$ which we call the *canonical space* on the Koszul complex $K_\bullet(A,\underline{T})$.

We have the following functorial property.

Lemma 3.2. *Let $f : A \longrightarrow B$ be a morphism of rings. Then, there is a natural isometry*

$$f^*\big(K_\bullet(A,\underline{T}),\Theta(A,\underline{T})\big) \xrightarrow{\;\cong\;} \big(K_\bullet(B,f(\underline{T})),\Theta(B,f(\underline{T}))\big)\,.$$

Proof. The natural isomorphism $K_\bullet(A,\underline{T})\otimes_A B \xrightarrow{\;\cong\;} K_\bullet(B,f(\underline{T}))$ is an isometry. $\qquad\square$

Remark 3.3. To define this canonical space on $K_\bullet(A,\underline{T})$, it is not necessary to use the product structure of the derived Witt groups. The advantage of this approach is that we see at once that the canonical n-space is a symmetric n-space, but for calculations in the sequel it might be useful to have a good description of the symmetric n-form $\Theta(A,\underline{T})$. We define an isomorphism

$$\rho\,:\,K_\bullet(A,\underline{T}) \xrightarrow{\;\cong\;} \mathcal{D}_A(K_\bullet(A,\underline{T}))[n]$$

following [7], Sect. 1.6. We fix for this an isomorphism $\omega : \bigwedge^n(A^n) \xrightarrow{\;\cong\;} A$, and define an A-bilinear pairing

$$b_i\,:\,K_i(A,\underline{T}) \times K_{n-i}(A,\underline{T}) \longrightarrow A$$

by $(x,y) \longmapsto \omega(x \wedge y)$ for all $0 \le i \le n$. This b_i induces a homomorphism $\varrho_i : K_i(A,\underline{T}) \longrightarrow \mathrm{Hom}_A(K_{n-i}(A,\underline{T}),A) = K_{n-i}(A,\underline{T})^\vee$ which is an isomorphism for all $0 \le i \le n$. It is straightforward (although a little cumbersome) to check that

$$d_{n-(i-1)}(A,\underline{T})^\vee \cdot \varrho_i = (-1)^{i-1}\varrho_{i-1} \cdot d_i(A,\underline{T})\,.$$

Consider the family of morphisms $(\rho_i)_{i\in\mathbb{Z}}$ defined by $\rho_i := (-1)^{\frac{i(i+1)}{2}+\frac{n(n-1)}{2}} \cdot \varrho_i$ for $0 \le i \le n$ and by $\rho_i = 0$ otherwise. This defines an isomorphism of complexes $\rho = \rho_\bullet\,:\,K_\bullet(A,\underline{T}) \longrightarrow \mathcal{D}_A(K_\bullet(A,\underline{T}))[n]$, which coincides with the morphism of complexes $\Theta(A,\underline{T})$ as a thrilling calculation using [10, Ex. 1.4, Rem. 1.9] shows.

By the following lemma this is easier to see if $\underline{T}$ is a regular sequence, which is the only interesting case for us here.

Lemma 3.4. *Assume that $\underline{T}$ is a regular sequence, i.e. $I := \sum_{i=1}^{n} A T_i \neq A$ and T_i is not a zero divisor in $A/ \sum_{j=1}^{i-1} A T_j$ for all $i = 1, \ldots, n$. Identify $A \cong \operatorname{Hom}_A(A, A)$ as usual. Then the following properties hold:*

   (i) *The Koszul complex $K_\bullet(A, \underline{T})$ and its n-dual $\mathcal{D}_A(K_\bullet(A, \underline{T}))[n]$ are A-free resolutions of A/I.*

   (ii) *We have $H_0(\Theta(A, \underline{T})) = (-1)^{\frac{n(n-1)}{2}} \operatorname{id}_{A/I}$.*

   (iii) *For any morphism in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$ between the Koszul complex and its n-dual*
$$\varsigma : K_\bullet(A, \underline{T}) \longrightarrow \mathcal{D}_A(K_\bullet(A, \underline{T}))[n],$$
   there exists an $s \in A$ such that $\varsigma = s \cdot \Theta(A, \underline{T})$ in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$.

   (iv) *In (iii), ς is an isomorphism in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$ if and only if $s + I$ is a unit in the quotient ring A/I.*

Proof. For (i), $\underline{T} = (T_1, \ldots, T_n)$ is a regular sequence by assumption and so the complex $K_\bullet(A, \underline{T})$ is an A-free resolution of A/I by [7, Cor. 1.6.14]. For (ii), see Remark 3.3 above. Point (iii) follows from (i) and (ii). Point (iv) is immediate from (iii). $\qquad\square$

Remark 3.5. It is clear that the restriction of $K_\bullet(A, \underline{T})$ becomes zero in the derived category $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_{A[T_j^{-1}]})$ for all $1 \leq j \leq n$. Hence the complex $K_\bullet(A, \underline{T})$ has support in the closed subset of $\operatorname{Spec}(A)$ defined by the ideal $I := \sum_{j=1}^{n} A T_j$. Therefore the symmetric n-space $\big(K_\bullet(A, \underline{T}), \Theta(A, \underline{T}) \big)$ defines an element in
$$\big[K_\bullet(A, \underline{T}), \Theta(A, \underline{T}) \big] \in \mathrm{W}_I^n(A).$$

Proposition 3.6. *Let $1 \leq i \leq n$. Define the ideal $I_i := \sum_{k \neq i} A T_k$ of A. Then*
$$\big[K_\bullet(A, \underline{T}), \Theta(A, \underline{T}) \big] = 0 \text{ in } \mathrm{W}_{I_i}^n(A).$$

Proof. The group $\mathrm{W}_{I_i}^{n-1}(A)$ obviously contains the element
$$y := [K_\bullet(A, T_1), \Theta(A, T_1)] \star \ldots \star [K_\bullet(A, T_{i-1}), \Theta(A, T_{i-1})]$$
$$\star [K_\bullet(A, T_{i+1}), \Theta(A, T_{i+1})] \star \ldots \star [K_\bullet(A, T_n), \Theta(A, T_n)].$$
Recall from 2.10 that the product also gives
$$\star : \mathrm{W}^1(A) \times \mathrm{W}_{I_i}^{n-1}(A) \longrightarrow \mathrm{W}_{I_i}^n(A).$$
Since the product is skew-commutative, we have
$$[K_\bullet(A, T_i), \Theta(A, T_i)] \star y = (-1)^{i-1} \big[K_\bullet(A, \underline{T}), \Theta(A, \underline{T}) \big],$$
where we consider $[K_\bullet(A, T_i), \Theta(A, T_i)]$ as an element of $\mathrm{W}^1(A)$. Therefore the result follows from the observation that this element is indeed zero in $\mathrm{W}^1(A)$. In fact, the complex $c_0(A) \in \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$ is a Lagrangian (*cf.* [2], Sect. 2) of the symmetric 1-space $(K_\bullet(A, T_i), \Theta(A, T_i))$:

$$
\begin{array}{ccccccccccc}
c_0(A) = & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow \cdots \\
& & & & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & \\
K_\bullet(A, T_i) = & \cdots & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{\cdot T_i} & A & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}
$$

and so $[K_\bullet(A, T_i), \Theta(A, T_i)] = 0$ in $\mathrm{W}^1(A)$. $\square$

In the above proof, note that the class $y \in \mathrm{W}_{I_i}^{n-1}(A)$ does not belong to $\mathrm{W}_I^{n-1}(A)$ and thus the argument can not be used to deduce that $[K_\bullet(A, \underline{T}), \Theta(A, \underline{T})] = 0$ in $\mathrm{W}_I^n(A)$. On the contrary, see Theorem 7.2.

Corollary 3.7. $\big[K_\bullet(A, \underline{T}), \Theta(A, \underline{T})\big] = 0$ *in* $\mathrm{W}^n(A)$.

Proof. Clear since $\mathrm{W}_I^n(A) \longrightarrow \mathrm{W}^n(A)$ factors via $\mathrm{W}_{I_1}^n(A)$ for instance. $\square$

4. Koszul cut in two

We want to "split" the Koszul complex of Section 3 into two pieces, dual to each other. This is easy to understand but a little tricky to write. Recall our running conventions of Section 1 that $r + 4q = n - 1$, see (2), and that $\ell := [\frac{n}{2}]$, see (3). Now, more precisely, we want to define a symmetric r-pair $\big(C_\bullet(A, \underline{T}), \Xi(A, \underline{T})\big)$, such that there is an isometry

$$\mathrm{cone}(C_\bullet(A, \underline{T}), \Xi(A, \underline{T})) \simeq \big(K_\bullet(A, \underline{T}), \Theta(A, \underline{T})\big)[-2q].$$

We abbreviate the canonical form on $K_\bullet(A, \underline{T}) =: K_\bullet$ by

$$\Theta := \Theta(A, \underline{T}),$$

and set

$$E = E(A, \underline{T}) := \mathrm{Coker}\big(K_{\ell+2}(A, \underline{T}) \xrightarrow{d_{\ell+2}(A, \underline{T})} K_{\ell+1}(A, \underline{T})\big). \tag{7}$$

Let $\mathrm{pr}_E = \mathrm{pr}_{E(A, \underline{T})} : K_{\ell+1} \longrightarrow E = \mathrm{Coker}\, d_{\ell+2}$ be the projection. Since $d_{\ell+1} d_{\ell+2} = 0$ there exists a unique morphism $\bar{d}_{\ell+1} = \bar{d}_{\ell+1}(A, \underline{T}) : E \longrightarrow K_\ell$, such that

$$d_{\ell+1}(A, \underline{T}) = \bar{d}_{\ell+1}(A, \underline{T}) \cdot \mathrm{pr}_E. \tag{8}$$

For each $j = 0, \dots, n$, we have $\mathrm{rank}_A(K_j) = \binom{n}{j}$. In particular, if $n = 2\ell + 1$ is odd, we have $\mathrm{rank}_A K_\ell = \mathrm{rank}_A K_{\ell+1}$ and life will be easy. When $n = 2\ell$ is even, K_ℓ has maximal (even) rank $\binom{2\ell}{\ell}$ and we need some preparatory considerations. In this case, the symmetric n-form $\Theta_\bullet : K_\bullet \xrightarrow{\cong} \mathcal{D}_A(K_\bullet)[n]$ gives an isomorphism

$$\Theta_\ell : K_\ell \longrightarrow K_\ell^\vee = \mathrm{Hom}_A(K_\ell, A)$$

which is symmetric if ℓ is even and skew-symmetric otherwise.

Lemma 4.1. *If* $n = 2\ell$ *is even, there exists two totally isotropic subspaces* L *and* M *of* (K_ℓ, Θ_ℓ), *of same rank* $\frac{1}{2}\binom{2\ell}{\ell}$, *such that* $K_\ell = L \oplus M$ *and such that* Θ_ℓ *becomes*

$$\Theta_\ell = \begin{pmatrix} 0 & (-1)^\ell \lambda^\vee \, \mathrm{can}_M \\ \lambda & 0 \end{pmatrix} \quad : \quad K_\ell = L \oplus M \longrightarrow L^\vee \oplus M^\vee = K_\ell^\vee,$$

where $\lambda : L \xrightarrow{\cong} M^\vee$ *is an isomorphism. Moreover, we have*

$$(-1)^\ell d_{\ell+1}^\vee \cdot (\mathrm{pr}_L)^\vee \cdot \lambda^\vee \cdot \mathrm{can}_M \cdot \mathrm{pr}_M \cdot d_{\ell+1} + d_{\ell+1}^\vee \cdot (\mathrm{pr}_M)^\vee \cdot \lambda \cdot \mathrm{pr}_L \cdot d_{\ell+1} = 0, \tag{9}$$

where $\mathrm{pr}_L : K_\ell \longrightarrow L$ *and* $\mathrm{pr}_M : K_\ell \longrightarrow M$ *denote the projections.*

Proof. Let $e_1, \ldots, e_n$ be a basis of A^n. The complementary subspaces of $K_\ell = \overset{\ell}{\bigwedge} A^n$

$$L := \bigoplus_{2 \le i_2 < i_3 < \ldots < i_\ell \le n} A \cdot e_1 \wedge e_{i_2} \wedge e_{i_3} \wedge \ldots \wedge e_{i_\ell}$$

and

$$M := \bigoplus_{2 \le i_1 < i_2 < \ldots < i_\ell \le n} A \cdot e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_\ell}$$

both have rank $\binom{2\ell-1}{\ell-1} = \binom{2\ell-1}{\ell} = \frac{1}{2}\binom{2\ell}{\ell}$. Now use the description of Θ_ℓ given in Remark 3.3. Let $\omega : \overset{n}{\bigwedge} A^n \overset{\simeq}{\longrightarrow} A$ be the isomorphism which sends $e_1 \wedge \ldots \wedge e_n$ to $1 \in A$. Then $\Theta_\ell(x)(y) = \pm\omega(x \wedge y)$. From this we easily get that both subspaces are totally isotropic: for L it is because $e_1 \wedge e_1 = 0$ and for M it is because two subsets with ℓ elements in $\{2, \ldots, n\}$ must intersect. Since Θ_ℓ is a $(-1)^\ell$-symmetric isomorphism, its decomposition in $L \oplus M$ must be as claimed in the lemma.

Note that $\Theta : K_\bullet \longrightarrow \mathcal{D}_A(K_\bullet)[n]$ is a morphism of complexes and we have

$$d_{\ell+1}^\vee \cdot \Theta_\ell \cdot d_{\ell+1} = 0 : \qquad K_{\ell+1} \xrightarrow{\ d_{\ell+1}\ } K_\ell \xrightarrow{\ \Theta_\ell\ } K_\ell^\vee \xrightarrow{\ d_{\ell+1}^\vee\ } K_{\ell+1}^\vee .$$

In the decomposition $K_\ell = L \oplus M$, the morphism $d_{\ell+1} : K_{\ell+1} \longrightarrow K_\ell$ becomes $\begin{pmatrix} \mathrm{pr}_L \cdot d_{\ell+1} \\ \mathrm{pr}_M \cdot d_{\ell+1} \end{pmatrix}$. Replacing this in $d_{\ell+1}^\vee \cdot \Theta_\ell \cdot d_{\ell+1} = 0$ gives equation (9) by a direct matrix multiplication. $\qquad\qquad\square$

Definition 4.2. As the above discussion shows, we will have to distinguish the cases where n is odd from those where n is even and the definition extends over 4.3‑4.6 below. We shall consider a sign

$$\epsilon_n \in \{-1, 1\}$$

which will be fixed later on, see 6.3.

We start with $n = 2\ell + 1$ odd.

4.3. *Case $\underline{r = 0}$.*

Then $\ell = 2q$ is even and $\big(C_\bullet(A, \underline{T}), \, \Xi(A, \underline{T}) \big)$ is defined to be the following symmetric 0-pair:

$$\text{deg }\ell \qquad\qquad\qquad\qquad \text{deg }0.$$

If $\underline{T}$ is a regular sequence then the Koszul complex $K_\bullet$ is exact and so we have the following quasi-isomorphism

$$C_\bullet(A,\underline{T}) = \quad 0 \longrightarrow K_n \xrightarrow{d_n} \cdots \longrightarrow K_{\ell+2} \xrightarrow{d_{\ell+2}} K_{\ell+1} \longrightarrow 0 \longrightarrow$$

$$p = p(A,\underline{T}) \Big\downarrow :=$$

$$c_0(E) = \quad 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \xrightarrow{\quad} E \longrightarrow 0 \longrightarrow$$

$$\deg \ell \qquad\qquad\qquad\qquad \deg 0.$$

with vertical maps including $\mathrm{pr}_E : K_{\ell+1} \to E$ at deg 0.

4.4. Case $\underline{r = 2}$.

Then $\ell = 2q+1$ is odd and $\big(C_\bullet(A,\underline{T}), \Xi(A,\underline{T})\big)$ is defined to be the following symmetric 2-pair:

$$0 \longrightarrow K_n \xrightarrow{d_n} \cdots \longrightarrow K_{\ell+2} \xrightarrow{d_{\ell+2}} K_{\ell+1} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow 0$$

with middle vertical map $\epsilon_n \Theta_\ell \cdot d_{\ell+1}$

$$0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow K_{\ell+1}^\vee \xrightarrow{d_{\ell+2}^\vee} K_{\ell+2}^\vee \longrightarrow \cdots \xrightarrow{d_n^\vee} K_n^\vee \longrightarrow 0$$

$$\deg \ell+1 \qquad\qquad\qquad\qquad \deg 1.$$

As above if $\underline{T}$ is a regular sequence the projection $\mathrm{pr}_E : K_{\ell+1} \longrightarrow E$ induces a quasi-isomorphism of complexes $p = p(A,\underline{T}) : C_\bullet(A,\underline{T}) \longrightarrow c_0(E)[1]$.

Now let $n = 2\ell$ be even.

We fix two totally isotropic subspaces L and M of K_ℓ and an isomorphism $\lambda : L \longrightarrow M^\vee$ as in Lemma 4.1 and keep notations as there. We set

$$h := \lambda^\vee \cdot \mathrm{can}_M \cdot \mathrm{pr}_M \cdot d_{\ell+1} \quad : \qquad K_{\ell+1} \longrightarrow L^\vee .$$

We now define the space $\big(C_\bullet(A,\underline{T}), \Xi(A,\underline{T})\big)$ for n even. It follows from equation (9) in Lemma 4.1 that both squares in the middle of the two diagrams below commute and so the morphism $\Xi(A,\underline{T})$ is really a morphism of complexes.

4.5. Case $\underline{r = -1}$.

Then $\ell = 2q$ is even and $\big(C_\bullet(A,\underline{T}), \Xi(A,\underline{T})\big)$ is defined to be the following symmetric (-1)-pair:

$$0 \longrightarrow K_n \xrightarrow{d_n} \cdots \longrightarrow K_{\ell+2} \xrightarrow{d_{\ell+2}} K_{\ell+1} \xrightarrow{\mathrm{pr}_L\, d_{\ell+1}} L \longrightarrow 0 \cdots \longrightarrow 0 \longrightarrow 0$$

with vertical maps $\epsilon_n h$ and $\epsilon_n h^\vee \mathrm{can}_L$

$$0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow L^\vee \xrightarrow{-(\mathrm{pr}_L\, d_{\ell+1})^\vee} K_{\ell+1}^\vee \xrightarrow{-d_{\ell+2}^\vee} \cdots \xrightarrow{-d_n^\vee} K_n^\vee \longrightarrow 0$$

$$\deg \ell-1 \qquad\qquad \deg 0 \qquad \deg -1$$

If the sequence $\underline{T}$ is regular the homology of $C_\bullet(A,\underline{T})$ is not concentrated in one degree (as in the case n odd) but there exists a "short" complex $F_\bullet(A,\underline{T})$ defined

as follows and which is quasi-isomorphic to $C_\bullet(A, \underline{T})$:

$$
\begin{array}{ccccccccccc}
C_\bullet(A, \underline{T}) = & & \cdots \longrightarrow K_{\ell+2} & \xrightarrow{d_{\ell+2}} & K_{\ell+1} & \xrightarrow{\mathrm{pr}_L \cdot d_{\ell+1}} & L \longrightarrow & 0 \longrightarrow & \cdots \\
{\scriptstyle p=p(A,\underline{T})} \Big\downarrow {\scriptstyle :=} & & \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{pr}_E} & & \Big\downarrow {\scriptstyle =} & \Big\downarrow & \\
F_\bullet(A, \underline{T}) := & & \cdots \longrightarrow 0 & \longrightarrow & E & \xrightarrow{\mathrm{pr}_L \cdot \bar{d}_{\ell+1}} & L \longrightarrow & 0 \longrightarrow & \cdots
\end{array}
$$

$$
\qquad\qquad\qquad \deg 0 \qquad\qquad \deg -1.
$$

4.6. *Case* $\underline{r = 1}$.

Then $\ell = 2q + 1$ is odd and $\big(C_\bullet(A, \underline{T}), \, \Xi(A, \underline{T}) \big)$ is defined to be the following symmetric 1-pair :

$$
\begin{array}{ccccccccccc}
0 \to K_n & \xrightarrow{d_n} & \cdots \longrightarrow K_{\ell+2} & \xrightarrow{d_{\ell+2}} & K_{\ell+1} & \xrightarrow{\mathrm{pr}_L \, d_{\ell+1}} & L & \longrightarrow 0 \cdots \longrightarrow & 0 \longrightarrow & 0 \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow {\scriptstyle \epsilon_n h} & & \Big\downarrow {\scriptstyle -\epsilon_n h^\vee \, \mathrm{can}_L} & & \Big\downarrow & \\
0 \to 0 & \longrightarrow & \cdots \longrightarrow 0 & \longrightarrow & L^\vee & \xrightarrow{-(\mathrm{pr}_L \, d_{\ell+1})^\vee} & K_{\ell+1}^\vee & \xrightarrow{-d_{\ell+2}^\vee} \cdots \longrightarrow & K_n^\vee \xrightarrow{-d_n^\vee} & 0
\end{array}
$$

$$
\qquad \deg \ell \qquad\qquad\qquad \deg 1 \qquad\qquad \deg 0
$$

As in the case $r = -1$, when $\underline{T}$ is a regular sequence, we have a quasi-isomorphism $p = p(A, \underline{T}) : C_\bullet(A, \underline{T}) \longrightarrow F_\bullet(A, \underline{T})$, where $F_\bullet(A, \underline{T})$ is now the complex

$$
\cdots \longrightarrow 0 \longrightarrow E \xrightarrow{\mathrm{pr}_L \cdot \bar{d}_{\ell+1}} L \longrightarrow 0 \longrightarrow \cdots
$$

$$
\qquad\qquad \deg 1 \qquad\qquad \deg 0.
$$

Lemma 4.7. *Let* $f : A \longrightarrow B$ *be a morphism of rings. There is a natural isometry*

$$
f^* \big(C_\bullet(A, \underline{T}), \, \Xi(A, \underline{T}) \big) \xrightarrow{\simeq} \big(C_\bullet(B, \underline{f(T)}), \, \Xi(B, \underline{f(T)}) \big).
$$

Proof. Straightforward, *cf.* Lemma 3.2. $\qquad\square$

Lemma 4.8. *The mapping cone of the morphism* $\Xi(A, \underline{T})$ *is isomorphic (as a complex) to* $K_\bullet(A, \underline{T})[-2q]$.

Proof. This is an easy direct computation, which we leave to the reader. It is clear in the cases where n is odd and it requires Lemma 4.1 for n even. In all four cases, we use the isomorphism Θ to replace the $K_j^\vee$ by K_{n-j} for $j \geq \ell + 1$. $\qquad\square$

Remark 4.9. Note that we do not claim that the *symmetric* cone of the *symmetric* r-form $\Xi(A, \underline{T})$ is the Koszul complex *with its canonical form* $\Theta(A, \underline{T})$. This would be true, however, with a suitable choice of the sign ϵ_n. Instead of going into these computations, we shall use a simplifying trick : see 6.3.

5. The Koszul symmetric space $\mathbf{K}_X^{(n)}$ over $\mathbb{A}_X^n$

Let R be a ring. We apply the constructions of Section 3 to $A := R[T_1, \ldots, T_n]$, the polynomial ring in n variables over R, and to the sequence $\underline{T} := (T_1, \ldots, T_n)$. The reader can think of $R = \mathbb{Z}$ or $R = \mathbb{Z}[1/2]$, since these are the important cases, from which the rest will be deduced.

Definition 5.1. The *Koszul symmetric n-space* $\mathbf{K}_R^{(n)} = \left(\mathrm{K}_R^{(n)}, \Theta_R^{(n)} \right)$ over $\mathbb{A}_R^n$ is the symmetric n-space where $\mathrm{K}_R^{(n)} := K_\bullet(A, \underline{T})$ will be called *the Koszul complex over* $\mathbb{A}_R^n$ and where the symmetric n-form $\Theta_R^{(n)} := \Theta(A, \underline{T})$ is the one of Definition 3.1.

Remark 5.2. Pay attention: $\mathbf{K}_R^{(n)}$ is a symmetric n-space defined over the ring $A = R[T_1, \ldots, T_n]$ and not over the ring R, as the notation might suggest.

It is clear that the Koszul symmetric n-space behaves well with respect to base-change. More precise, let $f : R \longrightarrow S$ be a morphism of rings and let

$$\alpha_f : R[T_1, \ldots, T_n] \longrightarrow S[T_1, \ldots, T_n]$$

be the obvious induced morphism. Then, by Lemma 3.2 there is a natural isometry

$$\alpha_f^*(\mathbf{K}_R^{(n)}) \xrightarrow{\simeq} \mathbf{K}_S^{(n)}.$$

In particular, $\mathbf{K}_R^{(n)}$ is extended from $\mathbf{K}_{\mathbb{Z}}^{(n)}$. This justifies the following extension of Definition 5.1.

Definition 5.3. Let X be a scheme. We define the symmetric n-space

$$\mathbf{K}_X^{(n)} := \alpha_X^*(\mathbf{K}_{\mathbb{Z}}^{(n)})$$

where $\alpha_X : \mathbb{A}_X^n \longrightarrow \mathbb{A}_{\mathbb{Z}}^n$ is the base-change morphism, see 1.4. We call $\mathbf{K}_X^{(n)}$ the *Koszul symmetric n-space* over $\mathbb{A}_X^n$. Like before, we denote the underlying complex of free $\mathcal{O}_{\mathbb{A}_X^n}$-modules and its symmetric n-form by

$$\mathrm{K}_X^{(n)} = \alpha_X^*\left(\mathrm{K}_{\mathbb{Z}}^{(n)} \right) \qquad \text{and} \qquad \Theta_X^{(n)} = \alpha_X^*\left(\Theta_{\mathbb{Z}}^{(n)} \right).$$

Remark 5.4. It is obvious from the definition that for any morphism of schemes $f : Y \longrightarrow X$ we have an isometry $\alpha_f^*\left(\mathbf{K}_X^{(n)} \right) \simeq \mathbf{K}_Y^{(n)}$ over $\mathbb{A}_Y^n$

Definition 5.5. By Remark 3.5, the complex $\mathrm{K}_X^{(n)}$ has support in the closed subset $\mathbb{A}_X^n \smallsetminus \mathbb{U}_X^n$ of $\mathbb{A}_X^n$. Therefore, the symmetric n-space $\mathbf{K}_X^{(n)}$ represents a Witt class

$$\kappa_X^{(n)} := [\mathbf{K}_X^{(n)}] \quad \in \mathrm{W}_{\mathbb{A}_X^n \smallsetminus \mathbb{U}_X^n}^n\left(\mathbb{A}_X^n \right).$$

Remark 5.6. Note that there are several choices of signs in the definition of the symmetric n-space $\mathbf{K}_X^{(n)}$. Other sign conventions give the same space or its negative, but the results of this work are clearly independent of such choices.

6. The half-Koszul symmetric space $\mathbf{B}_X^{(n)}$ over $\mathbb{U}_X^n$

Definition 6.1. Let R be a ring. We now apply the splitting of Section 4 to the space $\mathbf{K}_R^{(n)}$ of Section 5. As above, we put $A := R[T_1, \ldots, T_n]$ and $\underline{T} := (T_1, \ldots, T_n)$. We define

$$\mathrm{C}_R^{(n)} := C_\bullet(A, \underline{T}) \qquad \text{and} \qquad \Xi_R^{(n)} := \Xi(A, \underline{T})$$

as defined in 4.3 to 4.6. For any scheme X we define

$$\mathrm{C}_X^{(n)} := \alpha_X^*(\mathrm{C}_{\mathbb{Z}}^{(n)}) \qquad \text{and} \qquad \Xi_X^{(n)} := \alpha_X^*(\Xi_{\mathbb{Z}}^{(n)})$$

where $\alpha_X : \mathbb{A}_X^n \longrightarrow \mathbb{A}_{\mathbb{Z}}^n$ is the base-change morphism. This coincides with the above in the affine case by Lemma 4.7. For all $n \in \mathbb{N}$, the symmetric r-pair $(\mathrm{C}_X^{(n)}, \Xi_X^{(n)})$ on $\mathbb{A}_X^n$ will be denoted by $\mathbf{C}_X^{(n)}$.

Remark 6.2. We have the following facts:

(1) The pair $\mathbf{C}_X^{(n)}$ is a symmetric r-pair (see Definition 2.2) indeed.
(2) If $f : Y \longrightarrow X$ is a morphism of schemes then there is a natural isometry

$$\alpha_f^*(\mathbf{C}_X^{(n)}) \xrightarrow{\simeq} \mathbf{C}_Y^{(n)}.$$

6.3. *The symmetric cone of $\mathbf{C}_X^{(n)}$.*

Instead of calculating $\mathrm{cone}(\mathbf{C}_X^{(n)})$ directly (which is possible, but cumbersome) we take full advantage of Lemma 3.4. More precisely, we use the fact that any quasi-isomorphism $K_\bullet(\mathbb{Z}, \underline{T}) \longrightarrow \mathcal{D}_{\mathbb{Z}[T_1,\ldots,T_n]}(K_\bullet(\mathbb{Z}, \underline{T}))[n]$ is equal to the symmetric n-form $\pm \Theta_{\mathbb{Z}}^{(n)}$ in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_{\mathbb{Z}[T_1,\ldots,T_n]})$.

So let for a moment $R = \mathbb{Z}$ and $A = \mathbb{Z}[T_1, \ldots, T_n]$. We abbreviate $K_\bullet := K_\bullet(A, \underline{T})$ and $\Theta = \Theta(A, \underline{T})$. We get from Lemma 4.8 the following commutative diagram (where $\mathcal{D} = \mathcal{D}_A$)

$$
\begin{array}{ccccccc}
\mathrm{C}_{\mathbb{Z}}^{(n)} & \xrightarrow{\Xi_{\mathbb{Z}}^{(n)}} & \mathcal{D}(\mathrm{C}_{\mathbb{Z}}^{(n)})[r] & \xrightarrow{u} & K_\bullet[-2q] & \xrightarrow{v} & \mathrm{C}_{\mathbb{Z}}^{(n)}[1] \\
{\scriptstyle (-1)^{\frac{r(r+1)}{2}}\varpi_{\mathrm{C}_{\mathbb{Z}}^{(n)}}}\downarrow & & {\scriptstyle =}\downarrow & & & & \downarrow{\scriptstyle (-1)^{\frac{r(r+1)}{2}}\varpi_{\mathrm{C}_{\mathbb{Z}}^{(n)}}[1]} \\
\mathcal{D}\mathcal{D}(\mathrm{C}_{\mathbb{Z}}^{(n)}) & \xrightarrow[\mathcal{D}(\Xi_{\mathbb{Z}}^{(n)})[r]]{} & \mathcal{D}(\mathrm{C}_{\mathbb{Z}}^{(n)})[r] & \xrightarrow[-\mathcal{D}(v)[r+1]]{} & \mathcal{D}(K_\bullet[-2q])[r+1] & \xrightarrow[(-1)^r\mathcal{D}(u)[r+1]]{} & \mathcal{D}\mathcal{D}(\mathrm{C}_{\mathbb{Z}}^{(n)})[1]
\end{array}
$$

$$(10)$$

whose rows are exact triangles for all $n \in \mathbb{N}$ (the bottom row is the dual of the upper one, shifted r times). By the very basic properties of triangulated categories there exists an isomorphism

$$\varsigma : K_\bullet[-2q] \longrightarrow \mathcal{D}(K_\bullet[-2q])[r+1] = \big(\mathcal{D}(K_\bullet)[n]\big)[-2q],$$

in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$ such that diagram (10) commutes. By Lemma 3.4 the isomorphism ς is equal to $\pm\Theta[-2q]$. Replacing if necessary $\Xi_{\mathbb{Z}}^{(n)}$ by $-\Xi_{\mathbb{Z}}^{(n)}$, *i.e.* replacing ϵ_n by $-\epsilon_n$ in the definition of $\mathbf{C}_R^{(n)}$, we can assume that $\varsigma = \Theta[-2q]$ for all $n \in \mathbb{N}$, *i.e.* $(K_\bullet[-2q], \varsigma) = \mathbf{K}_{\mathbb{Z}}^{(n)}[-2q]$ for all $n \in \mathbb{N}$.

We fix ϵ_n as explained above.

We can now calculate $\mathrm{cone}(\mathbf{C}_X^{(n)})$ for any scheme X and any $n \in \mathbb{N}$. The pull-back via the base-change morphism $\alpha_X : \mathbb{A}_X^n \longrightarrow \mathbb{A}_{\mathbb{Z}}^n$ of diagram (10) above is a cone diagram for the symmetric r-form $\alpha_X^*(\mathbf{C}_{\mathbb{Z}}^{(n)})$. By Lemma 4.7 we have an isometry $\mathbf{C}_X^{(n)} \simeq \alpha_X^*(\mathbf{C}_{\mathbb{Z}}^{(n)})$ and so we get $\mathrm{cone}(\mathbf{C}_X^{(n)}) \simeq \alpha_X^*(\mathbf{K}_{\mathbb{Z}}^{(n)}[-2q]) \simeq \alpha_X^*(\mathbf{K}_{\mathbb{Z}}^{(n)})[-2q]$ (*cf.* Lemma B.1 for the later isometry). We have proven:

Theorem 6.4. *With this choice of ϵ_n, the cone of the symmetric pair $\mathbf{C}_X^{(n)}$ is the Koszul symmetric space shifted as follows:*

$$\mathrm{cone}(\mathbf{C}_X^{(n)}) = \mathbf{K}_X^{(n)}[-2q].$$

<div align="right">□</div>

In particular, $\Xi_X^{(n)}|_{\mathbb{U}_X^n}$ is an isomorphism because the homology of $K_\bullet|_{\mathbb{U}_X^n}$ vanishes.

Definition 6.5. Let X be a scheme. The symmetric r-space

$$\mathbf{B}_X^{(n)} := \mathbf{C}_X^{(n)}|_{\mathbb{U}_X^n}$$

will be called *the half-Koszul space* over the scheme X. Its Witt class is denoted by

$$\varepsilon_X^{(n)} := [\mathbf{B}_X^{(n)}] \quad \in \mathrm{W}^r(\mathbb{U}_X^n).$$

The following is obvious (*cf.* Remark 6.2).

Lemma 6.6. *Let $f : Y \longrightarrow X$ be a morphism of schemes. Then there is a natural isometry*

$$\upsilon_f^*(\varepsilon_X^{(n)}) \xrightarrow{\simeq} \varepsilon_Y^{(n)}.$$

<div align="right">□</div>

By the main result of [3] we know that $\mathbf{B}_X^{(n)}$ is Witt equivalent to a space living on a short complex (see 2.4). In fact, $\mathbf{B}_X^{(n)}$ is not only Witt equivalent, but isometric to such a space on a "short complex".

We use the notation of 4.3–4.6 with $R = \mathbb{Z}$, *i.e.* $A = \mathbb{Z}[T_1, \ldots, T_n]$ is the polynomial ring in n variables over $\mathbb{Z}$, $\underline{T} = (T_1, \ldots, T_n)$, and $K_\bullet = K_\bullet(A, \underline{T})$ is the Koszul complex of the sequence $\underline{T}$ over A. As in 4.3–4.6 we denote the differential of this Koszul complex by $d_\bullet$ and set

$$E = E(A, \underline{T}) = \mathrm{Coker}\left(K_{\ell+2} \xrightarrow{d_{\ell+2}} K_{\ell+1} \right).$$

Note that $\underline{T}$ is a regular sequence and so $K_\bullet$ is a finite free resolution of $\mathbb{Z} \simeq A/I$, where I is the ideal generated by $\underline{T}$. It follows that $K_\bullet(A, \underline{T})|_{\mathrm{Spec}\, A[T_i^{-1}]}$ is a split exact sequence and so

$$\mathcal{E}_{\mathbb{Z}}^{(n)} := \mathrm{Coker}\left(K_{\ell+2} \xrightarrow{d_{\ell+2}} K_{\ell+1} \right)\Big|_{\mathbb{U}_{\mathbb{Z}}^n} = E\big|_{\mathbb{U}_{\mathbb{Z}}^n} \simeq \mathrm{Ker}\, d_\ell\big|_{\mathbb{U}_{\mathbb{Z}}^n} \tag{11}$$

is a locally free $\mathcal{O}_{\mathbb{U}_{\mathbb{Z}}^n}$-module of rank $\sum_{i=0}^{\ell}(-1)^i \binom{n}{\ell-i} = \binom{n-1}{n-\ell-1}$. Clearly the same is true for the pull-back

$$\mathcal{E}_X^{(n)} := \upsilon_X^*(\mathcal{E}_{\mathbb{Z}}^{(n)}), \tag{12}$$

where $\upsilon_X : \mathbb{U}_X^n \longrightarrow \mathbb{U}_{\mathbb{Z}}^n$ is induced by base change, see (4). Note that we have

$$\mathcal{E}_X^{(n)} = \mathrm{Coker}\left(\mathrm{K}_{X,\ell+2}^{(n)} \longrightarrow \mathrm{K}_{X,\ell+1}^{(n)} \right)\Big|_{\mathbb{U}_X^n} \simeq \mathrm{Ker}\left(\mathrm{K}_{X,\ell}^{(n)} \longrightarrow \mathrm{K}_{X,\ell-1}^{(n)} \right)\Big|_{\mathbb{U}_X^n},$$

where $\mathrm{K}_X^{(n)} = \mathrm{K}_{X,\bullet}^{(n)} = \alpha_X^*(\mathrm{K}_{\mathbb{Z}}^{(n)})$ (see Definition 5.3).

We consider now the case n odd and n even separately.

6.7. *The space* $\mathbf{B}_X^{(n)}$ *if* $n = 2\ell + 1$ *is odd, i.e.* $r = 0$ *or* $r = 2$.

Since the functor $(-)^\vee = \mathrm{Hom}_A(\,-\,, A)$ is left exact we have $E^\vee = \mathrm{Ker}\,\bigl(K_{\ell+1}^\vee \xrightarrow{\ d_{\ell+2}^\vee\ } K_{\ell+2}^\vee \bigr)$ and hence a well defined homomorphism

$$\varphi_{\mathbb{Z}}^{(n)} := \epsilon_n \cdot \theta_\ell \cdot \bar{d}_{\ell+1}\big|_{\mathbb{U}_{\mathbb{Z}}^n} \; : \; \mathcal{E}_{\mathbb{Z}}^{(n)} \longrightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{U}_{\mathbb{Z}}^n}}(\mathcal{E}_{\mathbb{Z}}^{(n)}, \mathcal{O}_{\mathbb{U}_{\mathbb{Z}}^n}) = E^\vee\big|_{\mathbb{U}_{\mathbb{Z}}^n}$$

which is $(-1)^\ell$-symmetric, where $\bar{d}_{\ell+1}$ is the unique morphism $K_{\ell+1} \longrightarrow E$, such that $d_{\ell+1} = \bar{d}_{\ell+1} \cdot \mathrm{pr}_E$ (*cf.* (8)). We set

$$(\mathcal{F}_{\mathbb{Z}}^{(n)}, \phi_{\mathbb{Z}}^{(n)}) := \begin{cases} c_0(\mathcal{E}_{\mathbb{Z}}^{(n)}, \varphi_{\mathbb{Z}}^{(n)}) & \text{if } \ell \text{ is even} \\[2mm] c_0(\mathcal{E}_{\mathbb{Z}}^{(n)}, \varphi_{\mathbb{Z}}^{(n)})[1] & \text{if } \ell \text{ is odd.} \end{cases}$$

This is a symmetric r-pair. Recall now from 4.3–4.6 that the projection $\mathrm{pr}_E : K_{\ell+1} \longrightarrow E$ induces a quasi-isomorphism (hence an isomorphism in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_A)$)

$$p|_{\mathbb{U}_{\mathbb{Z}}^n} = p(\mathbb{Z}, \underline{T})|_{\mathbb{U}_{\mathbb{Z}}^n} \; : \; \mathrm{C}_{\mathbb{Z}}^{(n)} \xrightarrow{\simeq} c_0(\mathcal{E}_{\mathbb{Z}}^{(n)}) \qquad (\text{respectively } \mathrm{C}_{\mathbb{Z}}^{(n)} \xrightarrow{\simeq} c_0(\mathcal{E}_{\mathbb{Z}}^{(n)})[1]),$$

which is easily seen to be an isometry $\mathrm{B}_{\mathbb{Z}}^{(n)} \xrightarrow{\simeq} (\mathcal{F}_{\mathbb{Z}}^{(n)}, \phi_{\mathbb{Z}}^{(n)})$. It follows that $\phi_{\mathbb{Z}}^{(n)}$ is an isomorphism and so $(\mathcal{F}_{\mathbb{Z}}^{(n)}, \phi_{\mathbb{Z}}^{(n)})$ a symmetric r-space. In particular $(\mathcal{E}_{\mathbb{Z}}^{(n)}, \varphi_{\mathbb{Z}}^{(n)})$ is a $(-1)^\ell$-symmetric space over $\mathbb{U}_{\mathbb{Z}}^n$. Applying the pull-back v_X^* we get:

(i) *The pair*

$$(\mathcal{E}_X^{(n)}, \varphi_X^{(n)}) := v_X^*(\mathcal{E}_{\mathbb{Z}}^{(n)}, \varphi_{\mathbb{Z}}^{(n)})$$

is a $(-1)^\ell$-*symmetric space over* $\mathbb{U}_X^n$.

(ii) *The half Koszul space* $\mathbf{B}_X^{(n)}$ *is isometric to the short symmetric r-space:*

$$\mathbf{E}_X^{(n)} := (\mathcal{F}_X^{(n)}, \phi_X^{(n)}) := v_X^*(\mathcal{F}_{\mathbb{Z}}^{(n)}, \phi_{\mathbb{Z}}^{(n)}) = \begin{cases} c_0(\mathcal{E}_X^{(n)}, \varphi_X^{(n)}) & \text{if } \ell \text{ is even} \\[2mm] c_0(\mathcal{E}_X^{(n)}, \varphi_X^{(n)})[1] & \text{if } \ell \text{ is odd.} \end{cases}$$

Example 6.8. If $n = 1$ then $(\mathcal{E}_X^{(n)}, \varphi_X^{(n)})$ is the symmetric space

$$\mathcal{O}_{\mathbb{U}_X^n} \xrightarrow{\ \cdot T\ } \mathcal{O}_{\mathbb{U}_X^n} = \mathcal{H}om_{\mathcal{O}_{\mathbb{U}_X^n}}(\mathcal{O}_{\mathbb{U}_X^n}, \mathcal{O}_{\mathbb{U}_X^n}).$$

6.9. *The space* $\mathbf{B}_X^{(n)}$ *if* $n = 2\ell$ *is even, i.e.* $r = -1$ *or* $r = 1$.

We fix $L, M \subset K_\ell$ and $\lambda : L \xrightarrow{\simeq} M^\vee$ as in Lemma 4.1 (with $R = \mathbb{Z}$), and let $\mathrm{pr}_L : K_\ell \longrightarrow L$ and $\mathrm{pr}_M : K_\ell \longrightarrow M$ be the respective projections. We denote $\mathcal{L} := L|_{\mathbb{U}_{\mathbb{Z}}^n}$ and $\mathrm{pr}_{\mathcal{L}} := \mathrm{pr}_L|_{\mathbb{U}_{\mathbb{Z}}^n} : K_\ell|_{\mathbb{U}_{\mathbb{Z}}^n} \longrightarrow \mathcal{L}$.

On the complex $\mathcal{F}_{\mathbb{Z}}^{(n)} := F_\bullet(A, \underline{T})\big|_{\mathbb{U}_{\mathbb{Z}}^n}$ we have the following symmetric r-form:

$$\begin{array}{ccc}
\text{if } r = -1 & \deg 0 & \deg -1 \\[2mm]
\text{if } r = 1 & \deg 1 & \deg 0
\end{array}$$

where $\bar{h} = \lambda \cdot \mathrm{can}_M \cdot \mathrm{pr}_M \cdot \bar{d}_{\ell+1}$. Since $d_{\ell+1}|_{\mathbb{U}_\mathbb{Z}^n} = (\bar{d}_{\ell+1} \cdot \mathrm{pr}_E)|_{\mathbb{U}_\mathbb{Z}^n}$ we see that the quasi-isomorphism $p|_{\mathbb{U}_\mathbb{Z}^n} : \mathcal{F}_\mathbb{Z}^{(n)} \xrightarrow{\simeq} C_\mathbb{Z}^{(n)}$ is an isometry $\mathbf{B}_\mathbb{Z}^{(n)} \xrightarrow{\simeq} (\mathcal{F}_\mathbb{Z}^{(n)}, \phi_\mathbb{Z}^{(n)})$, and so $\phi_\mathbb{Z}^{(n)}$ is an isomorphism in $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_\mathbb{Z})$. Therefore $(\mathcal{F}_\mathbb{Z}^{(n)}, \phi_\mathbb{Z}^{(n)})$ is a symmetric r-space over $\mathbb{U}_\mathbb{Z}^n$. Applying the pull-back υ_X^* we get:

The half Koszul space $\mathbf{B}_X^{(n)}$ is isometric to the short symmetric r-space:

$$\mathbf{E}_X^{(n)} := (\mathcal{F}_X^{(n)}, \phi_X^{(n)}) := \upsilon_X^*(\mathcal{F}_\mathbb{Z}^{(n)}, \phi_\mathbb{Z}^{(n)}).$$

Remark 6.10. We give in Appendix A a proof of the following fact. If $n \geq 3$ then the locally free $\mathcal{O}_{\mathbb{U}_X^n}$-module $\mathcal{E}_X^{(n)}$ can not be extended to a locally free $\mathcal{O}_{\mathbb{A}_X^n}$-module, and hence in particular is not extended from X. Of course this is wrong for $n = 1, 2$, since then $\ell = 0$ and therefore $\mathcal{E}_\mathbb{Z}^{(n)} \simeq K_n(\mathbb{Z}, \underline{T})\big|_{\mathbb{U}_\mathbb{Z}^n}$ which is free.

It follows already from this that it is impossible to extend the symmetric r-space $\mathbf{E}_X^{(n)}$ to $\mathbb{A}_X^n$ as long as $n \geq 3$. We will see in Theorem 8.2 that even more is true. It is impossible to extend $\mathbf{E}_X^{(n)}$ up to Witt equivalence to $\mathbb{A}_X^n$, i.e. $[\mathbf{E}_X^{(n)}]$ is not in the image of $\mathrm{W}^r(\mathbb{A}_X^n) \xrightarrow{\iota_X^*} \mathrm{W}^r(\mathbb{U}_X^n)$, and this for any $n \geq 1$.

7. Witt groups of the punctured affine space

Recall the notations of Section 1, like formula (4), defining $r \in \{-1, 0, 1, 2\}$ by $n = 4q + r + 1$. We begin with an easy application of triangular Witt theory:

Theorem 7.1. *Let X be a regular scheme containing $\frac{1}{2}$. There exists a split short exact sequence*

$$0 \longrightarrow \mathrm{W}^i(X) \xrightarrow{\sigma_X^*} \mathrm{W}^i(\mathbb{U}_X^n) \xrightarrow{\partial} \mathrm{W}_{\mathbb{A}_X^n \smallsetminus \mathbb{U}_X^n}^{i+1}(\mathbb{A}_X^n) \longrightarrow 0,$$

for all $i \in \mathbb{Z}$, where ∂ is the connecting homomorphism of the localization $\mathbb{U}_X^n \subset \mathbb{A}_X^n$. This sequence is natural in X in the obvious way. Moreover, a left inverse to σ_X^* is given by $\gamma^* : \mathrm{W}^i(\mathbb{U}_X^n) \longrightarrow \mathrm{W}^i(X)$ for any X-point $\gamma : X \to \mathbb{U}_X^n$, i.e. any morphism $\gamma : X \to \mathbb{U}_X^n$ such that $\sigma_X \circ \gamma = \mathrm{id}_X$.

Proof. Consider the commutative (plain) diagram:

$$\cdots \xrightarrow{\partial} \mathrm{W}_{\mathbb{A}_X^n \smallsetminus \mathbb{U}_X^n}^i(\mathbb{A}_X^n) \longrightarrow \mathrm{W}^i(\mathbb{A}_X^n) \xrightarrow{\iota_X^*} \mathrm{W}^i(\mathbb{U}_X^n) \xrightarrow{\partial} \mathrm{W}_{\mathbb{A}_X^n \smallsetminus \mathbb{U}_X^n}^{i+1}(\mathbb{A}_X^n) \longrightarrow \cdots$$

The long sequence is exact by localization, see 2.9, and the homomorphism π_X^* is an isomorphism by homotopy invariance [4, Thm. 3.1]. Now, for any X-point $\gamma : X \to \mathbb{U}_X^n$, for instance $(1, \ldots, 1)$, since $\sigma_X \circ \gamma = \mathrm{id}_X$, the homomorphism σ_X^* is split injective with the wanted left inverse. Hence the homomorphism ι_X^* is also split injective and the unlabeled morphism in the above diagram must be equal to zero, for all $i \in \mathbb{Z}$, in particular for $i + 1$. This, in turn, gives the surjectivity of ∂ and the result follows easily. $\square$

We want to apply "dévissage" to the relative groups $W^{i+1}_{\mathbb{A}^n_X \smallsetminus \mathbb{U}^n_X}(\mathbb{A}^n_X)$ and we will need

Theorem 7.2. *Let X be a regular $\mathbb{Z}[1/2]$-scheme of finite Krull dimension. Consider the structure morphism $\pi_X : \mathbb{A}^n_X \longrightarrow \operatorname{Spec} X$. Then, the homomorphism*

$$\vartheta^{(n)}_X \; : \; W^{i-n}(X) \longrightarrow W^i_{\mathbb{A}^n_X \smallsetminus \mathbb{U}^n_X}(\mathbb{A}^n_X), \qquad w \longmapsto \pi_X{}^*(w) \, \star \, \kappa^{(n)}_X$$

is an isomorphism for all $i \in \mathbb{Z}$.

Proof. The affine case $X = \operatorname{Spec}(R)$ is [9, Thm. 9.3].

Since the homomorphism $\vartheta^{(n)}_X$ is given by the product with a "universal" Witt class $\kappa^{(n)}_X := [\mathbf{K}^{(n)}_X]$, we can deduce the global statement from the affine case by Mayer-Vietoris as follows. If a regular scheme

$$V_0 = V_1 \cup V_2$$

is covered by two open subschemes V_1 and V_2, we also have a covering of the corresponding affine spaces $\mathbb{A}^n$, in a compatible way with the closed subsets $\mathbb{A}^n \smallsetminus \mathbb{U}^n$. So, we get a Mayer-Vietoris long exact sequence [4] with supports, given in the first row of the diagram below, where we use the abbreviations $\mathbb{Y}^n(X) := \mathbb{A}^n_X \smallsetminus \mathbb{U}^n_X$ for all schemes X and $V_3 := V_1 \cap V_2$:

$$\cdots \to W^i_{\mathbb{Y}^n(V_1)}(\mathbb{A}^n_{V_1}) \oplus W^i_{\mathbb{Y}^n(V_2)}(\mathbb{A}^n_{V_2}) \longrightarrow W^i_{\mathbb{Y}^n(V_3)}(\mathbb{A}^n_{V_3}) \xrightarrow{\ \partial\ } W^{i+1}_{\mathbb{Y}^n(V_0)}(\mathbb{A}^n_{V_0}) \to \cdots$$

$$\uparrow{\vartheta^{(n)}_{V_1} \oplus \vartheta^{(n)}_{V_2}} \qquad\qquad \uparrow{\vartheta^{(n)}_{V_3}} \qquad \ddagger \qquad \uparrow{\vartheta^{(n)}_{V_0}}$$

$$\cdots \longrightarrow W^{i-n}(V_1) \oplus W^{i-n}(V_2) \longrightarrow W^{i-n}(V_3) \xrightarrow[(-1)^n \partial]{} W^{i-n+1}(V_0) \to \cdots$$

The second line is exact by the Mayer-Vietoris exact sequence for $V_0 = V_1 \cup V_2$. We claim that the diagram commutes. This is easy to check for the unmarked squares. For the square marked ($\ddagger$) this follows from the following calculation. Let $w \in W^{i-n}(V_3)$, then

$$\partial\Big(\pi^*_{V_3}(w) \star [\mathbf{K}^{(n)}_{V_3}]\Big) \; = \; \partial\Big(\pi^*_{V_3}(w) \star [\mathbf{K}^{(n)}_{V_0}]\Big) \; = \; (-1)^n \partial\big(\pi^*_{V_3}(w)\big) \star [\mathbf{K}^{(n)}_{V_0}]$$

The first equality holds without ∂ and uses the following fact: before computing the product of the class $[\mathbf{K}^{(n)}_{V_0}] \in W^n_{\mathbb{Y}^n(V_0)}(\mathbb{A}^n_{V_0})$ with $\pi^*_{V_3}(w) \in W^{i-n}(\mathbb{A}^n_{V_3})$, we can as well restrict $[\mathbf{K}^{(n)}_{V_0}]$ to $\mathbb{A}^n_{V_3}$ and then multiply; this restriction of $[\mathbf{K}^{(n)}_{V_0}]$ is precisely $[\mathbf{K}^{(n)}_{V_3}]$ by Remark 5.4. The second equality is a consequence of (6). It then suffices to apply naturality of the localization sequence to replace $\partial\big(\pi^*_{V_3}(w)\big)$ by $\pi^*_{V_0}\big(\partial(w)\big)$ and we have the claimed commutativity of ($\ddagger$). The usual Five-Lemma gives the statement by induction on the number of open subschemes in an affine covering of the regular scheme X (recall Convention 1.3). $\qquad\square$

Remark 7.3. We do not now whether $\vartheta^{(n)}_X$ is an isomorphism for more general schemes X, like *e.g.* regular (affine) schemes of infinite Krull dimension. The proof of [9, Thm. 9.3] uses coherent Witt groups and therefore only applies to regular rings of finite Krull dimension.

Theorem 7.4. *Let X be a $\mathbb{Z}[1/2]$-scheme. Let $1 \leq i \leq n$ be an integer and consider the X-point $\gamma_i : X \to \mathbb{U}^n_X \subset \mathbb{A}^n_X$ corresponding to $T_i = 1$ and $T_j = 0$ for all $j \neq i$.*

If $n \geq 2$, then the evaluation at this point of the Witt class $\varepsilon_X^{(n)} \in \mathrm{W}^r(\mathbb{U}_X^n)$ of the half-Koszul space is zero

$$\gamma_i^*(\varepsilon_X^{(n)}) = 0 \quad in \quad \mathrm{W}^r(X) \,.$$

Proof. Also denote by $\gamma_i : \mathrm{Spec}(\mathbb{Z}[1/2]) \longrightarrow \mathbb{U}_{\mathbb{Z}[1/2]}^n$ the "same" point. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{W}^r(\mathbb{U}_{\mathbb{Z}[1/2]}^n) & \xrightarrow{\gamma_i^*} & \mathrm{W}^r(\mathbb{Z}[1/2]) \\
{\scriptstyle \upsilon_X^*} \downarrow & & \downarrow \\
\mathrm{W}^r(\mathbb{U}_X^n) & \xrightarrow[\gamma_i^*]{} & \mathrm{W}^r(X)
\end{array}
$$

with the obvious morphisms and we know from 6.6 that $\upsilon_X^*(\varepsilon_{\mathbb{Z}[1/2]}^{(n)}) = \varepsilon_X^{(n)}$. Therefore, it suffices to prove the result for $\mathbb{Z}[1/2]$.

If $r \neq 0$ this is trivially true because in this case $\mathrm{W}^r(\mathbb{Z}[1/2]) = 0$. In fact, since $\mathbb{Z}[1/2]$ is a Dedekind domain, we have by [6, Thm. 10.1] that $\mathrm{W}^{-1}(\mathbb{Z}[1/2]) = \mathrm{W}^2(\mathbb{Z}[1/2]) = 0$ and $\mathrm{W}^1(\mathbb{Z}[1/2]) \simeq \mathrm{Coker}\left(\mathrm{W}(\mathbb{Z}[1/2]) \xrightarrow{\sum \partial_p} \bigoplus_{p \neq 2} \mathrm{W}(\mathbb{Z}/\mathbb{Z}p)\right)$, where ∂_p is a second residue homomorphism associated with the prime number p. But this cokernel is also zero by the classical calculation of the Witt group of $\mathbb{Q}$, *cf. e.g.* [14, Thm. VI.6.11].

If $r = 0$, *i.e.* $n = 2\ell + 1$ with $\ell \neq 0$ even this follows from the following lemma. $\square$

Lemma 7.5. *Let R be a ring and $n = 2\ell + 1$ with $\ell \geq 2$ even. Then $\gamma_i^*(\mathbf{B}_R^{(n)}) \simeq \gamma_i^*(\mathcal{E}_R^{(n)}, \varphi_R^{(n)})$ is trivial in $\mathrm{W}(R) \simeq \mathrm{W}^0(R)$.*

Proof. After renumbering we may assume $i = 1$. Let $K_\bullet = K_\bullet(A, \underline{T})$ be the Koszul complex of the regular sequence $\underline{T} = (T_1, \ldots, T_n)$ over $A = R[T_1, \ldots, T_n]$, and $\Theta = \Theta(A, \underline{T}) : K_\bullet \xrightarrow{\cong} \mathcal{D}_A(K_\bullet)[n]$ the canonical symmetric n-form. Recall that the differential $d_s : K_s \longrightarrow K_{s-1}$ is then given by

$$e_{i_1} \wedge \ldots \wedge e_{i_s} \longmapsto \sum_{j=1}^s (-1)^{j-1} T_{i_j} \cdot e_{i_1} \wedge \ldots \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge \ldots \wedge e_{i_s} \,,$$

where $e_1, \ldots, e_n$ constitute a basis of A^n. We denote (*cf.* 1.4) ι_R the open immersion $\mathbb{U}_R^n \hookrightarrow \mathbb{A}_R^n$. Then we have

$$(C', \Xi') := \gamma_1^*(\mathbf{B}_R^{(n)}) = \gamma_1^* \iota_R^*(\mathbf{C}_R^{(n)})$$

and $K_\bullet' = \gamma_1^* \iota_R^*(K_\bullet)$ is the Koszul complex $K_\bullet(R, (1, 0, \ldots, 0))$ for the sequence $(1, 0, \ldots, 0) \subset R$. We denote the differential of this Koszul complex by d'. Note that this complex is split exact. The isomorphism of complexes $\Theta' := \gamma_1^* \iota_R^*(\Theta)$ is a symmetric n-form on $K_\bullet'$ and (C', Ξ') is the following symmetric 0-space:

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & K_n' & \xrightarrow{d_n'} & K_{n-1}' & \longrightarrow & \cdots & \longrightarrow & K_{\ell+2}' & \xrightarrow{d_{\ell+2}'} & K_{\ell+1}' & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \downarrow & & {\scriptstyle \epsilon_n \Theta_\ell' \cdot d_{\ell+1}'} \downarrow & & & & & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & K_{\ell+1}'^\vee & \xrightarrow[d_{\ell+2}'^\vee]{} & K_{\ell+2}'^\vee & \longrightarrow & \cdots & \xrightarrow[d_n'^\vee]{} & K_n'^\vee & \longrightarrow & 0
\end{array}
$$

$$\qquad \deg \ell \qquad\qquad\qquad\qquad\qquad \deg 0$$

over R, where we have set $K_i'^{\vee} = \mathrm{Hom}_R(K_i', R)$. We give now a direct summand E' of $K_{\ell+1}'$, such that the projection $K_{\ell+1}' \longrightarrow E'$ induces a quasi-isomorphism $C_{\bullet}' \xrightarrow{\simeq} c_0(E')$ ($c_0 : \mathrm{VB}_R \longrightarrow \mathbb{D}^b(\mathrm{VB}_R)$ the natural embedding).

The elements $v_i = 1 \otimes e_i$ $(i = 1, \ldots, n)$ are a basis of $R \otimes_A A^n = \gamma_1^* \iota^*(A^n)$, and so the exterior products $v_{i_1} \wedge \ldots \wedge v_{i_s}$ $(1 \leq i_1 < \ldots < i_s \leq n)$ are free generators of $K_s' \simeq \wedge^s R^n$. The differential d_s' acts on them as follows:

$$d_s'(v_{i_1} \wedge \ldots \wedge v_{i_s}) = \begin{cases} 0 & 2 \leq i_1 < \ldots < i_s \leq n \\ v_{i_2} \wedge \ldots \wedge v_{i_s} & 1 = i_1 < i_2 < \ldots < i_s \leq n, \end{cases}$$

therefore the R-module $E' := v_1 \wedge \left(\overset{\ell}{\bigwedge} R^n \right) \subset K_{\ell+1}'$ is isomorphic to $\mathrm{Coker}\, d_{\ell+2}'$. Hence $c_0(E') \simeq C_{\bullet}'$ because $C_{\bullet}'$ has non vanishing homology only in degree 0, and we have an isometry $c_0(E', \varphi') \simeq (C', \Xi')$, where $\varphi' := \epsilon_n(\Theta_l' \cdot d_{\ell+1}')|_{E'}$.

Consider now the following free submodule of E':

$$M' := (v_1 \wedge v_2) \wedge \left(\overset{\ell-1}{\bigwedge} R^n \right).$$

We claim that M' is a totally isotropic subspace of (E', φ'). From this the lemma follows because $\mathrm{rank}\, M' = \frac{1}{2} \mathrm{rank}\, E'$ and so (E', φ') is hyperbolic by [1, I Thm. 4.6].

To see this we use the description of Θ given in Remark 3.3. Let $\omega : \wedge^n A^n \xrightarrow{\simeq} A$ be as in this remark and $\omega' := \mathrm{id}_R \otimes \omega$. Then $\omega'(v_1 \wedge \ldots \wedge v_n) = 1$ and $\Theta_{\ell}'(x)(y) = \pm \omega'(x \wedge y)$ for all $x \in K_{\ell}'$ and $y \in K_{\ell+1}'$. If now $x, y \in M'$ then $y = v_1 \wedge v_2 \wedge y'$ and $d_{\ell+1}'(x) = v_2 \wedge x'$ for some $x', y' \in \wedge^{\ell-1} R^n$, and so $\pm \varphi'(x)(y) = \omega'(d_{\ell+1}'(x) \wedge y) = 0$ since $v_2 \wedge v_2 = 0$. $\qquad\square$

Theorem 7.6. *Let X be a regular $\mathbb{Z}[1/2]$-scheme. The composition of the connecting homomorphism with the 4-periodicity isomorphism:*

$$\mathrm{W}^r(\mathbb{U}_X^n) \xrightarrow{\ \partial\ } \mathrm{W}_{\mathbb{A}_X^n \smallsetminus \mathbb{U}_X^n}^{r+1}(\mathbb{A}_X^n) \xrightarrow[\simeq]{\ \tau^q\ } \mathrm{W}_{\mathbb{A}_X^n \smallsetminus \mathbb{U}_X^n}^n(\mathbb{A}_X^n),$$

maps the Witt class $\varepsilon_X^{(n)} \in \mathrm{W}^r(\mathbb{U}_X^n)$ of the half-Koszul space to the Witt class $\kappa_X^{(n)} \in \mathrm{W}_{\mathbb{A}_X^n \smallsetminus \mathbb{U}_X^n}^n(\mathbb{A}_X^n)$ of the Koszul space over $\mathbb{A}_X^n$.

Proof. Recall that we always have $n = 4q + r + 1$. The statement is a direct consequence of Theorem 6.4, using the definition of the connecting homomorphism ∂ via the symmetric cone 2.9 and the fact that $\varepsilon_X^{(n)} = \left[\mathbf{C}_X^{(n)}|_{\mathbb{U}_X^n} \right]$ by Definition 6.5. $\qquad\square$

Theorem 7.7. *Let X be a regular $\mathbb{Z}[1/2]$-scheme. For all $i \in \mathbb{Z}$, define the following homomorphism:*

$$\rho_X^{(n)} : \ \mathrm{W}^{i-r}(X) \longrightarrow \mathrm{W}^i(\mathbb{U}_X^n), \qquad w \longmapsto \sigma_X^*(w) \star \varepsilon_X^{(n)}.$$

Then the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{W}^{i-r}(X) & \xleftarrow[\simeq]{\ \tau^q\ } & \mathrm{W}^{i+1-n}(X) \\[4pt]
{\scriptstyle \rho_X^{(n)}} \downarrow & & \downarrow {\scriptstyle \simeq}\, {\scriptstyle \vartheta_X^{(n)}} \\[4pt]
\mathrm{W}^i(\mathbb{U}_X^n) & \xrightarrow{\ \partial\ } & \mathrm{W}_{\mathbb{A}_X^n \smallsetminus \mathbb{U}_X^n}^{i+1}(\mathbb{A}_X^n)
\end{array}
$$

for all $i \in \mathbb{Z}$, where the isomorphism $\vartheta_X^{(n)}$ is the one of Theorem 7.2 and where τ is the 4-periodicity isomorphism.

Proof. Recall of course that $1 - n + 4q = r$ by (2). We have to show that

$$\partial \rho_X^{(n)}([x]) = \vartheta_X^{(n)}([x[-2q]]) \tag{13}$$

for all $[x] \in \mathrm{W}^{i-r}(X)$. Using the fact that $\sigma_X : \mathbb{U}_X^n \longrightarrow \mathrm{Spec}\, X$ composes

$$\mathbb{U}_X^n \xrightarrow{\iota_X} \mathbb{A}_X^n \xrightarrow{\pi_X} \mathrm{Spec}\, X\,,$$

we get:

$$\begin{aligned}
\partial \rho_X^{(n)}([x]) &= \pi_X^*([x]) \star \partial(\varepsilon_X^{(n)}) && \text{by equation (5)} \\
&= \pi_X^*([x]) \star \tau^{-q}(\kappa_X^{(n)}) && \text{by Theorem 6.4.}
\end{aligned}$$

But this is equal to the right hand side of (13) because:

$$\begin{aligned}
\vartheta_X^{(n)}([x[-2q]]) &= \pi_X^*([x])[-2q] \star \kappa_X^{(n)} && \text{by Lemma B.1} \\
&= \pi_X^*([x]) \star \tau^{-q}(\kappa_X^{(n)}) && \text{by Lemma B.3.}
\end{aligned}$$

$\square$

Corollary 7.8. *Let X be a regular $\mathbb{Z}[1/2]$-scheme. We have an isomorphism*

$$(\sigma_X^*, \rho_X^{(n)}) :\ \mathrm{W}^i(X) \oplus \mathrm{W}^{i-r}(X) \xrightarrow{\cong} \mathrm{W}^i(\mathbb{U}_X^n)\,.$$

for all $i \in \mathbb{Z}$.

Proof. From Theorem 7.1, it suffices to show that the homomorphism

$$\rho_X^{(n)} : \mathrm{W}^{i-r}(X) \longrightarrow \mathrm{W}^i(\mathbb{U}_X^n)$$

has the two following properties: first $\partial \cdot \rho_X^{(n)}$ is an isomorphism; secondly that $\gamma^* \cdot \rho_X^{(n)}$ is zero. The first one follows from Theorem 7.7 and the second one from the definition of $\rho_X^{(n)}$ and Theorem 7.4, since γ^* is a morphism of graded rings by [10, Thm. 3.2]. $\square$

Remark 7.9. Note that this result generalizes the well-known calculation of the Witt group of a Laurent ring (*cf. e.g.* [13]).

As for the Laurent ring it is likely that the result is true for a bigger class of rings, *e.g.* all regular rings. But already in the Laurent ring case it fails to be true for all (commutative) rings as *loc. cit.* Examples 8.1 and 8.2 show.

To understand the ring structure on $\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n)$, we need some properties of the symmetric spaces $\mathbf{K}_X^{(n)}$ and $\mathbf{B}_X^{(n)}$, which can be proven for non necessarily regular schemes as well. The case $n = 1$, *i.e.* the "Laurent scheme" case, is well-known, so we have to deal with $n \geq 2$.

Theorem 7.10. *Let X be a $\mathbb{Z}[1/2]$-scheme. If $n \geq 2$ then the symmetric r-space $\mathbf{B}_X^{(n)}$ is locally trivial, i.e. for any $x \in \mathbb{U}_X^n$ we have $[(\mathbf{B}_X^{(n)})_x] = 0$ in $\mathrm{W}^r(\mathcal{O}_{\mathbb{U}_X^n, x})$.*

Proof. Define for all $i \in \{1, \ldots, n\}$ the principal open $\mathbb{V}_X^n(i)$ of $\mathbb{A}_X^n$ given by the equation $T_i \neq 0$. Let $J \subseteq \{1, \ldots, n\} \subset \mathbb{N}$. We define

$$\mathbb{V}_X^n(J) := \bigcup_{j \in J} \mathbb{V}_X^n(j) \subseteq \mathbb{U}_X^n = \mathbb{V}_X^n(\{1, \ldots, n\})$$

and denote $\iota_J : \mathbb{V}_X^n(J) \longrightarrow \mathbb{U}_X^n$ the corresponding open immersion. Since $n \geq 2$, we can cover $\mathbb{U}_X^n$ with the open subschemes $\mathbb{V}_X^n(J)$ with $|J| \leq n - 1$. So it suffices to prove the following stronger result. $\square$

Theorem 7.11. *With the above notations, if $|J| \leq (n-1)$ then $[\mathbf{B}_X^{(n)}|_{\mathbb{V}_X^n(J)}] = 0$ in* $\mathrm{W}^r(\mathbb{V}_X^n(J))$.

Proof. We easily reduce to the case $X = \operatorname{Spec} \mathbb{Z}[1/2]$. In this case we argue as follows. For brevity we set $R := \mathbb{Z}[1/2]$.

For J empty, the result is trivial since $\mathbb{V}_R^n(J) = \varnothing$ and its Witt group is zero. So we assume that $J \neq \varnothing$. Consider the closed complement $\mathbb{Y}_R(J) := \mathbb{A}_R^n \smallsetminus \mathbb{V}_R^n(J)$ of $\mathbb{V}_R^n(J) \subset \mathbb{U}_R^n$. Note that $\mathbb{A}_R^n \smallsetminus \mathbb{U}_R^n \subset \mathbb{Y}_R(J)$. By Theorem 7.1, we have the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{W}^r(R) & \xrightarrow{\sigma_R^*} & \mathrm{W}^r(\mathbb{U}_R^n) & \xrightarrow{\partial} & \mathrm{W}_{\mathbb{A}_R^n \smallsetminus \mathbb{U}_R^n}^{r+1}(\mathbb{A}_R^n) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \iota_J^*} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{W}^r(R) & \xrightarrow{\iota_J^* \sigma_R^*} & \mathrm{W}^r(\mathbb{V}_R^n(J)) & \xrightarrow{\partial} & \mathrm{W}_{\mathbb{Y}_R(J)}^{r+1}(\mathbb{A}_R^n) & \longrightarrow & 0 \,.
\end{array}
$$

We get from the right-hand commutative square, from Theorem 7.6, and from Proposition 3.6 that $\partial\big(\iota_J^*(\varepsilon_R^{(n)})\big) = 0$. Therefore, by exactness of the above second row, there exists a unique class $w \in \mathrm{W}^r(R)$ such that $\iota_J^*(\mathbf{B}_R^{(n)}) = \iota_J^*(\sigma_R^*(w))$. In fact, $w = \gamma^*(\iota_J^*(\mathbf{B}_R^{(n)}))$ for any R-point $\gamma : \operatorname{Spec}(R) \to \mathbb{V}_R^n(J)$, which exists by assumption $J \neq \varnothing$. Choose $j \in J$ and define the R-point $\gamma : \operatorname{Spec}(R) \to \mathbb{V}_R^n(J)$ to be given by $T_j = 1$ and $T_i = 0$ for $i \neq j$. Since $w = \gamma^*(\iota_J^*(\mathbf{B}_R^{(n)})) = (\iota_J \cdot \gamma)^*(\mathbf{B}_R^{(n)}))$ and since $\iota_J \cdot \gamma : \operatorname{Spec}(R) \to \mathbb{U}_R^n$ is simply the R-point γ_j of Theorem 7.4, we conclude from it that $w = 0$. Hence $\iota_J^*(\mathbf{B}_R^{(n)}) = 0$ as wanted. $\square$

Remark 7.12. The statement of Theorem 7.10 is obviously not true for $n = 1$. The proof fails for $n = 1$ because then $\mathbb{V}_X^n(J) = \varnothing$ for any J such that $|J| \leq n - 1 = 0$ and hence we can not cover $\mathbb{U}_X^n$ with these.

It follows from this theorem above and [5, Thm. 4.2] that if $n \geq 2$ the space $\varepsilon_X^{(n)}$ is nilpotent in $\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n)$. We prove a more precise result.

Theorem 7.13. *Let X be a $\mathbb{Z}[\frac{1}{2}]$-scheme. Assume that $n \geq 2$ then*

$$
(\varepsilon_X^{(n)})^2 \;=\; \varepsilon_X^{(n)} \star \varepsilon_X^{(n)} \;=\; 0
$$

in $\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n)$. If $n = 1$ then $(\varepsilon_X^{(n)})^2 = 1$ in $\mathrm{W}^{\mathrm{tot}}(\mathbb{U}_X^n)$.

Proof. If $n = 1$ this is an obvious consequence of Example 6.8.

So let $n \geq 2$. Since $\alpha_X^* : \mathrm{W}^{\mathrm{tot}}(\mathbb{Z}[1/2]) \longrightarrow \mathrm{W}^{\mathrm{tot}}(X)$ is a morphism of graded rings (*cf.* [10, Thm. 3.2]) it is enough to prove this for the affine scheme $X = \operatorname{Spec} \mathbb{Z}[1/2]$.

Because we assume $n \geq 2$ there exists non-empty subsets $J_1, J_2 \subset \{1, \ldots, n\}$ with $J_1 \neq J_2$ and $J_1 \cup J_2 = \{1, \ldots, n\}$. We define $\mathbb{V}_{\mathbb{Z}[1/2]}^n(J_i) \subseteq \mathbb{U}_{\mathbb{Z}[1/2]}^n$ as in the proof of Theorem 7.10 above and let $\mathbb{Y}_{\mathbb{Z}[1/2]}^n(J_i) := \mathbb{U}_{\mathbb{Z}[1/2]}^n \setminus \mathbb{V}_{\mathbb{Z}[1/2]}^n(J_i)$ be the complement ($i = 1, 2$). Note that $J_1 \cup J_2 = \{1, \ldots, n\}$ implies $\mathbb{Y}_{\mathbb{Z}[1/2]}^n(J_1) \cap \mathbb{Y}_{\mathbb{Z}[1/2]}^n(J_2) = \emptyset$.

By Theorem 7.11 we know that $[\mathbf{B}_{\mathbb{Z}[1/2]}^{(n)}|_{\mathbb{V}_{\mathbb{Z}[1/2]}^n(J_i)}] = 0$ for $i = 1, 2$. Therefore by the localization sequence there exists $x_i \in \mathrm{W}_{\mathbb{Y}_{\mathbb{Z}[1/2]}^n(J_i)}^r(\mathbb{U}_{\mathbb{Z}[1/2]}^n)$ with $x_i = \varepsilon_{\mathbb{Z}[1/2]}^{(n)}$ in $\mathrm{W}^r(\mathbb{U}_{\mathbb{Z}[1/2]}^n)$ for $i = 1, 2$, and so $x_1 \star x_2 = (\varepsilon_{\mathbb{Z}[1/2]}^{(n)})^2$ in $\mathrm{W}^{2r}(\mathbb{U}_{\mathbb{Z}[1/2]}^n)$. But the space $x_1 \star x_2$ lives on a complex with support in $\mathbb{Y}_{\mathbb{Z}[1/2]}^n(J_1) \cap \mathbb{Y}_{\mathbb{Z}[1/2]}^n(J_2) = \emptyset$ and so $(\varepsilon_{\mathbb{Z}[1/2]}^{(n)})^2 = 0$. $\square$

Denote $W^{tot}(X)[\varepsilon]$ the graded skew polynomial ring in one variable ε of degree r over the graded ring $W^{tot}(X)$. Recall that this means that $c \cdot \varepsilon = (-1)^{r \deg c}(\varepsilon \cdot c)$ for a homogeneous element $c \in W^{tot}(X)$. We have a homogeneous homomorphism of graded rings given by

$$W^{tot}(X)[\varepsilon] \longrightarrow W^{tot}(\mathbb{U}_X^n) \qquad \sum_{i=0}^{m} c_i \varepsilon^i \longmapsto \sum_{i=0}^{m} \sigma_X^*(c_i) \star (\varepsilon_X^{(n)})^i .$$

Using this morphism we can restate Corollary 7.8 and Theorem 7.13 as follows

Theorem 7.14. *Let X be a regular scheme of finite Krull dimension over $\mathbb{Z}[1/2]$. Then we have an isomorphism of graded rings*

$$\left. \begin{array}{ll} \text{if } n \geq 2 : & W^{tot}(X)[\varepsilon] \Big/ (\varepsilon^2) \\[2mm] \text{if } n = 1 : & W^{tot}(X)[\varepsilon] \Big/ (\varepsilon^2 - 1) \end{array} \right\} \xrightarrow{\cong} W^{tot}(\mathbb{U}_X^n) .$$

8. Witt non-triviality of the (half) Koszul spaces

Theorem 8.1. *Let X be a scheme which is not of equicharacteristic 2. Then the Witt class of the symmetric n-space $\mathbf{K}_X^{(n)}$ is non-trivial in the Witt group with support $W_{\mathbb{A}_X^n \smallsetminus \mathbb{U}_X^n}^n (\mathbb{A}_X^n)$.*

Proof. By assumption, there is a point $x \in X$ whose residue field $k(x)$ has characteristic different from 2. By specialization at x (see Remark 5.4 for naturality), it suffices to prove the result for the regular $\mathbb{Z}[1/2]$-scheme $X := \operatorname{Spec}(k(x))$. Here, we apply Theorem 7.2 with $i := n$ and $w := 1 \in W^0(X)$, the unit of the Witt ring. $\square$

Theorem 8.2. *Let X be any scheme which is not of equicharacteristic 2. Then the Witt class $\varepsilon_X^{(n)}$ of the symmetric r-space $\mathbf{B}_X^{(n)}$ is not in the image of the natural homomorphism $W^r(\mathbb{A}_X^n) \longrightarrow W^r(\mathbb{U}_X^n)$. In particular, $\mathbf{B}_X^{(n)}$ can not be extended to the whole affine space $\mathbb{A}_X^n$.*

Proof. Again, by specialization at a point x with $\operatorname{char}(k(x)) \neq 2$, we are reduced to prove the result for the $\mathbb{Z}[1/2]$-regular scheme $X := \operatorname{Spec}(k(x))$. In this case, the following composition vanishes:

$$W^r(\mathbb{A}_X^n) \xrightarrow{\iota_X^*} W^r(\mathbb{U}_X^n) \xrightarrow{\partial} W_{\mathbb{A}_X^n \smallsetminus \mathbb{U}_X^n}^r (\mathbb{A}_X^n) .$$

Here, the connecting homomorphism ∂ is, for instance, as in Theorem 7.6, where we proved that $\partial(\varepsilon_X^{(n)})$ coincides with $[\mathbf{K}_X^{(n)}]$, up to 4-periodicity. So, $\varepsilon_X^{(n)}$ can not be extended to $\mathbb{A}_X^n$ since $[\mathbf{K}_X^{(n)}] \neq 0 \in W_{\mathbb{A}_X^n \smallsetminus \mathbb{U}_X^n}^n (\mathbb{A}_X^n)$ by Theorem 8.1. Note that we have to pass via the regular case otherwise the connecting homomorphism ∂ is not defined. $\square$

Appendix A. The locally free module $\mathcal{E}_X^{(n)}$

We use the notation of the main part of the text. We want to prove:

Theorem A.1. *Let X be a noetherian scheme and $n \geq 3$. Then there does not exist a locally free $\mathcal{O}_{\mathbb{A}_X^n}$-module $\mathcal{F}$, such that*

$$\mathcal{F}\Big|_{\mathbb{U}_X^n} \;\simeq\; \mathcal{E}_X^{(n)}\,.$$

In particular $\mathcal{E}_X^{(n)}$ is not a free $\mathcal{O}_{\mathbb{U}_X^n}$-module.

Let $x \in X$ and $\operatorname{Spec} k(x) \xrightarrow{f} X$ be the corresponding point. If there exists a locally free $\mathcal{O}_{\mathbb{A}_X^n}$-module $\mathcal{F}$, such that $\mathcal{F}|_{\mathbb{U}_X^n} \simeq \mathcal{E}_X^{(n)}$, then

$$\mathcal{E}_{k(x)}^{(n)} \simeq \upsilon_f^*(\mathcal{E}_X^{(n)}) \simeq \upsilon_f^*(\mathcal{F}|_{\mathbb{U}_X^n}) \simeq \alpha_f^*(\mathcal{F})|_{\mathbb{U}_{k(x)}^n}\,,$$

and so it is enough to show the theorem for $X = \operatorname{Spec} R$ with R a field. Similarly, localizing $R[T_1, \ldots, T_n]$ at the origin, we are reduced to the local case which follows from the following result of commutative algebra.

Theorem A.2. *Let $(A, \mathfrak{m})$ be a regular local ring, $\underline{T} = (T_1, \ldots, T_n)$ a regular system of parameters (see [7, Def. 2.2.1]), and $\mathbb{U} = \bigcup\limits_{i=1}^{n} \operatorname{Spec} A_{T_i} = \operatorname{Spec} A \setminus \{\mathfrak{m}\}$ the punctured spectrum of A. Assume that $\dim A = n \geq 3$. Then*

$$\mathcal{S}_j \;:=\; \operatorname{Ker}\big(\, K_j(A, \underline{T}) \xrightarrow{d_j(A,\underline{T})} K_{j-1}(A, \underline{T})\,\big)\Big|_{\mathbb{U}}$$

can not be extended to a free A-module if $n > j \geq 2$.

Let in the following $\mathcal{I}_j = \operatorname{Ker} d_j(A, \underline{T})$, i.e. $\mathcal{S}_j = \mathcal{I}_j|_{\mathbb{U}}$. Recall also that $(-)^\vee = \operatorname{Hom}_A(\,-\,, A)$. For the proof we need:

Proposition A.3.   (i) *Let $j \geq 2$. Then the A-module $\mathcal{I}_j$ is reflexive, i.e. the natural morphism $\operatorname{can} : \mathcal{I}_j \longrightarrow \mathcal{I}_j^{\vee\vee}$ is an isomorphism.*
 (ii) *If M and N are finitely generated A-modules, such that $M|_{\mathbb{U}} \simeq N|_{\mathbb{U}}$, and both $M|_{\mathbb{U}}$ and $N|_{\mathbb{U}}$ are locally free, then $M^\vee \simeq N^\vee$.*

Proof. By assumption $\mathcal{I}_j$ is a second Syzygie and so (i) is a consequence of [8, Thm. 3.6]. For (ii), by [11, Thm. 6.9.17] there exists $c \geq 0$, such that the given isomorphism $M|_{\mathbb{U}} \xrightarrow{\simeq} N|_{\mathbb{U}}$ is the restriction of a morphism $\mathfrak{m}^c M \longrightarrow N$. Therefore we can assume that there exists $g : M \longrightarrow N$, such that $g|_{\mathbb{U}}$ is an isomorphism, *i.e.* $\operatorname{Ker} g$ and $\operatorname{Coker} g$ have finite length.

Now we use the following fact (see [7, Thm. 1.2.8]). Since $\dim A \geq 2$ and A is regular (hence in particular Cohen-Macaulay) we have $\operatorname{Ext}_A^i(G, A) = 0$ for any finite length module G and $i = 0, 1$.

This and the exact sequences

$$0 \longrightarrow \operatorname{Ker} g \longrightarrow M \longrightarrow \operatorname{Im} g \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im} g \longrightarrow N \longrightarrow \operatorname{Coker} g \longrightarrow 0$$

give $M^\vee \simeq (\operatorname{Im} g)^\vee \simeq N^\vee$. $\square$

Proof. (of Theorem A.2) Assume that P is a free A-module, such that $P|_{\mathbb{U}} \simeq \mathcal{S}_j$. We have $\mathcal{I}_j|_{\mathbb{U}} \simeq \mathcal{S}_j$, too, and so $\mathcal{I}_j^{\vee} \simeq P^{\vee}$ by Proposition A.3 (ii). Part (i) of this proposition tells us that $\mathcal{I}_j$ is reflexive and hence $\mathcal{I}_j \simeq \mathcal{I}_j^{\vee\vee} \simeq P^{\vee\vee}$ is free. But this is impossible, because

$$\mathrm{Tor}_{n-j}^A(\mathcal{I}_j, A/\mathfrak{m}) \simeq \mathrm{Tor}_n^A(A/\mathfrak{m}, A/\mathfrak{m}) \simeq A/\mathfrak{m} \neq 0$$

(note that we need here $j < n = \dim A$). We are done. $\square$

Appendix B. The product and 4-periodicity

We have used in this work the fact that the product commutes with the translation. This has not been established in [10]. For the sake of completeness we give here a proof but refer to *loc. cit.* for unexplained notations and definitions.

To start with let $\mathcal{A}^{(0)} = (\mathcal{A}, \mathcal{D}_{\mathcal{A}}, \delta_{\mathcal{A}}, \varpi^{\mathcal{A}})$ and $\mathcal{B}^{(0)} = (\mathcal{B}, \mathcal{D}_{\mathcal{B}}, \delta_{\mathcal{B}}, \varpi^{\mathcal{B}})$ be triangulated categories with $\delta_{\mathcal{A}}$- respectively $\delta_{\mathcal{B}}$-exact duality (like *e.g.* $\mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X)$ with the usual 1-exact duality as in the main part of this work). We denote the shift functor in these triangulated categories by $\Sigma_{\mathcal{A}}$ respectively $\Sigma_{\mathcal{B}}$ (to distinguish we do not use $X \mapsto X[1]$). A symmetric i-space in $\mathcal{A}^{(0)}$ is a pair (X, ψ) consisting of an object $X \in \mathcal{A}$ and a symmetric i-form $X \xrightarrow{\psi} \Sigma_{\mathcal{A}}^i \mathcal{D}_{\mathcal{A}} X$ which is an isomorphism, the symmetry of an i-form reads $\Sigma_{\mathcal{A}}^i \mathcal{D}_{\mathcal{A}}(\psi) \cdot \varpi_X^{\mathcal{A}} = (-1)^{\frac{i(i+1)}{2}} \delta_{\mathcal{A}}^i \cdot \psi$. As in the case of derived categories if (X, ψ) is a symmetric i-form then $(\Sigma_{\mathcal{A}}^2 X, \Sigma_{\mathcal{A}}^2(\psi))$ is a symmetric $(i + 4)$-form.

Let $(F, \rho): \mathcal{A}^{(0)} \longrightarrow \mathcal{B}^{(0)}$ be a duality preserving functor, *i.e.* $\rho: F\mathcal{D}_{\mathcal{A}} \xrightarrow{\simeq} \mathcal{D}_{\mathcal{B}} F$ is an isomorphism of functors satisfying some compatibility axioms. We will only use the following. Since F is a covariant exact functor between triangulated categories there exists a family of isomorphisms of functors $\theta^{(i)}: F\Sigma_{\mathcal{A}}^i \xrightarrow{\simeq} \Sigma_{\mathcal{B}}^i F$ ($i \in \mathbb{Z}$) which are related by the following formulas:

$$\theta^{(i+j)} = \Sigma_{\mathcal{B}}^i(\theta^{(j)}) \cdot \theta^{(i)}_{\Sigma_{\mathcal{A}}^j} \tag{14}$$

$(i, j \in \mathbb{Z})$. Then we have

$$\mathcal{D}_{\mathcal{B}} \Sigma_{\mathcal{B}}^{-1}(\theta^{(1)}_{\Sigma_{\mathcal{A}}^{-1}}) \cdot \rho_{\Sigma_{\mathcal{A}}^{-1}} = (\delta_{\mathcal{A}} \delta_{\mathcal{B}}) \cdot \Sigma_{\mathcal{B}}(\rho) \cdot \theta^{(1)}_{\mathcal{D}_{\mathcal{A}}} \tag{15}$$

(*cf. loc. cit.* Definition 1.8). This axioms are made such that if (X, ψ) is a symmetric i-space in $\mathcal{A}^{(0)}$ then

$$(F, \rho)_*(X, \psi) := \left(FX, (\delta_{\mathcal{A}} \delta_{\mathcal{B}})^i \Sigma_{\mathcal{B}}^i(\rho_X) \cdot \theta^{(i)}_{\mathcal{D}_{\mathcal{A}} X} \cdot F(\psi) \right)$$

is a symmetric i-space in $\mathcal{B}^{(0)}$.

Lemma B.1. *Let (X, ψ) be a symmetric i-space in $\mathcal{A}$. Then there is an isometry*

$$(F, \rho)_*(\Sigma_{\mathcal{A}}^2 X, \Sigma_{\mathcal{A}}^2(\psi)) \xrightarrow{\simeq} \Sigma_{\mathcal{B}}^2((F, \rho)_*(X, \psi)).$$

Proof. We claim that $\theta_X^{(2)}: F\Sigma_{\mathcal{A}}^2 X \simeq \Sigma_{\mathcal{B}}^2 FX$ is an isometry, *i.e.* we have to show

$$(\delta_{\mathcal{A}} \delta_{\mathcal{B}})^{i+4} \Sigma_{\mathcal{B}}^{i+4}(\rho_{\Sigma_{\mathcal{A}}^2 X}) \cdot \theta^{(i+4)}_{\mathcal{D}_{\mathcal{A}} \Sigma_{\mathcal{A}}^2 X} \cdot F\Sigma_{\mathcal{A}}^2(\psi)$$

$$= (\delta_{\mathcal{A}} \delta_{\mathcal{B}})^i \Sigma_{\mathcal{B}}^{i+4} \mathcal{D}_{\mathcal{B}}(\theta_X^{(2)}) \cdot \Sigma_{\mathcal{B}}^{i+2}(\rho_X) \cdot \Sigma_{\mathcal{B}}^2(\theta^{(i)}_{\mathcal{D}_{\mathcal{A}} X}) \cdot \Sigma_{\mathcal{B}}^2 F(\psi) \cdot \theta_X^{(2)} \tag{16}$$

We observe first that

$$\Sigma_{\mathcal{B}}^2 F(\psi) \cdot \theta_X^{(2)} = \theta_{\Sigma_{\mathcal{A}}^i \mathcal{D}_{\mathcal{A}} X}^{(2)} \cdot F\Sigma_{\mathcal{A}}^2(\psi)$$

since $\theta^{(2)}$ is a natural transformation. By using (14) three times we get (recall that by definition $\Sigma_{\mathcal{A}}\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}}^{-1}$)

$$\theta_{\mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}}^2 X}^{(i+4)} \;=\; \Sigma_{\mathcal{B}}^{i+3}(\theta_{\mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}}^2 X}^{(1)}) \cdot \Sigma_{\mathcal{B}}^{i+2}(\theta_{\mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}} X}^{(1)}) \cdot \Sigma_{\mathcal{B}}^2(\theta_{\mathcal{D}_{\mathcal{A}} X}^{(i)}) \cdot \theta_{\Sigma_{\mathcal{A}}^i \mathcal{D}_{\mathcal{A}} X}^{(2)}$$

and so (16) is equivalent to

$$\Sigma_{\mathcal{B}}^{i+4}(\rho_{\Sigma_{\mathcal{A}}^2 X}) \cdot \Sigma_{\mathcal{B}}^{i+3}(\theta_{\mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}}^2 X}^{(1)}) \cdot \Sigma_{\mathcal{B}}^{i+2}(\theta_{\mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}} X}^{(1)}) \;=\; \Sigma_{\mathcal{B}}^{i+4}\mathcal{D}_{\mathcal{B}}(\theta_X^{(2)}) \cdot \Sigma_{\mathcal{B}}^{i+2}(\rho_X)\,.$$

Hence the result follows from the following calculation:

$$\Sigma_{\mathcal{B}}^{i+3}(\Sigma_{\mathcal{B}}(\rho_{\Sigma_{\mathcal{A}}^2 X}) \cdot \theta_{\mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}}^2 X}^{(1)}) \cdot \Sigma_{\mathcal{B}}^{i+2}(\theta_{\mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}} X}^{(1)})$$

$$= (\delta_{\mathcal{A}}\delta_{\mathcal{A}})\, \Sigma_{\mathcal{B}}^{i+3}(\mathcal{D}_{\mathcal{B}}\Sigma_{\mathcal{B}}^{-1}(\theta_{\Sigma_{\mathcal{A}} X}^{(1)}) \cdot \rho_{\Sigma_{\mathcal{A}} X}) \cdot \Sigma_{\mathcal{B}}^{i+2}(\theta_{\mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}} X}^{(1)}) \quad \text{by Eq. (15)}$$

$$= (\delta_{\mathcal{A}}\delta_{\mathcal{A}})\, \Sigma_{\mathcal{B}}^{i+4}(\mathcal{D}_{\mathcal{B}}\theta_{\Sigma_{\mathcal{A}} X}^{(1)}) \cdot \Sigma_{\mathcal{B}}^{i+2}(\Sigma_{\mathcal{B}}(\rho_{\Sigma_{\mathcal{A}} X}) \cdot \theta_{\mathcal{D}_{\mathcal{A}}\Sigma_{\mathcal{A}} X}^{(1)})$$

$$= \Sigma_{\mathcal{B}}^{i+4}\mathcal{D}_{\mathcal{B}}(\Sigma_{\mathcal{B}}\theta_X^{(1)} \cdot \theta_{\Sigma_{\mathcal{A}} X}^{(1)}) \cdot \Sigma_{\mathcal{B}}^{i+2}(\rho_X) \qquad\qquad \text{by Eq. (15)}$$

$$= \Sigma_{\mathcal{B}}^{i+4}\mathcal{D}_{\mathcal{B}}(\theta_X^{(2)}) \cdot \Sigma_{\mathcal{B}}^{i+2}(\rho_X)$$

since $\theta_X^{(2)} = \Sigma_{\mathcal{B}}\theta_X^{(1)} \cdot \theta_{\Sigma_{\mathcal{A}} X}^{(1)}$ by Eq. (14). $\qquad\qquad\qquad\square$

Assume now we have a third triangulated category with duality, say $\mathcal{C}^{(0)} = (\mathcal{C}, \mathcal{D}_{\mathcal{C}}, \delta_{\mathcal{C}}, \varpi^{\mathcal{C}})$, and a dualizing pairing (*loc. cit.* Definition 1.11)

$$\boxtimes \;:\; \mathcal{A}^{(0)} \times \mathcal{B}^{(0)} \;\longrightarrow\; \mathcal{C}^{(0)}\,.$$

Example B.2. Let X be a scheme and $Z \subseteq X$ a closed subset. Then the (derived) tensor product

$$\otimes_{\mathcal{O}_X} \;:\; \mathbb{D}^{\mathrm{b}}(\mathrm{VB}_X) \times \mathbb{D}_Z^{\mathrm{b}}(\mathrm{VB}_X) \;\longrightarrow\; \mathbb{D}_Z^{\mathrm{b}}(\mathrm{VB}_X)$$

is a dualizing pairing. Note that in this case $\delta_{\mathcal{A}} = \delta_{\mathcal{B}} = \delta_{\mathcal{C}} = 1$.

Let (X, ψ) be a symmetric i-space in $\mathcal{A}^{(0)}$ and (Y, ϕ) a symmetric j-space in $\mathcal{B}^{(0)}$. The left product $(X, \psi) \star_l (Y, \phi)$ is then defined by considering $X \boxtimes -$ as duality preserving functor with the aid of a duality transformation $\mathcal{L}(\psi)$ which depends on ψ, i.e. the left product $(X, \psi) \star_l (Y, \phi)$ of these spaces is by definition $(X \boxtimes -, \mathcal{L}(\psi))_*(Y, \phi)$. The right product $\star_r$ is defined analogous by making the functor $- \boxtimes Y$ duality preserving using the symmetric j-form ϕ. Both products are related by the following isometry:

$$(X, \psi) \star_l (Y, \phi) \;\simeq\; (\delta_{\mathcal{A}}\delta_{\mathcal{C}})^j \cdot (\delta_{\mathcal{B}}\delta_{\mathcal{C}})^i \cdot (-1)^{ij} \cdot \big((X, \psi) \star_r (Y, \phi)\big)\,. \qquad (17)$$

(*loc. cit.* Theorem 2.9). From this we easily deduce

Lemma B.3. *There is an isometry*

$$(\Sigma_{\mathcal{A}}^2 X, \Sigma_{\mathcal{A}}^2(\psi)) \star_l (Y, \phi) \;\simeq\; (X, \psi) \star_l (\Sigma_{\mathcal{B}}^2 Y, \Sigma_{\mathcal{B}}^2(\phi))\,,$$

and the same is true for the right product.

Proof. From Lemma B.1 we get isometries $(X, \psi) \star_l (\Sigma_{\mathcal{B}}^2 Y, \Sigma_{\mathcal{B}}^2(\phi)) \simeq \Sigma_{\mathcal{C}}^2 \big((X, \psi) \star_l (Y, \phi)\big)$ and $(\Sigma_{\mathcal{A}}^2 X, \Sigma_{\mathcal{A}}^2(\psi)) \star_r (Y, \phi) \simeq \Sigma_{\mathcal{C}}^2 \big((X, \psi) \star_r (Y, \phi)\big)$. Hence the theorem follows by applying (17) twice. $\square$

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