Isotropy of quadratic forms over the function field of a quadric in characteristic 2

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Abstract

We extend to characteristic 2 a theorem by the first author which states that if φ and ψ are anisotropic quadratic forms over a field F such that $\dim\varphi \leq 2^n < \dim\psi$ for some nonnegative integer n, then φ stays anisotropic over the function field $F(\psi)$ of ψ. The case of singular forms is systematically included. We give applications to the characterization of quadratic forms with maximal splitting. We also prove a characteristic 2 version of a theorem by Izhboldin on the isotropy of φ over $F(\psi)$ in the case $\dim\varphi = 2^n + 1 \leq \dim\psi$.

Key words: Quadratic forms, function field of a quadratic form, Pfister forms, Pfister neighbors, dominated quadratic forms, standard splitting of a quadratic form, maximal splitting.
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1 Introduction

An important problem in the algebraic theory of quadratic forms over a field F is to classify anisotropic quadratic forms ψ for which a given anisotropic quadratic form φ becomes isotropic over $F(\psi)$, the function field of ψ.

In characteristic different from 2, the first author proved a general result on this problem which asserts that φ stays anisotropic over $F(\psi)$ if $\dim\varphi \leq 2^n < \dim\psi$ for some integer $n \geq 1$ [9, Th. 1]. A partial generalization of this result to characteristic 2 was given by the second author and Mammone [20], more precisely it was shown that φ stays anisotropic over $F(\psi)$ in the case $\dim\varphi + \dim \mathrm{ql}(\varphi) \leq 2^n < \dim\psi$, where $\mathrm{ql}(\varphi)$ denotes the quasilinear part of φ. In particular, this gives a complete generalization in the case of nonsingular quadratic forms, i.e. quadratic forms φ with $\dim \mathrm{ql}(\varphi) = 0$. Recall that in characteristic different from 2, the result [9, Th. 1] is also a consequence of Rost's degree formula [23], or of the very general results by Karpenko [15] on the first Witt index of quadratic forms and by Karpenko and Merkurjev on the essential dimension of quadrics [16].

The main result of this paper is the following theorem which completely extends [9, Th. 1] to characteristic 2, and thus gives a positive answer to the question asked at the end of [20]:

Theorem 1.1 *Let F be a field of characteristic 2. Let φ, ψ be anisotropic quadratic forms over F (possibly singular) such that $\dim\varphi \leq 2^n < \dim\psi$ for some integer $n \geq 1$. Then, φ stays anisotropic over $F(\psi)$.*

One of the consequences of Theorem 1.1 is that an anisotropic quadratic form φ of dimension $2^n + m$ with $0 < m \leq 2^n$ satisfies $i_t(\varphi_{F(\varphi)}) \leq m$, where $i_t(\psi)$ denotes the total index of a quadratic form ψ, i.e. the maximal dimension of a totally isotropic subspace inside the underlying vector space of ψ (cf. also Definition 2.4 and Lemma 4.1). When $i_t(\varphi_{F(\varphi)}) = m$ we say that φ has maximal splitting. Examples of such quadratic forms are Pfister neighbors and any quadratic form φ with $\dim\varphi = 2^n+1$ for some integer $n \geq 0$ (cf. Section 4 for more details). Our main result on quadratic forms with maximal splitting is the following theorem:

Theorem 1.2 *Let F be a field of characteristic 2. Let ψ be an anisotropic not totally singular quadratic form over F such that $\dim\psi \leq 2^{n+1}$ and such that one of the following conditions holds:*

(1) $\dim\psi \geq 2^{n+1}-2$ *with* $n \geq 2$*;*
(2) $\dim\psi = 2^{n+1}-3$ *with* $n \geq 2$ *if* $\dim \mathrm{ql}(\psi) = 1$*, and* $n \geq 3$ *otherwise;*
(3) $\dim\psi = 2^{n+1}-4$ *with* $n \geq 3$*;*
(4) $\dim\psi = 2^{n+1}-5$ *with* $n \geq 3$ *and (*$\dim \mathrm{ql}(\psi) = 1$ *or* $\mathrm{ql}(\psi)$ *is similar to* $\langle 1, a, b, ab\rangle \perp \langle c\rangle$ *for some* $a, b, c \in F^*$*);*
(5) $\dim\psi = 2^{n+1}-6$ *and* $\psi \in I^2W_q(F)$ *with* $n \geq 3$.

If ψ *has maximal splitting, then it is a Pfister neighbor.*

We also generalize to characteristic 2 a theorem by Izhboldin [12, Th. 0.2] except in the case of totally singular quadratic forms:

Theorem 1.3 *Let* F *be a field of characteristic* 2*, and let* φ*,* ψ *be anisotropic quadratic forms over* F *such that* φ *is not totally singular and* $\dim\varphi = 2^n+1$*. If* $\dim\psi > 2^n$ *and* $\varphi_{F(\psi)}$ *is isotropic, then* ψ *is not totally singular and* $\psi_{F(\varphi)}$ *is also isotropic.*

Again, it should be remarked that in characteristic not 2, Izhboldin's theorem can easily be deduced from Karpenko and Merkurjev's results in [15], [16]. It would be very interesting to know in how far their methods and results carry over to the case of characteristic 2 and possibly singular forms.

We combine Theorems 1.2, 1.3 and Proposition 4.6 to get:

Corollary 1.4 *Let* F *be field of characteristic* 2*, and let* φ*,* ψ *be anisotropic quadratic forms over* F *such that* φ *is not totally singular,* $\dim\varphi = 2^n+1$ *for some integer* $n \geq 1$*, and* ψ *satisfies one of the conditions (1)–(5) of Theorem 1.2 for this integer* n*. If* $\varphi_{F(\psi)}$ *is isotropic, then* φ *and* ψ *are Pfister neighbors of the same* $(n+1)$*-fold Pfister form.*

This paper is organized as follows. The next section is devoted to defining the notions and providing some preliminary results which we will need in the proofs. This includes Witt decomposition of a quadratic form, the domination relation (which is introduced since we don't exclude singular forms), the analogue of the Cassels-Pfister subform theorem, some facts about Pfister neighbors and the standard splitting of a quadratic form. Most of this can also be found in this or similar form in earlier articles by the authors [18], [19], [10], but since many facts in characteristic 2 are not too widely known and in an attempt to keep the paper as self-contained as possible, we decided to give a fairly detailed account of the notions, facts, and preliminary results which we will use throughout the paper.

Section 3 is devoted to the proofs of Theorem 1.1 and Theorem 1.3.

Before proving Theorem 1.1 at the end of subsection 3.1, we first provide the main ingredient (Proposition 3.1) for the proof of the theorem. This proposition is based on the idea that an anisotropic quadratic form becomes dominated by a Pfister form after extending scalars to a suitable extension of the ground field. This was used originally by the first author in his proof of [9, Th. 1]. However, in our case we have to apply special care as we include singular quadratic forms without any further hypotheses on their quasilinear parts, contrary to what was made in [20], and thus some known results in characteristic different from 2, like Witt cancellation, cannot be carried over to characteristic 2. To avoid this difficulty we use a new idea from [10] called "completion lemma" (cf. subsection 2.4) which allows us to extend an isometry between singular forms to one between suitable nonsingular forms which dominate them.

The proof of 1.3 in subsection 3.2 is based on the same idea already employed by Izhboldin [12] in characteristic $\neq 2$. More precisely, we will show and use the fact that an anisotropic not totally singular quadratic form φ of dimension $2^n + 1$ will be dominated by an anisotropic $(n+1)$-fold Pfister form after extending scalars to a suitable field extension E/F with the property that $E(\varphi)/F(\varphi)$ is unirational.

Various general and rather straightforward results concerning quadratic forms with maximal splitting are given in section 4.

In section 5, we prove Theorem 1.2. This is done case by case, using an important result on Pfister neighbors (due to Knebusch in characteristic $\neq 2$), and a result which extends to characteristic 2 a theorem due to Fitzgerald ([10, Th. 6.6], [10, Th. 5.1, Cor. 5.3], see Theorem 2.19 in the present paper), some results on division algebras like the index reduction theorem, some descent techniques developed before by Kahn [14], the completion lemma when the dimension of the quasi-linear part is ≥ 1, and the excellence property of an extension given by the function of a 2-fold Pfister form.

2 Preliminaries

From now on, we assume that F is of characteristic 2.

2.1 Notations and definitions

A quadratic form over F of dimension n is a pair (V, φ) where V is an F-vector space of dimension n and $\varphi : V \longrightarrow F$ is a map satisfying

(1) $\varphi(\alpha v) = \alpha^2 \varphi(v)$ for any $\alpha \in F$ and $v \in V$.

(2) The map $B_\varphi : V \times V \longrightarrow F$ given by $B_\varphi(v,w) = \varphi(v+w) - \varphi(v) - \varphi(w)$ is bilinear (symmetric).

The *radical* of B_φ is the F-vector space $\mathrm{rad}(B_\varphi) = \{v \in V \mid B_\varphi(v,V) = 0\}$. Since B_φ is alternating, i.e. $B_\varphi(v,v) = 0$ for any $v \in V$, the integer $n - \dim_F \mathrm{rad}(B_\varphi)$ is even. If we set $\dim_F \mathrm{rad}(B_\varphi) = s$ and $n - s = 2r$, then after a choice of an F-basis of V, the quadratic form φ can be written up to isometry:

$$\varphi \simeq [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \langle c_1 \rangle \perp \cdots \perp \langle c_s \rangle \ ,$$

where $[a,b]$ (resp. $\langle c \rangle$) denotes the quadratic form $ax^2 + xy + by^2$ (resp. the quadratic form cx^2). The quadratic form $\langle c_1 \rangle \perp \cdots \perp \langle c_s \rangle$ is nothing but the restriction of φ to its radical. It is therefore unique up to isometry, and thus the pair (r,s) is also unique. We call this form (resp. the pair (r,s)) the *quasilinear part* of φ and denote it by $\mathrm{ql}(\varphi)$ (resp. the *type* of φ).

To shorten notations, we will write $\langle c_1, \cdots, c_s \rangle$ instead of $\langle c_1 \rangle \perp \cdots \perp \langle c_s \rangle$.

Definition 2.1 A quadratic form φ is called:

(1) *nonsingular* if $\mathrm{ql}(\varphi) = 0$, and *singular* otherwise;
(2) *totally singular* if $\varphi = \mathrm{ql}(\varphi)$;
(3) *nondefective* if it is nonsingular or singular with anisotropic quasilinear part.

Let $W_q(F)$ (resp. $W(F)$) denote the Witt group of nonsingular quadratic forms (resp. the Witt ring of regular symmetric bilinear forms).

For a quadratic form φ, we denote by $C(\varphi)$ its Clifford algebra. If φ is nonsingular, then $C(\varphi)$ is a central simple algebra over F which, up to isomorphism, only depends on the isometry class of φ, and the center $Z(\varphi)$ of the even Clifford algebra $C_0(\varphi)$ is a separable quadratic algebra over F, i.e. there exists $\delta \in F$ with $Z(\varphi) = F[x]/(x^2 + x + \delta)$. If we put $\wp(a) = a^2 + a$, then $\wp(F) = \{\wp(a) \mid a \in F\}$ is an additive subgroup of F, and the class of δ modulo $\wp(F)$ is an invariant of the isometry class of φ called the *Arf-invariant* of φ and denoted by $\Delta(\varphi)$. More explicitly, if $\varphi \simeq a_1[1,b_1] \perp \cdots \perp a_n[1,b_n]$, then $\Delta(\varphi) = b_1 + \cdots + b_n \in F/\wp(F)$, and $C(\varphi) = [b_1, a_1) \otimes_F \cdots \otimes_F [b_n, a_n)$, where $[b,a)$ ($a \in F^*$, $b \in F$) denotes the quaternion F-algebra generated by two elements u, v over F subject to the relations: $u^2 = a$, $v^2 + v = b$, $uv = (v+1)u$. By abuse of notation, we will often identify $\Delta(\varphi)$ with a representative in F of its class in $F/\wp(F)$.

For a quadratic form φ over F and a field extension K/F, we denote by $D_F(\varphi)$ the set of scalars in F^* represented by φ, and by φ_K the quadratic form $\varphi \otimes K$. Two quadratic forms φ and ψ are called similar if $\varphi \simeq a\psi$ for some scalar $a \in F^*$.

2.2 Witt decomposition and Witt equivalence

For an integer $n \geq 0$ and a quadratic form φ, we denote by $n \times \varphi$ the quadratic form $\underbrace{\varphi \perp \cdots \perp \varphi}_{n \text{ times}}$. A quadratic form φ is called split if $\varphi \simeq r \times \mathbb{H} \perp s \times \langle 0 \rangle$ where $\mathbb{H} = [0,0]$ is the hyperbolic plane. A hyperbolic form is a split nonsingular quadratic form.

A basic notion that we will use in the case of singular forms is the existence of the anisotropic part:

Proposition 2.2 *([10, Prop. 2.4]) If φ is a quadratic form which is not split, then there exists up to isometry a unique anisotropic quadratic form φ_{an} such that*

$$\varphi \simeq \varphi_{\mathrm{an}} \perp r \times \mathbb{H} \perp s \times \langle 0 \rangle \, .$$

This proposition is a consequence of Witt cancellation concerning nonsingular forms and split totally singular forms:

Proposition 2.3 *([17, Prop. 1.2] for (1); [10, Lem. 2.6] for (2)) Let ψ and ψ' be two quadratic forms (singular or not) such that* $\dim \psi = \dim \psi'$. *Assume that one of the following conditions holds:*

(1) $\varphi \perp \psi \simeq \varphi \perp \psi'$ for some $\varphi \in W_q(F)$;
(2) ψ and ψ' are nondefective and $\psi \perp s \times \langle 0 \rangle \simeq \psi' \perp s \times \langle 0 \rangle$ for some integer $s \geq 0$.

Then, $\psi \simeq \psi'$.

Definition 2.4 With the same notations as in Proposition 2.2, we call

(1) φ_{an} the *anisotropic part* of φ;
(2) $r \times \mathbb{H} \perp s \times \langle 0 \rangle$ the *split part* of φ;
(3) $\varphi_{\mathrm{an}} \perp r \times \mathbb{H}$ the *nondefective part* of φ, denoted by φ_{nd};
(4) r (resp. s) the *Witt index* of φ, denoted by $i_W(\varphi)$ (resp. the *defect index* or simply *defect* of φ, denoted by $i_d(\varphi)$);
(5) $i_W(\varphi) + i_d(\varphi)$ the *total index* of φ, denoted by $i_t(\varphi)$.

Furthermore, we call two forms φ and ψ *Witt-equivalent*, $\varphi \sim \psi$, if $\varphi_{\mathrm{an}} \simeq \psi_{\mathrm{an}}$ (cf. [10]).

Remark 2.5 (i) One readily verifies that $i_t(\varphi)$ is in fact the dimension of a maximal totally isotropic subspace of the underlying vector space V of φ, where a subspace $W \subset V$ is said to be totally isotropic if $\varphi(x) = 0$ for all $x \in W$.

(ii) If φ and ψ are forms with $i_d(\varphi) = i_d(\psi)$ (so in particular if both forms are nondefective), then $\varphi \sim \psi$ iff $\varphi \perp m \times \mathbb{H} \simeq \psi \perp n \times \mathbb{H}$ for some nonnegative integers m, n. $\square$

2.3 The function field of a quadratic form

Let φ be a nonzero quadratic form over F with underlying vector space V of dimension $n \geq 1$, and let $P_\varphi \in F[X_1, \cdots, X_n]$ be the homogeneous polynomial of degree 2 corresponding to φ after a choice of a basis of V. The polynomial P_φ is reducible if and only if φ_{nd} is either of type $(0,1)$ or of type $(1,0)$ and $\varphi_{\mathrm{nd}} \simeq \mathbb{H}$ [22, Prop. 3]. When P_φ is irreducible, we define the function field $F(\varphi)$ of φ to be the quotient field of $F[X_1, \cdots, X_n]/(P_\varphi)$. We set $F(\varphi) = F$ if P_φ is reducible or $\varphi = 0$. The extension $F(\varphi)/F$ is purely transcendental if and only if φ_{nd} is isotropic (i.e. $i_W(\varphi) \geq 1$). The polynomial P_φ is absolutely irreducible if and only if φ is of type (r, s) with $r \geq 1$ and $\dim \varphi_{\mathrm{nd}} \geq 3$ [1]. If φ is of this shape, then we will say that φ is *nice*. Therefore, a nice form φ over F has the property that $\dim(\varphi_K)_{\mathrm{nd}} \geq 3$ for any field extension K/F, i.e. the property of being a nice form is invariant under field extensions, and the free compositum $K \cdot F(\varphi)$ can be identified with $K(\varphi)$, the function field of φ_K over K. We also note that $F(\varphi)$ can in an obvious way be considered as a purely transcendental extension of transcendence degree $i_d(\varphi)$ over $F(\varphi_{\mathrm{nd}})$.

The function field of a regular symmetric bilinear form B is defined as the function field of the totally singular form $v \mapsto B(v, v)$.

Proposition 2.6 *([18, Cor. 3.3]) Let φ and ψ be anisotropic quadratic forms over F. If φ is totally singular and ψ is not totally singular, then $\varphi_{F(\psi)}$ is anisotropic.*

As a corollary we get:

Corollary 2.7 *Let φ, ψ be anisotropic quadratic forms over F.*

(1) If both φ and ψ are not totally singular, and if $\varphi_{F(\psi)}$ is isotropic, then $i_W(\varphi_{F(\psi)}) \geq 1$.

(2)

$$i_t(\varphi_{F(\varphi)}) = \begin{cases} i_W(\varphi_{F(\varphi)}) & \textit{if } \varphi \textit{ is not totally singular,} \\ i_d(\varphi_{F(\varphi)}) & \textit{otherwise.} \end{cases}$$

Proof. (1) If $\mathrm{ql}(\varphi) = 0$, then there is nothing to prove. If not, Proposition 2.6 implies that $\mathrm{ql}(\varphi)_{F(\psi)}$ is anisotropic. By the uniqueness of the quasilinear part, we necessarily have $i_W(\varphi_{F(\psi)}) \geq 1$.

(2) is a direct consequence of Proposition 2.6 and the uniqueness of the quasilinear part. □

2.4 The domination relation and the subform theorem

The subform relation refers to one form being isometric to an orthogonal summand of another, and it is an important tool in the algebraic theory of quadratic forms in characteristic $\neq 2$, but it turns out to be too restrictive in characteristic 2 if one wants to study also forms which are singular. It therefore becomes necessary to introduce what we call the domination relation which basically says that one form is simply the restriction to a subspace of the underlyinhg vector space of another form.

Definition 2.8 *Let φ and ψ be quadratic forms over F with underlying vector spaces V and W, respectively.*

(1) φ is a subform of ψ, denoted by $\varphi \subset \psi$, if $\psi \simeq \varphi \perp \varphi'$ for some quadratic form φ'.
(2) φ is dominated by ψ, denoted by $\varphi \prec \psi$, if there exists an injective isometry $t : (\varphi, V) \longrightarrow (\psi, W)$ (i.e. t is an injective F-linear map $V \rightarrow W$ with $\psi(t(v)) = \varphi(v)$ for all $v \in V$).
(3) ψ is a nonsingular completion of φ if ψ is nonsingular, $\varphi \prec \psi$ and $\dim \psi = 2r + 2s$ where (r, s) is the type of φ.
(4) φ is weakly dominated by ψ if $a\varphi \prec \psi$ for some $a \in F^*$.

In the following lemma we give an equivalent description of the domination relation.

Lemma 2.9 *([10, Lem. 3.1]) With the same notation as in Definition 2.8, the following are equivalent:*

(1) $\varphi \prec \psi$;
(2) There exist nonsingular forms φ_r and ψ', nonnegative integers $s' \leq s \leq s''$, $c_i \in F$ $(1 \leq i \leq s'')$ and $d_j \in F$ $(1 \leq j \leq s')$ such that $\varphi \simeq \varphi_r \perp \langle c_1, \cdots, c_s \rangle$ and

$$\psi \simeq \varphi_r \perp \psi' \perp [c_1, d_1] \perp \cdots \perp [c_{s'}, d_{s'}] \perp \langle c_{s'+1}, \cdots, c_{s''} \rangle .$$

Remark 2.10 (1) Clearly, if $\varphi \prec \psi$, $\dim \varphi \geq 2$, then $\psi_{F(\varphi)}$ is isotropic.

(2) Lemma 2.9 implies that if $\varphi \prec \psi$ and $\varphi \in W_q(F)$, then $\varphi \subset \psi$.

(3) As was mentioned in [10, Rem. 3.2], the form φ_r in Lemma 2.9 can be any nonsingular form with the property that $\varphi \simeq \varphi_r \perp \mathrm{ql}(\varphi)$. $\square$

The following lemma is well-known in characteristic different from 2 for subforms (cf. [9, Lem. 3]).

Lemma 2.11 *Let φ and ψ be quadratic forms such that $i_t(\varphi) \geq 1$ and $\psi \prec \varphi$. If $\dim \psi \geq \dim \varphi - i_t(\varphi) + 1$, then ψ is isotropic.*

Proof. Let V be the underlying vector space of φ. We may assume that ψ is given by the restriction of φ to some subspace U of V with $\dim U = \dim \psi$. By assumption, V contains a totally isotropic subspace W of dimension $i_t(\varphi)$, and also $\dim U + \dim W > \dim V$. Hence U intersects W non trivially, and thus ψ is isotropic. $\square$

The next lemma will be used frequently in our proofs:

Lemma 2.12 *Let S be a totally singular form of dimension s.*

(1) If S' is any nonsingular completion of S, then $S \perp S' \simeq S \perp s \times \mathbb{H}$.
(2) $s \times \mathbb{H}$ is a nonsingular completion of S.

Proof. We use the facts that $\langle c \rangle \perp [c, d] \simeq \langle c \rangle \perp \mathbb{H}$ and $\langle c \rangle \prec [c, 0] \simeq \mathbb{H}$ for any $c, d \in F$. $\square$

The analogue of the Cassels-Pfister subform theorem is as follows:

Proposition 2.13 *([18], [10, Th. 4.2]) Let φ, ψ be quadratic forms over F such that φ is nonsingular anisotropic, and ψ is nondefective (singular or not). If $\varphi_{F(\psi)}$ is hyperbolic, then $\alpha\beta\psi \prec \varphi$ for each $\alpha \in D_F(\psi)$ and $\beta \in D_F(\varphi)$. In particular, $\dim \varphi \geq \dim \psi$.*

The following result is very useful in the proof of Theorem 1.1 since it allows us to avoid the use of Witt cancellation which generally does not hold for singular forms.

Completion Lemma 2.14 *([10, Cor.3.10]) Let φ, ψ be nonsingular quadratic forms over F such that $\dim \varphi = \dim \psi$ and $\varphi \perp \sigma \perp \tau \simeq \psi \perp \sigma \perp \tau$ for some totally singular forms σ and τ. Let ρ be a nonsingular completion of σ. Then, there exists a nonsingular completion ρ' of σ such that $\varphi \perp \rho \perp \tau \simeq \psi \perp \rho' \perp \tau$.*

Corollary 2.15 *Let φ, ψ be nonsingular quadratic forms over F, and let σ be a totally singular form such that $\varphi \perp \sigma \simeq (\dim \sigma) \times \mathbb{H} \perp \psi \perp \sigma$. Then, $\psi \perp \sigma \prec \varphi$.*

Proof. The form $(\dim \sigma) \times \mathbb{H}$ is a nonsingular completion of σ. By the completion lemma there exists a nonsingular completion ρ of σ such that $\varphi \perp (\dim \sigma) \times \mathbb{H} \simeq (\dim \sigma) \times \mathbb{H} \perp \psi \perp \rho$. Hence, by Witt cancellation (Proposition 2.3(1)) we get $\varphi \simeq \psi \perp \rho$, and thus $\psi \perp \sigma \prec \varphi$. $\square$

The following corollary can be also deduced from a more general statement in [10, Prop. 3.11]. For the reader's convenience we give a proof independent of [10].

Corollary 2.16 *Let φ and ψ be quadratic forms with φ nonsingular. Then, $\psi \prec \varphi$ if and only if $i_W(\varphi \perp \psi) \geq \dim \psi$. If furthermore φ is anisotropic, then $\psi \prec \varphi$ if and only if $i_W(\varphi \perp \psi) = \dim \psi$.*

Proof. Let R be nonsingular such that $\psi \simeq R \perp \mathrm{ql}(\psi)$. Suppose that $\psi \prec \varphi$ and let ψ' be nonsingular and S be a nonsingular completion of $\mathrm{ql}(\psi)$ such that $\varphi \simeq R \perp S \perp \psi'$. Since $R \perp R \simeq (\dim R) \times \mathbb{H}$ and $\mathrm{ql}(\psi) \perp S \sim \mathrm{ql}(\psi)$ we get

$$\varphi \perp \psi \simeq (\dim \psi) \times \mathbb{H} \perp \mathrm{ql}(\psi) \perp \psi'$$

and thus $i_W(\varphi \perp \psi) \geq \dim \psi$. If furthermore φ is anisotropic, then necessarily $i_W(\varphi \perp \psi) \leq \dim \psi$ by Lemma 2.11, and we have in fact equality.

Conversely, suppose that

$$\varphi \perp \psi \simeq (\dim \psi) \times \mathbb{H} \perp \mathrm{ql}(\psi) \perp \psi'$$

for some quadratic form $\psi' \in W_q(F)$, then

$$\varphi \perp R \perp \psi \simeq (\dim \psi) \times \mathbb{H} \perp R \perp \mathrm{ql}(\psi) \perp \psi'.$$

By Witt cancellation (Proposition 2.3(1)), we get

$$\varphi \perp \mathrm{ql}(\psi) \simeq (\dim \mathrm{ql}(\psi)) \times \mathbb{H} \perp \psi \perp \psi'.$$

It follows from Corollary 2.15 that $\psi \prec \varphi$. $\square$

2.5 Pfister neighbors

It is well-known that $W_q(F)$ is endowed with a $W(F)$-module structure as follows. If (V, φ) is a nonsingular quadratic form and B is a regular symmetric bilinear form defined over an F-vector space W, we define a nonsingular quadratic form $B \otimes \varphi$ on $W \otimes_F V$ by $B \otimes \varphi(w \otimes v) = B(w,w)\varphi(v)$ whose associated symmetric bilinear form is $B \otimes B_\varphi$ [7].

Let us write $\langle a_1, \cdots, a_n\rangle_b$ for the bilinear form $\sum_{i=1}^n a_i x_i y_i$, where $a_1, \cdots, a_n \in F^*$. An n-fold bilinear Pfister form is a form of type $\langle 1, a_1\rangle_b \otimes \cdots \otimes \langle 1, a_n\rangle_b$. These forms generate the ideal $I^nF \subset WF$, where $I^nF = (IF)^n$ is the n-th power of the ideal of even-dimensional symmetric bilinear forms in WF.

An $(n+1)$-fold (quadratic) Pfister form is a nonsingular quadratic form of type $\langle 1, a_1\rangle_b \otimes \cdots \otimes \langle 1, a_n\rangle_b \otimes [1, b]$ for some $a_i \in F^*$, $b \in F$. We write $\langle\langle a_1, \cdots, a_n, b]]$ for short. $\langle\langle b]] = [1, b]$ is thus a 1-fold Pfister form. The set of forms isometric (resp. similar) to n-fold Pfister forms will be denoted by P_nF (resp. GP_nF). We denote the WF-submodule of W_qF generated by n-fold Pfister forms by I^nW_qF, so that $I^nW_qF = (I^{n-1}F)W_qF$, and we obtain a filtration $W_qF = IW_qF \supset I^2W_qF \supset \cdots$.

Recall that the Hauptsatz of Arason-Pfister asserts that if $\varphi \in I^nW_qF$ is anisotropic, then $\dim \varphi \geq 2^n$, and if $\dim \varphi = 2^n$, then $\varphi \in GP_nF$ [6].

A quadratic form φ over F is called a *Pfister neighbor* if there exist $\pi \in P_nF$ (with n satisfying $2^{n-1} < \dim \varphi \leq 2^n$) and some $a \in F^*$ such that $a\varphi \prec \pi$, in which case we say that φ is a Pfister neighbor of π. Below, we collect some standard results on Pfister neighbors which we will need in the sequel (see [18, Prop. 3.1]).

Proposition 2.17 *(1) A totally singular quadratic form cannot be a Pfister neighbor.*

(2) If φ is a Pfister neighbor of π, then
- *(a) π is unique up to isometry;*
- *(b) For any field extension K/F, φ_K is isotropic if and only if π_K is also isotropic. In particular, if $\dim \varphi \geq 2$, then $\varphi_{F(\pi)}$ and $\pi_{F(\varphi)}$ are isotropic.*

(3) If $\pi \in P_nF$ is anisotropic, then φ is a Pfister neighbor of π if and only if $\dim \varphi > 2^{n-1}$ and $\pi_{F(\varphi)}$ is isotropic.

Definition 2.18 Let K/F be a field extension.

(1) A quadratic form φ over K is *defined over F* if there exists a quadratic form ψ over F such that $\varphi \simeq \psi_K$. In this case, we say that φ is defined by ψ (in general the form ψ is not unique).

(2) We say that K/F is *excellent* if for any quadratic form φ over F the quadratic form $(\varphi_K)_{\mathrm{an}}$ is defined over F.

In Section 5 we will see that an extension given by the function field of a quadratic form of dimension 2 or of type $(1,1)$ is excellent. In characteristic

$\neq 2$, the corresponding result on 2-dimensional forms is well-known, and for 3-dimensional forms it is due to Arason [8] (see also Rost [25]).

In certain cases, Pfister neighbors can be characterized by their behaviour over their own function field. Also, Pfister forms can be characterized (up to similarity) as being those forms which become hyperbolic over the function field of a form which they dominate provided the dimension of this dominated form is large enough. These are the contents of the following results which will be crucial in the proof of Theorem 1.2:

Theorem 2.19 *Let φ be an anisotropic quadratic form of type (r,s) over F.*

(i) ([10, Th. 6.6]) Suppose that $(\varphi_{F(\varphi)})_{\mathrm{an}}$ is defined over F and that $2r > s$. If $s \leq 4$, or if $s = 5$ and $\mathrm{ql}(\varphi)$ is similar to $\langle 1, a, b, ab\rangle \perp \langle c\rangle$ for some $a, b, c \in F^*$, then φ is a Pfister neighbor.

(ii) ([10, Cor. 5.3]) Let q be a nonsingular and anisotropic form over F. Suppose that $3\dim\varphi + s > \dim q$. If $q_{F(\varphi)}$ is hyperbolic, then q is similar to a Pfister form. In particular, if $2\dim\varphi > \dim q$, then φ is a Pfister neighbor.

2.6 The standard splitting of a quadratic form

For a quadratic form φ with $\dim\varphi_{\mathrm{an}} \geq 2$, we define its standard splitting tower $(F_i, \varphi_i)_{0\leq i\leq h}$ as follows: $F_0 = F$, $\varphi_0 = \varphi_{\mathrm{an}}$ and for $n \geq 1$, we define by induction $F_n = F_{n-1}(\varphi_{n-1})$ and $\varphi_n = ((\varphi_{n-1})_{F_n})_{\mathrm{an}}$. The smallest integer h for which $\dim\varphi_h \leq 1$ is called the standard height of φ and denoted by $\mathrm{h}(\varphi)$.

Suppose now that φ is nonsingular. Then the degree $\deg(\varphi)$ of φ is defined as follows. We have $\mathrm{h}(\varphi) \geq 1$ and $\varphi_{\mathrm{h}(\varphi)-1}$ becomes hyperbolic over its own function field, which implies that $\varphi_{\mathrm{h}(\varphi)-1}$ is similar to an n-fold Pfister form π over $F_{\mathrm{h}(\varphi)-1}$ for some $n \geq 1$. We then put $\deg(\varphi) = n$ and we call π the leading form of φ. In the case of a split nonsingular form, we set $\deg(\varphi) = \infty$.

The set $J_nF = \{\varphi \in W_qF \mid \deg(\varphi) \geq n\}$ is a WF-submodule of W_qF with $I^nW_qF \subset J_nF$. Aravire and Baeza have shown in [4], [5] that in fact $J_nF = I^nW_qF$. It should be noted that the corresponding result $I^nF = J_nF$ in characteristic different from 2 is known to be true for $n \leq 5$ due to the work on the Milnor Conjecture in small degrees by Merkurjev-Suslin and Rost (see, e.g. [13, Th. 2.8, Remarque]), and for all n due to Orlov-Vishik-Voevodsky [24] (see also [3, Th. 1.5]).

For some results concerning quadratic forms with maximal splitting we will need a generic property of the standard splitting tower of a form which is not totally singular.

Proposition 2.20 *Let φ be a form over F which is anisotropic and not totally singular, and let $(F_i, \varphi_i)_{0\leq i\leq \mathrm{h}(\varphi)}$ be its standard splitting tower. Let K/F be a field extension such that $i_W(\varphi_K) \geq 1$. Then, there exists $j_0 \in [1, \mathrm{h}(\varphi)]$ such that*

(1) $i_W(\varphi_{F_{j_0}}) = i_W(\varphi_K)$;
(2) Any anisotropic quadratic form over F which is anisotropic over K stays anisotropic over F_{j_0}.

Proof. Set $k = i_W(\varphi_K)$. Without loss of generality we may suppose $\dim\varphi \geq 3$, so that φ is a nice form (cf. subsection 2.3). Since $k \geq 1$, the extension $K \cdot F_1/K$ is purely transcendental. Let $j_0 \in [1, \mathrm{h}(\varphi)]$ be maximal such that the extension $K \cdot F_{j_0}/K$ is purely transcendental. Since $i_W(\varphi_{K\cdot F_{j_0}}) = i_W(\varphi_K)$, we get $i_W(\varphi_{F_{j_0}}) \leq i_W(\varphi_{K\cdot F_{j_0}}) = i_W(\varphi_K) = k$. If $k > i_W(\varphi_{F_{j_0}})$, then the extension $K \cdot F_{j_0+1}/K \cdot F_{j_0}$ is purely transcendental, and thus $K \cdot F_{j_0+1}/K$ is purely transcendental too, a contradiction to the choice of j_0. Hence, $k = i_W(\varphi_{F_{j_0}})$. Now, if ψ is an anisotropic quadratic form over F which is anisotropic over K, then it stays anisotropic over $K \cdot F_{j_0}$, and thus it is also anisotropic over F_{j_0} since $F_{j_0} \subset K \cdot F_{j_0}$. $\square$

Corollary 2.21 *With the same notations and hypotheses as in Proposition 2.20, we have $i_W(\varphi_K) \geq i_W(\varphi_{F(\varphi)})$.*

3 Domination of quadratic forms by Pfister forms and proofs of Theorems 1.1 and 1.3

3.1 Proof of Theorem 1.1

This proof is mainly based on the following proposition which generalizes [20, Prop. 3] to characteristic 2:

Proposition 3.1 *Let $n \geq 1$ be an integer and let φ be an anisotropic quadratic form of type (r, s) such that*

(A) $\dim\varphi \leq 2^n$, *or*
(B) $\dim\varphi = 2^n + 1$ *and $r \geq 1$ (i.e. φ is not totally singular).*

Then there exists a field extension K/F with an $\pi \in P_{n+1}K$ anisotropic such that:

(i) $\varphi_K \prec \pi$;

(ii) *Any anisotropic quadratic form over F stays anisotropic over $K(\pi)$ if we are in case (A).*

Proof. Let R be a nonsingular form over F such that $\varphi = R \perp \mathrm{ql}(\varphi)$. By assumption, $\dim R = 2r$, $\dim \mathrm{ql}(\varphi) = s$. Let $L = F(x_1, \cdots, x_{n+1})$ be the rational function field in the variables $x_1, \cdots, x_{n+1}$ over F, and $\pi = \langle\langle x_1, \cdots, x_n, x_{n+1}]]$. Now π is a nice form (cf. subsection 2.3), and since $1 \leq i_W(\pi_{L(\sqrt{x_1})})$, we have that $L(\sqrt{x_1})(\pi)/L(\sqrt{x_1})$ is purely transcendental. Clearly the extension $L(\sqrt{x_1})/F$ is also purely transcendental. Hence, any anisotropic quadratic form over F stays anisotropic over $L(\pi)$.

Let E/L be a field extension satisfying the conditions:

(C1) π_E is anisotropic;
(C2) Any anisotropic quadratic form over F stays anisotropic over $E(\pi)$ if we are in case (A) of the proposition;
(C3) φ_E is anisotropic if we are in case (B) of the proposition.

Obviously the field L satisfies (C1) and we have seen that it also satisfies (C2) resp. (C3).

From the uniqueness of the quasilinear part, we get $\mathrm{ql}(\pi_E \perp \varphi_E) \simeq \mathrm{ql}(\varphi)_E$ for any field extension E/F. Hence, if E satisfies (C2) resp. (C3), then $\mathrm{ql}(\varphi)_E$ is anisotropic (and in fact also $\mathrm{ql}(\varphi)_{E(\pi)}$), and thus $i_t(\pi_E \perp \varphi_E) = i_W(\pi_E \perp \varphi_E)$. So for such a field extension, let $m(E) = i_W(\pi_E \perp \varphi_E)$ and

$$m = \max\{\, m(E) \mid E/L \text{ satisfies (C1) and [(C2) resp. (C3)]}\}.$$

Let K/L be a field extension for which $m(K) = m$. One has $m \leq \dim \varphi$, otherwise π_K would be isotropic by Lemma 2.11. If we prove $m = \dim \varphi$, then Corollary 2.16 implies $\varphi_K \prec \pi_K$, and the proof is complete.

Assume now that $m < \dim \varphi$ and set

$$\pi_K \perp R_K \perp \mathrm{ql}(\varphi)_K \simeq \nu \perp \mathrm{ql}(\varphi)_K \perp m \times \mathbb{H} \tag{1}$$

for some nonsingular form ν over K and with $\alpha := \nu \perp \mathrm{ql}(\varphi)_K$ anisotropic. We have $2^n + 1 \geq \dim \varphi > m$, which, together with (1), implies :

$$\dim \alpha = 2^{n+1} + \dim \varphi - 2m > 2^{n+1} - m > 2^n - 1 \geq 1\ ,$$

hence, $\dim \alpha \geq 3$. Furthermore, $\dim \nu > 0$ because otherwise, again by the relation (1), we would get $2m = 2^{n+1} + 2r$, hence $m \geq 2^n \geq \dim \varphi$ in case (A), and $m \geq 2^n + 1 \geq \dim \varphi$ in case (B), a contradiction in both cases. This shows that α is a nice form.

By Corollary 2.7, one gets $i_W(\alpha_{K(\alpha)}) \geq 1$ and thus $m(K(\alpha)) = m(K) + i_W(\alpha_{K(\alpha)}) \geq m+1 > m$. To get a contradiction it suffices to prove that the field $K(\alpha)$ satisfies conditions (C1) and [(C2) resp. (C3)].

Property (C1). Adding π_K on both sides in relation (1), we get that $\pi_K \perp \alpha \sim \varphi_K$, and comparing dimensions shows that $\pi_K \perp \alpha$ is isotropic. Then, by the anisotropy of π_K and α, there exists $r \in D_K(\pi_K) \cap D_K(\alpha)$. Now suppose that $\pi_{K(\alpha)}$ is isotropic and hence hyperbolic. By the subform theorem (Proposition 2.13) and Remarks 2.10(3), we obtain $r^2\alpha \simeq \alpha \prec \pi_K$, and thus

$$\pi_K \simeq \nu \perp S \perp \mu \tag{2}$$

for some nonsingular form μ over K and a nonsingular completion S of $\mathrm{ql}(\varphi)_K$. We substitute the relation (2) in the relation (1), we use the fact that $S \perp \mathrm{ql}(\varphi)_K \simeq s \times \mathbb{H} \perp \mathrm{ql}(\varphi)_K$ (Lemma 2.12(1)), and we use Witt cancellation to get

$$\mathrm{ql}(\varphi)_K \perp m \times \mathbb{H} \simeq s \times \mathbb{H} \perp \mu \perp \varphi_K \ .$$

Since $s \times \mathbb{H}$ is a nonsingular completion of $\mathrm{ql}(\varphi)_K$ (Lemma 2.12(2)), the Completion Lemma yields a nonsingular completion S' of $\mathrm{ql}(\varphi)_K$ such that

$$(m+s) \times \mathbb{H} \simeq s \times \mathbb{H} \perp \mu \perp R_K \perp S' \ .$$

In particular, we have $\dim(\mu \perp R_K \perp S') = 2m$ and $i_W(\mu \perp R_K \perp S') = i_t(\mu \perp R_K \perp S') = m$. But $\varphi_K \prec \mu \perp R_K \perp S'$ and $\dim \varphi_K > m = \dim(\mu \perp R_K \perp S') - i_t(\mu \perp R_K \perp S')$, hence φ_K is isotropic by Lemma 2.11, a contradiction. Consequently, we have that $\pi_{K(\alpha)}$ is anisotropic.

Property (C2) in case (A). Let ψ be an anisotropic quadratic form over F. We extend (1) to the field $K(\pi)$. Since $\pi_{K(\pi)}$ is hyperbolic, i.e. $i_W(\pi_{K(\pi)}) = 2^n > m$, we can use Witt cancellation to get that $i_W(\alpha_{K(\pi)}) \geq 1$. Now α is also nice, hence $K(\pi)(\alpha)/K(\pi)$ is purely transcendental. If $\psi_{K(\alpha)}$ is isotropic, then also $\psi_{K(\alpha)(\pi)}$ and hence $\psi_{K(\pi)}$ is isotropic too, a contradiction to property (C2) for K.

Property (C3) in case (B). We have $\dim \varphi = 2^n + 1$, so that φ is singular, and we can write $\varphi \simeq \eta \perp \langle a \rangle$ for some form η over F and some $a \in F^*$. Using the proof from case (A), there exists an extension M/L such that π_M is anisotropic and $\eta_M \prec \pi_M$, so that $i_W(\pi_M \perp \varphi_M) \geq i_W(\pi_M \perp \eta_M) = 2^n$ (Corollary 2.16). On the other hand, the maximality of m and the assumption that $m < \dim \varphi = 2^n + 1$ therefore imply that $m = 2^n$. Hence, we obtain

$$\pi_{K(\alpha)} \perp \varphi_{K(\alpha)} \simeq \alpha_{K(\alpha)} \perp 2^n \times \mathbb{H} \ ,$$

and since $i_W(\alpha_{K(\alpha)}) \geq 1$, we have $i_W(\pi_{K(\alpha)} \perp \varphi_{K(\alpha)}) \geq 2^n + 1 = \dim \varphi$. By Corollary 2.16, $\varphi_{K(\alpha)} \prec \pi_{K(\alpha)}$, implying in particular that $\varphi_{K(\alpha)}$ is anisotropic because $K(\alpha)$ has property (C1). $\square$

Proof of Theorem 1.1. Let φ and ψ be as in Theorem 1.1. Assume that there exists an integer $n \geq 1$ such that $\dim \varphi \leq 2^n < \dim \psi$. By Proposition 3.1, there exists a field extension K/F and an anisotropic quadratic form $\pi \in P_{n+1}K$ such that: $\varphi_K \prec \pi$ and any anisotropic quadratic form over F stays anisotropic over $K(\pi)$. If $\varphi_{F(\psi)}$ is isotropic, then $\pi_{K(\psi)}$ is also isotropic. Since $\dim \psi > 2^n$, it follows from Proposition 2.17(3) that ψ_K is a Pfister neighbor of π, and Proposition 2.17(2) implies that $\psi_{K(\pi)}$ is isotropic, a contradiction. Hence, $\varphi_{F(\psi)}$ is anisotropic. $\square$

3.2 Proof of Theorem 1.3

To prove this theorem, we need the following result which is a more precise version of case (B) in Proposition 3.1.

Proposition 3.2 *Let φ be an anisotropic not totally singular quadratic form of dimension 2^n+1 for some integer $n \geq 1$. Then, there exists a field extension E/F and an anisotropic $\pi \in P_{n+1}E$ such that*

(1) $\varphi_E \prec \pi$, and
(2) the extension $E(\varphi)/F(\varphi)$ is unirational (i.e. there exists a purely transcendental extension $M/F(\varphi)$ such that $E(\varphi) \subset M$).

Proof. Let $x_1, \cdots, x_{n+1}$ be variables over $F(\varphi)$, $L = F(x_1, \cdots, x_{n+1})$ and $\pi = \langle\langle x_1, \cdots, x_{n+1}]]$. Let $\delta := \pi \perp \varphi_L$ and $(\delta_i, L_i)_{0 \leq i \leq \mathrm{h}(\delta)}$ its standard splitting tower. By Proposition 3.1, there exists a field extension K/L such that π_K is anisotropic and $\varphi_K \prec \pi_K$. It follows from Corollary 2.16 that $i_W(\pi_K \perp \varphi_K) = \dim \varphi$. By Proposition 2.20 there exists $j_0 \in [1, \mathrm{h}(\delta)]$ such that

- $i_W(\delta_{L_{j_0}}) = 2^n + 1$, and
- any anisotropic quadratic form over L which is anisotropic over K stays anisotropic over $E := L_{j_0}$.

In particular, π_E and φ_E are anisotropic. By Corollary 2.16, the condition $i_W(\delta_E) = 2^n + 1$ implies $\varphi_E \prec \pi_E$. To finish the proof it suffices to show that the extension $E(\varphi)/F(\varphi)$ is unirational. In fact, for $N := L(\sqrt{x_1})(\varphi) = F(\varphi)(\sqrt{x_1}, x_2, \cdots, x_{n+1})$, we have $i_W(\delta_N) \geq 2^n+1$ since $i_W(\varphi_{F(\varphi)}) \geq 1$ (Corollary 2.7) and π_N is isotropic and therefore hyperbolic. Hence

$$\dim(\delta_N)_{\mathrm{an}} \leq 2^n - 1 \tag{3}$$

Moreover, if $j \leq j_0 - 1$, then $\mathrm{ql}(\varphi)_{L_j}$ is anisotropic as $\mathrm{ql}(\varphi)_E$ is anisotropic, and we clearly have $i_W(\delta_{L_j}) < i_W(\delta_E) = 2^n + 1$. Hence, for $j \leq j_0 - 1$, we get

$$\dim(\delta_{L_j})_{\mathrm{an}} = \dim \delta_j \geq 2^n + 1 \tag{4}$$

Let $M_j = N \cdot L_j$ for $j \leq j_0$. It follows from (3) and (4) that $i_W((\delta_j)_{M_j}) \geq 1$ for every $j \leq j_0 - 1$, and thus the tower $M_0 \subset M_1 \subset \cdots \subset M_{j_0}$ is a succession of purely transcendental extensions. Now $M_0 = N \cdot L_0 = N \cdot L = F(\varphi)(\sqrt{x_1}, x_2, \cdots, x_{n+1})$, which is purely transcendental over $F(\varphi)$. Thus, M_{j_0} is purely transcendental over $F(\varphi)$, and since $E(\varphi) \subset M_{j_0}$ we get the claim by taking $M = M_{j_0}$. $\square$

Proof of the Theorem 1.3. Let E/F and π be as in Proposition 3.2 and suppose that $\varphi_{F(\psi)}$ is isotropic. Then, $\varphi_{E(\psi)}$ is isotropic, and thus $\pi_{E(\psi)}$ is hyperbolic. By Proposition 2.17 ψ_E is a Pfister neighbor of π. In particular, ψ is not totally singular. Since $\varphi_E \prec \pi$, the form $\psi_{E(\varphi)}$ is isotropic. Then, $\psi_{F(\varphi)}$ is isotropic since the extension $E(\varphi)/F(\varphi)$ is unirational. $\square$

4 Quadratic forms with maximal splitting

Lemma 4.1 *Let φ be an anisotropic quadratic form of dimension $2^n + m$ with $0 < m \leq 2^n$. Then, $i_t(\varphi_{F(\varphi)}) \leq m$.*

Proof. Let $\psi \prec \varphi$ with $\dim \psi = \dim \varphi - i_t(\varphi_{F(\varphi)}) + 1$. By Lemma 2.11 $\psi_{F(\varphi)}$ is isotropic, and thus Theorem 1.1 implies that $\dim \varphi - i_t(\varphi_{F(\varphi)}) + 1 > 2^n$, i.e. $i_t(\varphi_{F(\varphi)}) \leq m$. $\square$

Definition 4.2 Let φ be an anisotropic quadratic form of dimension $2^n + m$ with $0 < m \leq 2^n$. φ is said to have *maximal splitting* if $i_t(\varphi_{F(\varphi)}) = m$.

In view of Corollary 2.7, the following lemma is nothing but a reformulation of the definition of maximal splitting:

Lemma 4.3 *Let φ be an anisotropic quadratic form of dimension $2^n + m$ with $0 < m \leq 2^n$ and type (r, s). Let (r', s') be the type of $\varphi_1 = (\varphi_{F(\varphi)})_{\mathrm{an}}$. Then, the following are equivalent:*

(1) φ has maximal splitting;

(2)

$$\dim(\varphi_{F(\varphi)})_{\mathrm{an}} = \begin{cases} 2^n - m & \text{if } \varphi \text{ is not totally singular,} \\ 2^n & \text{otherwise;} \end{cases}$$

(3) $(r', s') = (r - m, s)$ *if* φ *is not totally singular (i.e.* $r \geq 1$*), and* $s' = 2^n$ *otherwise.*

Here are some examples of quadratic forms with maximal splitting:

Proposition 4.4 *Let* $\varphi = R \perp \mathrm{ql}(\varphi)$ *be anisotropic.*

(1) If $\dim \varphi = 2^n + 1$ *form some* $n \geq 0$*, then* φ *has maximal splitting.*
(2) If φ *is a Pfister neighbor, and if* S *is a nonsingular completion of* $\mathrm{ql}(\varphi)$ *and* φ' *is a nonsingular form over* F *such that* $R \perp S \perp \varphi'$ *is similar to a Pfister form, then:*
  (i) $(\varphi_{F(\varphi)})_{\mathrm{an}} \simeq (\mathrm{ql}(\varphi) \perp \varphi')_{F(\varphi)}$*, and*
  (ii) φ *has maximal splitting.*

Proof. (1) This is a direct consequence of Lemma 4.1.

(2) (i) [10, Prop. 6.1].

(2) (ii) Set $\dim \varphi = 2^n + m$ with $0 < m \leq 2^n$. Since φ is not totally singular $i_t(\varphi_{F(\varphi)}) = i_W(\varphi_{F(\varphi)})$. Statement (2)(i) implies that $2i_W(\varphi_{F(\varphi)}) = \dim R - \dim \varphi'$. Since φ is a Pfister neighbor of an $(n+1)$-fold Pfister form, $\dim \varphi' = 2^{n+1} - \dim R - 2 \dim \mathrm{ql}(\varphi)$. Hence, $i_W(\varphi_{F(\varphi)}) = m$. $\square$

Let φ be an anisotropic form over F. Statement (2)(i) above shows that if φ a Pfister neighbor, then $(\varphi_{F(\varphi)})_{\mathrm{an}}$ is defined over F. In characteristic not 2, the converse is known to be true due to a result by Knebusch, i.e. if $(\varphi_{F(\varphi)})_{\mathrm{an}}$ is defined over F, then φ is a Pfister neighbor. In characteristic 2, however, the converse fails to be true in general. A discussion of this problem with various counterexamples can be found in [10].

Now we give two principal results concerning the behaviour of maximal splitting over the function field of a suitable quadratic form:

Lemma 4.5 *Let* φ *be an anisotropic not totally singular quadratic form of dimension* $2^n + m$ *with* $0 < m \leq 2^n$*. Let* ψ *be an anisotropic not totally singular quadratic form of dimension* $> 2^n$*. Then:*

(1) Suppose φ *is of type* (r, s) *with* $2r \geq m$*. Then* φ *does not have maximal splitting if and only if there exists a field extension* K/F *such that* $i_W(\varphi_K) < m$*.*

(2) *If K/F is a purely transcendental extension, then φ has maximal splitting if and only if φ_K has maximal splitting.*
(3) *If $\varphi_{F(\psi)}$ is anisotropic, then φ has maximal splitting if and only if $\varphi_{F(\psi)}$ has maximal splitting.*

Proof. (2) is rather obvious and the proofs of (1) and (3) can essentially be copied from those given for the analogous statements in characteristic $\neq 2$, see [9, Lem. 4 and 5] (where (1) is proved first and then used to prove (3)), using the fact that if K/F is a field extension such that $i_W(\varphi_K) \geq 1$, then $i_W(\varphi_K) \geq i_W(\varphi_{F(\varphi)})$ (Corollary 2.21). $\square$

Proposition 4.6 *Let φ, ψ be two anisotropic not totally singular quadratic forms of respective dimensions $2^n + m$ and $2^n + l$ with $0 < m, l \leq 2^n$. Suppose that φ has maximal splitting and $\varphi_{F(\psi)}$ is isotropic. Then ψ also has maximal splitting and $\psi_{F(\varphi)}$ is isotropic as well.*

Proof. Let $\varphi' \prec \varphi$ be of dimension $2^n + 1$. Since $i_W(\varphi_{F(\varphi)}) = m$ and $\varphi_{F(\psi)}$ is isotropic, it follows from Corollary 2.21 that $i_W(\varphi_{F(\psi)}) \geq m$ and thus $\varphi'_{F(\psi)}$ is isotropic. Hence, φ' is not totally singular.

By the proof of Proposition 3.2 there exists a field extension E/F which is obtained as a purely transcendental extension followed by a succession of function fields of quadratic forms of dimension $> 2^n$, such that φ'_E is a Pfister neighbor of an anisotropic form $\pi \in P_{n+1}E$. Since $\varphi'_{E(\psi)}$ is isotropic, it follows that $\pi_{E(\psi)}$ is hyperbolic, and thus ψ_E is a Pfister neighbor of π. By Proposition 4.4 ψ_E has maximal splitting, and inductively by Lemma 4.5(2) and (3), ψ has maximal splitting.

Now $\varphi'_{F(\varphi)}$ is isotropic by Lemma 2.11, hence $i_W(\varphi'_{F(\varphi)}) \geq 1$ by Corollary 2.7, therefore $F(\varphi)(\varphi')/F(\varphi)$ is purely transcendental. To complete the proof, we note that by Theorem 1.3, the isotropy of $\varphi'_{F(\psi)}$ implies that of $\psi_{F(\varphi')}$. But then ψ is obviously isotropic over $F(\varphi)(\varphi')$, thus also over $F(\varphi)$ because $F(\varphi)(\varphi')/F(\varphi)$ is purely transcendental. $\square$

5 Proof of Theorem 1.2

5.1 Preliminary results

First of all we need a weak version of a theorem by Aravire and Baeza [5]. For the reader's convenience we give a proof:

Proposition 5.1 *Let $\varphi \in I^{n+1}W_q(F)$, and let K be the function field of an anisotropic bilinear Pfister form $\psi = \langle 1, a_1\rangle_b \otimes \cdots \otimes \langle 1, a_n\rangle_b$ for $a_1, \cdots, a_n \in F^*$. If φ_K becomes hyperbolic, then $\varphi \equiv \langle\langle a_1, \cdots, a_n, b]] \pmod{I^{n+2}W_q(F)}$ for some $b \in F$.*

Proof. Let ψ_1 be the totally singular quadratic form defined by $\psi_1(v) = \psi(v, v)$. Without loss of generality, we may assume that φ is anisotropic. The condition $\varphi_K \sim 0$ implies that $x_1\psi_1 \prec \varphi$ for some $x_1 \in F^*$. Choose δ a not totally singular form of dimension $2^n + 1$ such that $x_1\psi_1 \prec \delta \prec \varphi$. One readily verifies that δ is a Pfister neighbor of a Pfister form $\pi_1 = \langle\langle a_1, \cdots, a_n, b_1]]$ for some $b_1 \in F$.

Since δ is anisotropic, one easily checks that $i_t(\delta \perp \delta) = \dim \delta$, hence $i_t(\varphi \perp x_1\pi_1) = i_W(\varphi \perp x_1\pi_1) \geq 2^n + 1 = \dim \delta$ because $\delta \prec \varphi$ and $\delta \prec x_1\pi_1$. Now, let $\varphi_1 = (\varphi \perp x_1\pi_1)_{\mathrm{an}}$. We thus get $\dim \varphi_1 = \dim \varphi + \dim \pi - 2i_W(\varphi \perp x_1\pi_1) < \dim \varphi$. Moreover, $(\varphi_1)_K \sim 0$. By induction on the dimension of φ, we obtain $\varphi \sim \perp_{i=1}^m x_i \langle\langle a_1, \cdots, a_n, b_i]]$ for some integer m and some $x_i \in F^*$, $b_i \in F$. Since $\langle\langle a_1, \cdots, a_n, b_i]] \equiv x_i \langle\langle a_1, \cdots, a_n, b_i]] \pmod{I^{n+2}W_q(F)}$, we get the claim by taking $b = \sum_{i=1}^m b_i$. $\square$

Remark 5.2 (i) This proof shows that the Witt kernel $W_q(K/F)$ is generated as F-module by Pfister forms of type $\langle\langle a_1, \cdots, a_n, b]]$, or equivalently $W_q(K/F) = \psi \otimes W_q(F)$ (ψ and K as in the proposition).

(ii) For the Witt kernel $W_q(L/F)$ where $L = F(\pi)$ for an anisotropic *quadratic* Pfister form $\pi \in P_nF$, one obtains $W_q(L/F) = \{\rho \otimes \pi \mid \rho \in W(F)\}$. More precisely, if φ is a nonsingular anisotropic quadratic form with φ_L hyperbolic, then $\varphi \simeq \perp_{i=1}^m c_i\pi$ for some m and some $c_i \in F^*$. This can easily be deduced from Proposition 2.13, using the fact that π_L is hyperbolic. $\square$

Lemma 5.3 *Let A be a finite dimensional central simple F-algebra, and let L be a quadratic extension given by $F[x]/(x^2+u)$ ($u \in F^*$) resp. $F[x]/(x^2+x+u)$ such that* $\operatorname{ind} A_L \leq 2$. *Then there exists a quaternion F-algebra Q and $a \in F$ resp. $b \in F^*$ such that $A = [a, u) \otimes_F Q$ resp. $A = [u, b) \otimes_F Q \in \mathrm{Br}(F)$, according as L/F is inseparable or separable.*

Proof. The condition $\operatorname{ind} A_L \leq 2$ implies $\operatorname{ind} A \leq 4$.

If $\operatorname{ind} A \leq 2$, then A is Brauer-equivalent to a quaternion algebra Q and the lemma is true by taking for a (resp. for b) any element in $\wp(F)$ (resp. in F^{*2}).

If $\operatorname{ind} A = 4$, then there exists an Albert form φ such that $A = C(\varphi) \in \mathrm{Br}(F)$ (an Albert form is a 6-dimensional nonsingular quadratic form of trivial Arf-invariant). By [21], the form φ is anisotropic and φ_L is isotropic. It follows from [18, Th. 1.1] that there exists φ' of dimension 2, $a, c \in F$ and $b \in F^*$ such that $b\varphi \simeq [1,c] \perp u[1,a] \perp \varphi'$ or $b[1,u] \subset \varphi$. To get the desired claim we take the Clifford algebra and we use the fact that $C(\varphi) = C(b\varphi) \in \mathrm{Br}(F)$. $\square$

Lemma 5.4 *Let φ be a quadratic form which is not split, and let L be a quadratic extension. Then, $(\varphi_L)_{\mathrm{an}}$ is defined over F.*

Proof. Let V be the underlying vector space of φ. We distinguish between separable and inseparable quadratic extension. Without loss of generality, we may assume that φ is anisotropic. There is nothing to prove if φ_L is anisotropic. So suppose that φ_L is isotropic.

Suppose first that L/F is separable, and let $a \in F^*$, $\alpha \in L^*$ be such that $L = F(\alpha)$ and $\alpha^2+\alpha+a = 0$. In this case φ is not totally singular as anisotropic totally singular forms stay anisotropic over separable extensions (see, e.g. [10, Prop. 8.7]). The isotropy of φ_L implies the existence of $v, w \in V$, not both zero, such that $\varphi(v + \alpha w) = 0$, hence $\varphi(v) = a\varphi(w)$ and $B_\varphi(v, w) = \varphi(w)$. Since φ is anisotropic, we get $\varphi(v) \neq 0$ and $\varphi(w) \neq 0$. It follows that $\varphi(w)[1, a] \subset \varphi$. If we set $\varphi \simeq \varphi(w)[1, a] \perp \varphi'$, it follows that $\varphi_L \sim \varphi'_L$. Since $\dim \varphi' < \dim \varphi$ the claim follows by induction on $\dim \varphi$.

Suppose finally that L/F is inseparable, and let $b \in F^*$ be such that $L = F(\sqrt{b})$. In this case the isotropy of φ_L implies the existence of $v', w' \in V$, not both zero, such that $\varphi(v') = b\varphi(w')$ and $B_\varphi(v', w') = 0$ (also $\varphi(v') \neq 0$ and $\varphi(w') \neq 0$ since φ is anisotropic). It follows that $\varphi(w') \langle 1, b\rangle \prec \varphi$. By Lemma 2.9 there exists $x, y, z, t \in F^*$ such that $\varphi(w') \langle 1, b\rangle \simeq \langle x, y\rangle$ and one of the three following forms is a subform of φ: $\langle x, y\rangle$, $[x, z] \perp \langle y\rangle$, or $[x, z] \perp [y, t]$. Since $xy \in L^{*2}$, we have $\langle x, y\rangle_L \simeq \langle x, 0\rangle_L$, $([x, z] \perp \langle y\rangle)_L \simeq \mathbb{H} \perp \langle y\rangle_L$, and $([x, z] \perp [y, t])_L \simeq [x, z + (yt/x)]_L \perp \mathbb{H}$. Hence, one can easily see that in the first case $\varphi_L \simeq \langle 0\rangle \perp \varphi'_L$, and in the other two cases $\varphi_L \simeq \mathbb{H} \perp \varphi'_L$ for some quadratic form φ' defined over F. In this case we also conclude by induction on $\dim \varphi$. $\square$

Remark 5.5 The excellence of quadratic extensions in characteristic 2 (separable or inseparable) has previously been shown in [2]. $\square$

For the excellence of an extension given by the function field of a 2-fold Pfister form we need the following proposition which generalizes [25, Proposition] to characteristic 2:

Proposition 5.6 *Let K be the quotient field of $F[s,t]/(s^2 + st + at^2 + b)$, and let φ be a quadratic form (singular or not). Then there exists an integer p, quadratic forms φ_i, ψ_i for $0 \le i \le p$, and scalars $c_1, \cdots, c_{p-1} \in F^*$ such that $\varphi_0 = \varphi$ and*

(1) $\varphi_i \simeq c_i[1,a] \perp \psi_i$ *for* $0 \leq i \leq p-1$*;*
(2) $\varphi_{i+1} \simeq bc_i[1,a] \perp \psi_i$ *for* $0 \leq i \leq p-1$*;*
(3) $((\varphi_p)_K)_{\mathrm{an}} \simeq ((\varphi_p)_{\mathrm{an}})_K$.

By the same reasoning as in [25, Corollary, end of page 511], we obtain the following corollary:

Corollary 5.7 *An extension given by the function field of a 2-fold Pfister form is excellent.*

Proof of Proposition 5.6. With the same notations as in [25], one can essentially reproduce the proof of [25, Proposition] after applying the following changes:

- For R we take the ring $F[s,t]/(s^2+st+at^2+b)$;
- In [25, Lemma, page 512], we take $[1,a]$ instead of $\langle 1,a\rangle$;
- The claim in [25, middle of page 512] becomes:

$$B_\varphi(v_n,w_n)=\varphi(v_n) \text{ and } \varphi(w_n)=-a\varphi(v_n);$$

- At the end of [25, page 512] we change L by $F[z]/(z^2+z+a)$, and clearly in [25, page 513] the conjugate of λ is given by $\alpha \mapsto \alpha+1$.

We leave the details to the reader. □

The following corollary is well-known in characteristic different from 2, due to Elman and Lam [8]:

Corollary 5.8 *Let* $\tau_1,\tau_2 \in P_2F$*, and let* φ *be a nonsingular quadratic form which becomes hyperbolic over* $F(\tau_1)(\tau_2)$*. Then, there exists symmetric bilinear forms* ρ_1 *and* ρ_2 *such that* $\varphi \sim \rho_1 \otimes \tau_1 \perp \rho_2 \otimes \tau_2$.

Proof. Essentially the same proof like the one of [8, Cor 2.12] by using the excellence property given by Corollary 5.7. □

Corollary 5.9 *With the same notations and hypotheses as in Corollary 5.8, and if* $\operatorname{ind} C(\varphi)=4$*, then there exist scalars* $\alpha_1,\alpha_2 \in F^*$ *such that* $\varphi \equiv \alpha_1\tau_1 \perp \alpha_2\tau_2 \pmod{I^4W_q(F)}$.

Proof. Let ρ_1,ρ_2 be as in Corollary 5.8. Now for $a,b \in F^*$, we have $\langle a,b\rangle_b \otimes \tau_i \in I^3W_qF$, and thus $C(\langle a,b\rangle_b \otimes \tau_i) \sim 0 \in \mathrm{Br}(F)$ (cf. [26]). Thus, by using the Clifford algebra, we deduce that ρ_1 and ρ_2 are of odd dimension, otherwise we would get $\operatorname{ind} C(\varphi) \leq 2$. For $\alpha_1, \alpha_2 \in F^*$ such that $\rho_1 \perp \langle \alpha_1 \rangle_b \in I^2F$ and $\rho_2 \perp \langle \alpha_2 \rangle_b \in I^2F$, we therefore get $\varphi \equiv \alpha_1\tau_1 \perp \alpha_2\tau_2 \pmod{I^4W_q(F)}$. $\square$

5.2 Proof of Theorem 1.2

Let ψ be as in Theorem 1.2, (r,s) its type, $\psi_1 = (\psi_{F(\psi)})_{\mathrm{an}}$ and $K = F(\psi)$. Set $\psi = \psi' \perp \mathrm{ql}(\psi)$ with $\psi' \in W_q(F)$, and set $\mathrm{ql}(\psi) = \langle c_1, \cdots, c_s \rangle$, where we may assume after scaling that $c_1 = 1$ if $s > 0$. By the uniqueness of the quasilinear part, Proposition 2.6 and Lemma 4.3, we have $\mathrm{ql}(\psi_1) = \mathrm{ql}(\psi)_K$ and $\dim \psi_1 \leq 6$ (where $s = 0$ if $\dim \psi_1 = 6$), and thus $s \leq 5$. Hence, $2r > s$ and the quasi-linear part of ψ satisfies the assumption made in Theorem 2.19(i). To conclude, it therefore suffices by Theorem 2.19 to prove that either ψ_1 is defined over F, or that there exists a nonsingular anisotropic form q over F with $2\dim \varphi > \dim q$ and $q_{F(\varphi)}$ hyperbolic.

By Lemma 4.3, $\dim \psi_1 = 0$ if and only if $\dim \psi = 2^{n+1}$, in which case we are done (a 0-dimensional form is clearly defined over F).

If $\dim \psi_1 > 0$ and ψ_1 is totally singular, then ψ_1 is defined over F by $\mathrm{ql}(\psi)$, and again, we are done. So we may suppose that ψ_1 is not totally singular. Thus, the hypotheses on ψ in Theorem 1.2 together with Lemma 4.3 imply that we are in one of the following cases (where ψ has maximal splitting by assumption):

(a) ψ_1 is of type $(1,0)$, or, equivalently, ψ is of dimension $2^{n+1}-2$ and of type $(2^n-1, 0)$;
(b) ψ_1 is of type $(1,1)$, or, equivalently, ψ is of dimension $2^{n+1}-3$ and of type $(2^n-2, 1)$;
(c) ψ_1 is of type $(2,0)$, or, equivalently, ψ is of dimension $2^{n+1}-4$ and of type $(2^n-2, 0)$;
(d) ψ_1 is of type $(1,2)$, or, equivalently, ψ is of dimension $2^{n+1}-4$ and of type $(2^n-3, 2)$;
(e) ψ_1 is of type $(2,1)$, or, equivalently, ψ is of dimension $2^{n+1}-5$ and of type $(2^n-3, 1)$;
(f) $\psi \in I^2W_qF$ and ψ_1 is of type $(3,0)$, or, equivalently, $\psi \in I^2W_qF$ and ψ is of dimension $2^{n+1}-6$ and of type $(2^n-3, 0)$.

Case (a). Set $\psi_1 = \alpha\,[1,\beta]$. By comparing Arf-invariants in the relation $\psi_K \sim \psi_1$, we may assume $\beta \in F$. We follow the argument given in [14, page 149]. Take $L = F[x]/(x^2+x+\beta)$ and $M = L(\psi)$. Since $C(\psi)_M = [\beta,\alpha)_M = 0 \in \mathrm{Br}(L)$ and $\dim \psi > 4$ (because $n \geq 2$), we get $C(\psi)_L = 0 \in \mathrm{Br}(K)$. Then $C(\psi)$ is Brauer-equivalent to a quaternion algebra $[\beta,\gamma)$ for some $\gamma \in F^*$. Hence, $\langle\langle \alpha,\beta]] \simeq (\langle\langle \gamma,\beta]])_K$, and by Witt cancellation $\alpha\,[1,\beta] \simeq (\gamma\,[1,\beta])_K$.

Case (b). Set $\psi_1 = \alpha\,[1,\beta] \perp \langle 1\rangle$ (recall that $c_1 = 1$ and $\mathrm{ql}(\psi_1) = \mathrm{ql}(\psi)_K$). We then get

$$C_0(\psi)_K = C(\psi')_K = [\beta,\alpha) \in \mathrm{Br}(K)$$

(see, e.g., [22, Lemma 2]). Since $\dim\psi \geq 5$ (because $n \geq 2$), the index reduction theorem [22] implies that $C_0(\psi)$ is Brauer-equivalent to a quaternion algebra $[d,e)$ for some $d \in F$ and $e \in F^*$. Hence,

$$[1,\beta] \perp \alpha\,[1,\beta] \simeq ([1,d] \perp e\,[1,d])_K\ .$$

Adding on both sides the form $\langle 1\rangle$, Witt cancellation yields $\langle 1\rangle \perp \alpha\,[1,\beta] \simeq (\langle 1\rangle \perp e\,[1,d])_K$. The claim then follows.

Case (c). We consider two cases depending on whether $\psi_1 \in GP_2K$ or not.

Suppose first that $\psi_1 \in GP_2K$. Then $\mathrm{ind}\,C(\psi)_K \leq 2$. Since $\dim\psi \geq 12$ (because $n \geq 3$), we get by the index reduction theorem the existence of $\tau \in P_2F$ such that $C(\psi)$ is Brauer-equivalent to $C(\tau)$. In particular, ψ_1 is similar to τ_K and thus $\psi_{K(\tau)} \sim 0$ and therefore $\psi_{F(\tau)(\psi)} \sim 0$. Also, τ is necessarily anisotropic.

If $\psi_{F(\tau)}$ is anisotropic (or, equivalently, if $i_W(\psi_{F(\tau)}) = 0$), then we have $i_W(\psi_{F(\tau)(\psi)}) = 2^n - 2 > 2^n - 4$, and $\dim\psi = 2^n + (2^n - 4)$, a contradiction to Lemma 4.1. Hence, $i_W(\psi_{F(\tau)}) > 0$ and $F(\tau)(\psi)/F(\tau)$ is purely transcendental. This implies that $\psi_{F(\tau)}$ is already hyperbolic. By Remark 5.2(ii), there exists a symmetric bilinear form ρ such that $\psi \simeq \rho \otimes \tau$. The dimension of ρ is necessarily odd since $C(\psi) = C(\tau) \neq 0 \in \mathrm{Br}(F)$. For $x \in F^*$ such that $\rho \perp \langle x\rangle_b \in I^2F$, we get $\psi \perp x\tau \in I^4W_q(F)$. We extend this relation to K and we use the Hauptsatz of Arason-Pfister to get $\psi_1 \simeq (x\tau)_K$.

Now suppose that $\psi_1 \notin GP_2K$ and let $k \in F^*$ be such that $\Delta(\psi) = k \pmod{\wp(F)}$. We follow the argument given in [14, page 149]. Let $N = F[x]/(x^2 + x + k)$. Note that $[N : F] = 2$ since otherwise $\psi_1 \in GP_2K$.

Since $\mathrm{ind}\,C(\psi)_{N(\psi)} \leq 2$, it follows from the index reduction theorem that $\mathrm{ind}\,C(\psi)_N \leq 2$. By Lemma 5.3, there exists $\rho := \langle\langle d,c]] \in P_2F$ and $r \in F^*$ such that $C(\psi)$ is Brauer-equivalent to $C(\rho) \otimes_F [k,r)$.

Now let $\nu = r([1,c+k] \perp d\,[1,c])$. It is clear that $\Delta(\nu) = \Delta(\psi)$ (i.e. $\Delta(\nu \perp \psi) = 0 \pmod{\wp(F)}$), and one readily checks that $C(\psi \perp \nu) = 0 \in \mathrm{Br}(F)$. By a result of Sah [26], this implies $\mu := \psi \perp \nu \in I^3W_q(F)$.

Also, $\nu_N \simeq r\rho_N$ and $C(\psi)_N = C(\rho)_N = C(\nu_N) \in \mathrm{Br}(N)$. Hence, $C(\psi_1)_{N(\psi)}$ is Brauer-equivalent to $C(\nu)_{N(\psi)}$, and we deduce by [21] that ν_K is similar to ψ_1. Therefore, $K(\psi_1) = K(\nu)$, and since $\dim(\psi_{K(\psi_1)})_{\mathrm{an}}$, $\dim(\nu_{K(\nu)})_{\mathrm{an}} \leq 2$, the Hauptsatz of Arason-Pfister implies that $\mu_{K(\nu)} \sim 0$.

Now $\dim(\psi_{F(\nu)})_{\mathrm{an}} \leq 4$ or $= 2^{n+1} - 4$ by Proposition 2.20. Furthermore, we have $\dim(\psi_{K(\psi_1)})_{\mathrm{an}} = \dim(\psi_{F(\psi)(\nu))})_{\mathrm{an}} = \dim(\psi_{F(\nu)(\psi))})_{\mathrm{an}} \leq 2$ by the above, and thus, we cannot have $\dim(\psi_{F(\nu)})_{\mathrm{an}} = 2^{n+1} - 4$ (otherwise, we would get a contradiction to Lemma 4.1). Hence, $\dim(\psi_{F(\nu)})_{\mathrm{an}} \leq 4$, and since $\dim(\nu_{F(\nu)})_{\mathrm{an}} \leq 2$, the Hauptsatz again implies that $\mu_{F(\nu)}$ is hyperbolic.

Now $\nu_{F(\rho)}$ is isotropic as ν weakly dominates the Pfister neighbor $\langle 1\rangle \perp d[1,c]$ of $\rho \in P_2F$. Hence, the extension $F(\rho)(\nu)/F(\rho)$ is purely transcendental and we therefore must have $\mu_{F(\rho)} \sim 0$. But $\nu \sim r\rho \perp r[1,k]$, hence, $(\psi \perp r\,[1,k])_{F(\rho)} \sim 0$. Since $C(\psi \perp r\,[1,k])$ is Brauer-equivalent to $C(\rho)$, there exists a symmetric bilinear form λ of odd dimension such that $\psi \perp r\,[1,k] \sim \lambda \otimes \rho$ (see Remark 5.2). Now, for $y \in F^*$ such that $\lambda \perp \langle y\rangle \in I^2F$, we get $\psi \perp r\,[1,k] \perp y\rho \in I^4W_q(F)$. We extend this relation to K to get $\psi_1 \perp (r\,[1,k] \perp y\rho)_K \in I^4W_q(K)$. The Hauptsatz of Arason-Pfister implies that $\psi_1 \sim (r\,[1,k] \perp y\rho)_K$. By comparing dimensions and by using Theorem 1.1 we conclude that $r\,[1,k] \perp y\rho$ is isotropic, and thus ψ_1 is defined over F.

Case (d). Let $\alpha, \beta \in K^*$ be such that

$$(\psi_K)_{\mathrm{an}} \simeq \alpha[1,\beta] \perp \langle 1, c_2\rangle_K \ .$$

Let $\psi'' = \psi' \perp \langle 1\rangle$. We then get $\psi_{K(\sqrt{c_2})} \sim \psi''_{K(\sqrt{c_2})} \sim (\alpha[1,\beta] \perp \langle 1\rangle)_{K(\sqrt{c_2})}$ and

$$C_0(\psi'')_{K(\sqrt{c_2})} = C(\psi')_{K(\sqrt{c_2})} = [\beta,\alpha)_{K(\sqrt{c_2})} \in \mathrm{Br}(K(\sqrt{c_2}))$$

(see, e.g., [22, Lemma 2]). Since $K(\sqrt{c_2})$ is a purely transcendental extension of $F(\sqrt{c_2})(\psi'')$ and $\dim\psi'' \geq 11$, we get by the index reduction theorem $\operatorname{ind} C_0(\psi'')_{F(\sqrt{c_2})} \leq 2$. By Lemma 5.3, there exist $v \neq 0, a, b \in F$ such that

$$C_0(\psi'') = C(\psi') = [a, c_2) \otimes [b, v) \in \mathrm{Br}(F).$$

Now let $\ell \in F$ be such that $\Delta(\psi') = l \,(\mathrm{mod}\,\wp(F))$, and let

$$\rho = \psi' \perp [1, a+b+\ell] \perp c_2[1,a] \perp v[1,b].$$

Clearly, $\psi \prec \rho$ and $\dim\rho = 2^{n+1}$. Furthermore, $\Delta(\rho) = 0\,(\mathrm{mod}\wp(F))$, and a quick computation yields $C(\rho) = 0 \in \mathrm{Br}(F)$, so that $\rho \in I^3W_qF$. Also, $\rho \not\sim 0$ since $\dim\psi > \frac{1}{2}\dim\rho$ and ψ is anisotropic. Note that ρ is a nonsingular completion of $\psi' \perp v[1,b] \perp \langle 1, c_2\rangle$. After passing to K and by invoking the completion lemma, there exist $x, y \in K$ such that for the K-form

$$\pi := \alpha[1,\beta] \perp [1,x] \perp c_2[1,y] \perp v[1,b]$$

we have $\rho_K \sim \pi$, and thus $\pi \in I^3W_qK$. Since $\dim\pi = 8$, the Hauptsatz of Arason-Pfister yields $\pi \in P_3K$ and $\delta := \langle 1, c_2, v\rangle \prec \pi$.

If δ is isotropic, then π is isotropic and hence hyperbolic, and thus $\rho_K \sim 0$. By Theorem 2.19(ii), ρ_{an} is similar to a Pfister form of which ψ is a neighbor. A simple dimension count then shows that indeed $\rho = \rho_{an}$, and it is a Pfister form since $1 \in D_F(\rho)$, i.e. $\rho \in P_{n+1}F$.

Now suppose that δ is anisotropic and let $\delta' := \delta \perp \langle vc_2 \rangle$, so that δ' is the totally singular form derived from the bilinear Pfister form $\langle 1, c_2 \rangle_b \otimes \langle 1, v \rangle_b$. It follows from [10, Prop. 8.9(iii)] that for any field extension M/F, we have that δ_M is isotropic iff δ'_M is isotropic. Hence, δ_K and thus also δ'_K are anisotropic (Proposition 2.6).

Let $E = F(\delta')$ and $L = K(\delta')$. Then δ_L is isotropic, and since $\delta_L \prec \pi_L$, it follows that π_L is hyperbolic, and thus $\rho_L = \rho_{E(\psi)} \sim 0$. Note that $\mathrm{ql}(\psi_E) \simeq \langle 1, c_2 \rangle_E$ is anisotropic by Theorem 1.1 as $\langle 1, c_2 \rangle$ is anisotropic over F, and $E = F(\delta')$ with δ' anisotropic of dimension 4. In particular, ψ_E is nondefective.

If $\dim(\rho_E)_{\mathrm{an}} > 0$, then ψ_E is necessarily anisotropic (otherwise L/E would be purely transcendental, a contradiction to $\rho_E \not\sim 0$). By comparing dimensions and by Theorem 2.19(ii), $(\rho_E)_{\mathrm{an}}$ is similar to a Pfister form of which ψ_E is a neighbor. This Pfister form is then necessarily of dimension 2^{n+1}. But ρ_E is isotropic since $\delta \prec \rho$ and δ_E is isotropic. Thus, $\dim(\rho_E)_{\mathrm{an}} \leq 2^{n+1} - 2$, a contradiction. Hence, $\rho_E \sim 0$, and by Proposition 5.1, there exists $c \in F$ such that

$$\rho \equiv \langle\langle c_2, v, c]] \pmod{I^4 W_q(F)}.$$

Let

$$\gamma = \psi' \perp [1, a + b + c + \ell] \perp c_2[1, a + c] \perp v[1, b + c] \perp c_2 v[1, c].$$

We have $\dim \gamma = 2^{n+1} + 2$, $\gamma \sim \rho \perp \langle\langle c_2, v, c]] \in I^4 W_q F$, and $\psi \prec \gamma$. Since ψ is anisotropic and $\dim \psi > \frac{1}{2} \dim \gamma$, the form γ is not hyperbolic. Let $\gamma' = \gamma_{\mathrm{an}}$. We have $\gamma'_K \equiv \pi \perp \langle\langle c_2, v, c]]_K \pmod{I^4 W_q(K)}$ and $\dim(\pi \perp \langle\langle c_2, v, c]]_K)_{\mathrm{an}} < 16$ since $\delta \prec \pi$ and $\delta \prec \langle\langle c_2, v, c]]$. Hence, by the Hauptsatz, $\gamma'_K \sim 0$. Since $2 \dim \psi > \dim \gamma'$, we get again by Theorem 2.19(ii) that γ' is similar to a Pfister form of which ψ is a neighbor.

The cases (e) and (f). Here, ψ_1 is of dimension 5 and type $(r, 1)$ in case (e) (in which $\mathrm{ql}(\psi) = \langle 1 \rangle$), or of dimension 6, nonsingular, with trivial Arf-invariant in case (f) (i.e., ψ_1 is an Albert form). Set $\psi_1 = \psi_2 \perp \mathrm{ql}(\psi)_K$. Let $\rho = \psi' \perp [1, \Delta(\psi')]$ if $\dim \psi = 2^{n+1} - 5$ (resp. $\rho = \psi = \psi'$ if $\dim \psi = 2^{n+1} - 6$). In particular, $\Delta(\rho)$ is trivial and thus $\rho \in I^2 W_q F$. We follow some arguments of the proof by Izhboldin [11]. Now one readily sees that $C(\rho) = C(\psi') \in \mathrm{Br}(F)$ ($= C_0(\psi)$ in case (e)), and thus $C(\rho)_K = C(\psi')_K = C(\psi_2) \in \mathrm{Br}(K)$. Hence, $\mathrm{ind}\, C(\rho)_K = \mathrm{ind}\, C(\psi_2) \leq 4$, and we get by the index reduction theorem that $\mathrm{ind}\, C(\rho) \leq 4$.

The subcase $\operatorname{ind} C(\rho) = 4$. Let $\tau_1, \tau_2 \in P_2F$ be such that $C(\rho) = C(\tau_1) \otimes_F C(\tau_2) \in \mathrm{Br}(F)$. Since $C(\rho)$ is split over $L := F(\tau_1)(\tau_2)$, we get by a result of Sah [26] that $\rho_L \in I^3W_q(L)$. Since $\dim \psi_1 \leq 6$ we get $\dim(\rho_{L(\psi)})_{\mathrm{an}} \leq 6$ (in fact, if $\dim \psi = 2^{n+1} - 5$ we get by the completion lemma $\rho_K \sim \psi_2 \perp [1, m]$ for some $m \in K$, and if $\dim \psi = 2^{n+1} - 6$ we also have $\dim(\rho_K)_{\mathrm{an}} \leq 6$).

By the Hauptsatz, $\rho_{L(\psi)}$ is hyperbolic. If ρ_L is not hyperbolic then ψ_L is anisotropic, and by Theorem 2.19(ii), $(\rho_L)_{\mathrm{an}}$ is similar to a Pfister form, necessarily of dimension 2^{n+1}, a contradiction to $\dim \rho < 2^{n+1}$. Hence, $\rho_L \sim 0$. By Corollary 5.9, there exist α_1, α_2 such that $\rho \perp \alpha_1\tau_1 \perp \alpha_2\tau_2 \in I^4W_q(F)$. Since $\lambda := (\alpha_1\tau_1 \perp \alpha_2\tau_2)_{\mathrm{an}}$ has dimension ≤ 8, we get $\dim(\rho_K)_{\mathrm{an}} + \dim(\lambda_K)_{\mathrm{an}} \leq 14 < 16$. It follows from the Hauptsatz that $\rho_K \sim \lambda_K$. Note that λ_K is anisotropic since $\dim \lambda \leq 8 < \dim \psi$ (Theorem 1.1), and thus $(\rho_K)_{\mathrm{an}} \simeq \lambda_K$ which implies $\dim \lambda \leq 6$. Note that in the case $\dim \psi = 2^{n+1} - 5$, we have $\rho \perp \langle 1 \rangle \sim \psi$, hence

$$(\rho \perp \langle 1 \rangle)_K \sim \psi_1 \sim (\lambda \perp \langle 1 \rangle)_K \ .$$

But the form $(\lambda \perp \langle 1 \rangle)_{\mathrm{an}}$ (which is of dimension $< 8 < \dim \psi$) stays also anisotropic over K by Theorem 1.1, and thus

$$(\psi_K)_{\mathrm{an}} \simeq \begin{cases} ((\lambda \perp \langle 1 \rangle)_{\mathrm{an}})_K & \text{if } \dim \psi = 2^{n+1} - 5 \ , \\ \lambda_K & \text{if } \dim \psi = 2^{n+1} - 6 \ . \end{cases}$$

The subcase $\operatorname{ind} C(\rho) \leq 2$. By the same arguments as before we get $\rho_{F(\tau)} \sim 0$ where $\tau \in P_2F$ satisfies $C(\rho) = C(\tau) \in \mathrm{Br}(F)$. By Remark 5.2(ii), there exists a symmetric bilinear form μ such that $\rho \sim \mu \otimes \tau$. Since $C(\rho) = C(\tau)$, the dimension of μ is necessarily odd and thus $\rho \perp z\tau \in I^4W_q(F)$ where $z \in F^*$ satisfies $\mu \perp \langle z \rangle_b \in I^2F$. Now it is clear that the proof can be completed as in the previous case. In fact, the form λ in the above proof will now be given by $z\tau_{\mathrm{an}}$, which shows in particular that this case cannot occur in the case $\dim \psi = 2^n - 6$, and that necessarily $\operatorname{ind} C(\rho) = 2$ if $\dim \psi = 2^n - 5$. $\square$

5.3 Proof of Corollary 1.4

By Theorem 1.3, $\psi_{F(\varphi)}$ is also isotropic and ψ is not totally singular. Since φ has maximal splitting and $\dim \psi > 2^n$, we get by Proposition 4.6 that ψ has maximal splitting too, and thus by Theorem 1.2, ψ is a Pfister neighbor of an $(n+1)$-fold Pfister form π. Since $\psi_{F(\varphi)}$ is isotropic and $\dim \varphi > 2^n$ we deduce that φ is also a Pfister neighbor of π. $\square$

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