Supersoluble Crossed Product Criterion for Division Algebras

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Abstract

Let D be a finite dimensional F-central division algebra. A criterion is given for D to be a supersoluble (nilpotent) crossed product division algebra in terms of subgroups of the multiplicative group D^* of D. More precisely, it is shown that D is supersoluble (nilpotent) crossed product if and only if D^* contains an irreducible abelian-by-supersoluble (nilpotent) subgroup.

1 Introduction

Let D be a division algebra with center F and degree n, (i.e. $\dim_F D = n^2$). The algebra D is called *crossed product* if it contains a maximal subfield K such that K/F is Galois. D is said to be *supersoluble* crossed product if $Gal(K/F)$ is supersoluble. We also recall that a subgroup G of D^* is *irreducible* if $F[G] = D$. When $n = p$, a prime, it is shown in [1] that D is cyclic if and only if D^* contains a nonabelian soluble subgroup. Here we generalize this result to a division algebra of arbitrary degree n. To be more precise, it is proved that D is supersoluble crossed product if and only if D^* contains an irreducible abelian-by-supersoluble subgroup. We then present a criterion for D to be nilpotent (abelian or cyclic) crossed product. In fact, it is shown that a noncommutative finite dimensional F-central division algebra D is nilpotent (abelian or cyclic) crossed product if and only if there exist an irreducible subgroup G of D^* and an abelian normal subgroup A of G such that G/A is nilpotent (abelian or cyclic). We recall that soluble subgroups of the multiplicative group of a division ring were first studied by Suprunenko in [4].

2 Notations and conventions

Let D be a division ring with center F and G be a subgroup of D^*. We denote by $F[G]$ the F-linear hull of G, i.e., the F-algebra generated by elements of G over F. We shall say that G is *irreducible* if $D = F[G]$. For any group G we denote its center by $Z(G)$. Given a subgroup H of G, $N_G(H)$ means the *normalizer* of H in G, and $\langle H, K\rangle$ the group generated by H and K, where K is a subgroup of G. We shall say that H is *abelian-by-finite (supersoluble)* if there is an abelian normal subgroup K of H such that H/K is finite (supersoluble). Let S be a subset of D, then the *centralizer* of S in D is denoted by $C_D(S)$. For notations and results, used in the text, on central simple algebras see [3].

3 Supersoluble Crossed product Division Algebras

This section deals with a few results on division algebras whose multiplicative groups contain certain subgroups. Using these results we eventually prove our main theorem which asserts that a division algebra D is supersoluble crossed product if and only if D^* contains an irreducible abelian-by-supersoluble subgroup. Further criteria are also given for when a division algebra is nilpotent (abelian or cyclic) crossed product. To be more precise, it is shown that a noncommutative finite dimensional F-central division algebra D is nilpotent (abelian or cyclic) crossed product if and only if there exist an irreducible subgroup G of D^* and an abelian normal subgroup A of G such that G/A is nilpotent (abelian or cyclic). we begin our study with the following:

LEMMA 3.1. *Let D be a finite dimensional F-central division algebra. If D is crossed product, then D^* contains an irreducible abelian-by-finite subgroup.*

PROOF. Let K be a maximal subfield of D such that K/F is Galois. By Skolem-Noether Theorem, for any $\sigma \in Gal(K/F)$ there exists an element $x \in N = N_{D^*}(K^*)$ such that $\sigma(k) = xkx^{-1}$, for all $k \in K$. Hence $N_{D^*}(K^*)/C_{D^*}(K^*) \simeq Gal(K/F)$. Since K is a maximal subfield of D, we have $C_{D^*}(K^*) = K^*$. Therefore, $N_{D^*}(K^*)$ is an abelian-by-finite subgroup of D^*. To complete the proof of the lemma, it is enough to show that N is irreducible, i.e., $F[N] = D$. Put $D_1 = F[N]$. We have $C_D(D_1) \subseteq C_D(K) = K$, and hence $C_D(D_1)$ is an intermediate field of the Galois extension K/F. By the fact that every element of $Gal(K/F)$ is the restriction of an inner automorphism of N we conclude that $C_D(D_1) \subseteq Fix(Gal(K/F))$. Therefore $C_D(D_1) = F$. Now, by Centralizer Theorem, we obtain $D = C_D(F) = C_D(C_D(D_1)) = D_1$, which completes the proof. $\square$

LEMMA 3.2. *Let D be a finite dimensional F-central division algebra. Suppose that K is a subfield of D containing F. If G is an irreducible subgroup of D^* such that $K^* \lhd G$, then K/F is Galois and $G/C_G(K^*) \simeq Gal(K/F)$.*

PROOF. Consider the homomorphism $\sigma : G \longrightarrow Gal(K/F)$ given by $\sigma(x) = f_x$, where $f_x(k) = xkx^{-1}$, for any $k \in K$. It is clear that $\ker \sigma = C_G(K)$. Now, we claim that $Fix(\text{im}\sigma) = F$. Choose an element $a \in Fix(\text{im}\sigma)$. For any $x \in G$ we have $f_x(a) = a$, and hence $xa = ax$. This shows that $a \in C_K(G) = F$ since G is irreducible. So $Fix(Gal(K/F)) \subseteq Fix(\text{im}\sigma) = F$, which implies that K/F is a Galois extension and σ is surjective. Therefore we have $G/C_G(K) \simeq Gal(K/F)$ and K/F is a Galois extension. $\square$

Lemma 3.3. *Let D be a finite dimensional F-central division algebra and let G be an irreducible subgroup of D^*. If K is a subfield of D containing F such that $[G : C_G(K^*)] = [K : F]$, then $C_D(K) = F[C_G(K^*)]$.*

Proof. Put $D_1 = C_D(K)$ and $D_2 = F[C_G(K^*)]$. It is clear that $D_2 \subseteq D_1$. By Centralizer Theorem, we have $D \otimes_F K \simeq M_t(F) \otimes_F C_D(K)$, where $t = [K : F]$. Therefore, by comparing dimensions of both sides of the last relation, we obtain $[D_1 : F][K : F] = [D : F] = [D : D_1][D_1 : F]$. Hence $[D : D_1] = [K : F]$. On the other hand any element of D can be written in the form $\sum_{i=1}^{s} f_i g_i$, where $g_i \in G$ and $f_i \in F$, for any $1 \le i \le s$. Now, let $\ell = [G : C_G(K^*)]$ and $G = \bigcup_{i=1}^{\ell} C_G(K^*)x_i$. Therefore, every element of D can be written in the form $\sum_{i=1}^{\ell} a_i x_i$, where $a_i \in D_2$, for any $1 \le i \le \ell$. So $[D : D_2] \le \ell$ and we obtain $[D : D_1] \le [D : D_2] \le [G : C_G(K^*)] = [K : F] = [D : D_1]$, and so $D_1 = D_2$, as desired. □

The following theorem provides us a criterion for an F-central division algebra to be supersoluble crossed product:

Theorem 3.4. *Let D be a noncommutative finite dimensional F-central division algebra. Then D is supersoluble crossed product if and only if there exist an irreducible subgroup G of D^* and an abelian normal subgroup A of G such that G/A is supersoluble.*

Proof. The "only if" part is clear from the proof of Lemma 3.1. Suppose that G is an irreducible subgroup of D^* and A is an abelian normal subgroup of G such that G/A is supersoluble. Take A maximal. Therefore, we have a maximal abelian normal subgroup A of G such that G/A is supersoluble. Set $G_1 = K^*G$, where $K = F(A)$. One may easily show that G_1 is irreducible and K^* is a maximal normal abelian subgroup of G_1 such that G_1/K^* is supersoluble. By Lemma 3.2, we conclude that K/F is Galois and $G_1/C_{G_1}(K^*) \simeq Gal(K/F)$. Therefore, K/F is supersoluble Galois. To complete the proof, it is enough to show that K is a maximal subfield of D. We know that G_1/K^* is supersoluble, and hence there exists a normal series $\langle e \rangle = N_t \subseteq \dots N_1 \subseteq N_0 = G_1/K^*$ such that N_i/N_{i+1} is cyclic. Set

$N = C_G(K^*)/K^*$. It is clearly seen that N is a normal subgroup of G_1/K^*. We observe that there exists a minimal natural number s such that $N \cap N_s$ is nontrivial and so $N \cap N_{s+1} = \langle e \rangle$. Therefore, we conclude that $N \cap N_s$ is a cyclic normal subgroup of G_1/K^*. Thus, there exists $x \in C_{G_1}(K^*) \backslash K^*$ such that $\langle K^*, x\rangle/K^*$ is a normal subgroup of G_1/K^*, and hence $\langle K^*, x\rangle \neq K^*$ is an abelian normal subgroup of G_1. This contradicts the maximality of K^* in G_1. Therefore, $C_{G_1}(K^*) = K^*$ and the claim is established. Now, by Lemma 3.3, we obtain $F[C_{G_1}(K^*)] = C_D(K)$, and hence $C_D(K) = K$. Thus, K is a maximal subfield of D and the proof is complete. □

The following theorem provides us a criterion for an F-central division algebra to be nilpotent crossed product:

Theorem 3.5. *Let D be a noncommutative finite dimensional F-central division algebra. Then D is nilpotent crossed product if and only if there exist an irreducible subgroup G of D^* and an abelian normal subgroup A of G such that G/A is nilpotent.*

Proof. The "only if" part is clear from the proof of Lemma 3.1. As in the proof of Theorem 3.4 D^* contains an irreducible subgroup G_1 and a maximal normal abelian subgroup K^* such that G_1/K^* is nilpotent. By Lemma 3.2, we conclude that K/F is Galois and $G_1/C_{G_1}(K^*) \simeq Gal(K/F)$. Therefore, K/F is nilpotent Galois. To complete the proof, it is enough to show that K is a maximal subfield of D. First, we claim that $C_{G_1}(K^*) = K^*$. Otherwise, since it is known that if G is a nilpotent group and $1 \neq N \lhd G$, then $Z(G) \bigcap N \neq 1$, there is an element $K^* \neq xK^* \in C_{G_1}(K^*)/K^* \bigcap Z(G_1/K^*)$. This implies that $\langle K^*, x\rangle/K^*$ is a normal subgroup of G_1/K^*, and hence $\langle K^*, x\rangle \neq K^*$ is an abelian normal subgroup of G_1. This contradicts the maximality of K^* in G_1. Thus, $C_{G_1}(K^*) = K^*$ and the claim is established. Now, by Lemma 3.3, we obtain $F[C_{G_1}(K^*)] = C_D(K)$, and hence $C_D(K) = K$. Thus, K is a maximal subfield of D, and the proof is complete. □

Corollary 3.6. *Let D be a noncommutative finite dimensional division algebra. If D^* contains an irreducible locally nilpotent subgroup G, then D is* *nilpotent crossed product.*

Proof. To prove the corollary, it is enough to choose a basis for the vector space D over $Z(D)$ from G, then consider the group generated by this basis and use Theorem 3.5. $\square$

The multiplicative group of the real quaternion division algebra contains the quaternion group which is an irreducible 2-group. Therefore, by Corollary 3.5, it is nilpotent and even cyclic. The following result says that if the multiplicative group of a noncommutative division algebra D contains an irreducible p-group, then it is nilpotent crossed product with $p = 2$ and $[D : F] = 2^m$ for some $m \in \mathbb{N}$.

Corollary 3.7. *Let D be a noncommutative finite dimensional F-central division algebra. If D^* contains an irreducible p-subgroup, then D is nilpotent crossed product with $[D : F] = 2^m$, for some $m \in \mathbb{N}$.*

Proof. Let G be an irreducible p-subgroup of D^*. Since G is locally nilpotent, by Corollary 3.6, we conclude that D is nilpotent crossed product. If p is odd, then by a theorem of [3, p. 45], G is abelian which contradicts the irreducibility of G . If $p = 2$, then by the proof of Theorem 3.4, there exists a maximal subfield K of D such that $GK^*/K^* \simeq Gal(K/F)$ and K/F is a Galois extension. Hence $[K : F]$ is a power of 2 and the result follows. $\square$

The next result gives us a criterion for when an F-central division algebra is abelian crossed product.

Theorem 3.8. *A noncommutative finite dimensional F-central division algebra D is abelian crossed product if and only if there exist an irreducible subgroup G of D^* and an abelian normal subgroup A of G such that G/A is abelian. Equivalently D^* contains an irreducible metabelian subgroup.*

Proof. The "only if" part is clear from the proof of Lemma 3.1. Suppose that G is an irreducible subgroup of D^* and A is an abelian normal subgroup of G such that G/A is abelian. Take A maximal and put $K = F(A)$. Consider the group $G_1 = K^*G$. One can easily show that K^* is a maximal abelian normal subgroup of G_1 such that G_1/K^* is abelian. Therefore, for the derived subgroup G_1' of G_1 we have $G_1' \subseteq K^*$. We claim that $C_{G_1}(K^*) = K^*$. On the contrary, if $x \in C_{G_1}(K^*)\backslash K^*$, then $\langle K^*, x\rangle$ is an abelian subgroup of G. Since $G_1' \subseteq \langle K^*, x\rangle$ we conclude that $\langle K^*, x\rangle$ is also normal in G_1. This contradicts the maximality of K^* in G_1, and hence the claim is established, i.e., $C_{G_1}(K^*) = K^*$. Now, by Lemma 3.2, we conclude that K/F is Galois and $G_1/K^* \simeq Gal(K/F)$ and so K/F is abelian Galois. To complete the proof, it is enough to show that K is a maximal subfield of D. To see this, by Lemma 3.3, we have $F[C_{G_1}(K^*)] = C_D(K)$ which means that $C_D(K) = K$. Hence K is a maximal subfield of D and so the proof is complete. □

Let D be an F-central division algebra of prime degree. Suppose that D^* contains an irreducible soluble subgroup. Using the fact that the degree of D is prime one may easily conclude that D^* contains an irreducible metabelian subgroup. Now using Theorem 3.8, we obtain the following corollary, which is the main result of [1].

COROLLARY 3.9. *Let D be an F-central division algebra of prime degree p. Then D is cyclic if and only if D^* contains a nonabelian soluble subgroup.*

Finally, we present a criterion for a finite dimensional F-central division algebra to be cyclic.

THEOREM 3.10. *A noncommutative finite dimensional F-central division algebra D is cyclic if and only if there exist an irreducible subgroup G of D^* and an abelian normal subgroup A of G such that G/A is cyclic.*

PROOF. The "only if" part is clear from the proof of Lemma 3.1. So, assume that G is an irreducible subgroup of D^* and A is an abelian normal subgroup of G such that G/A is cyclic. Put $K = F(A)$ and consider the isomorphism $K^*G/K^* \simeq G/K^* \cap G$. This implies that K^*G/K^* is a cyclic group. Set $H = K^*G$. Obviously H is an irreducible subgroup of D^* and $K^* \lhd H$. Therefore, by Lemma 3.2, we conclude that K/F is Galois and $H/C_H(K^*) \simeq Gal(K/F)$. Now, since H/K^* is cyclic and $K^* \subseteq C_H(K^*)$ we conclude that K/F is cyclic. To complete the proof of the theorem, it is enough to show that K is a maximal subfield of D. We know that $C_H(K^*)/K^*$ is cyclic and $K^* \subseteq Z(C_H(K^*))$. Thus, we conclude that $C_H(K^*)/Z(C_H(K^*))$ is cyclic, i.e., $C_H(K^*)$ is abelian. Now, by Lemma 3.3, we obtain $F[C_H(K^*)] = C_D(K)$ and hence $C_D(K)$ is a field. Thus, K is a maximal subfield of D, and the proof is complete. □

The following example shows that working with certain subgroups of D^* may be sometimes more useful than maximal subfields.

EXAMPLE 1. Let L/K be a cyclic field extension of degree n, and denote by σ the generator of $Gal(L/K)$. Let $D = L((T, \sigma))$ be the division algebra of formal Laurent series. Although it is not hard to show that $L((T^n))/K((T^n))$ is a cyclic extension and therefore D is a cyclic division algebra, we would like to show its cyclicity by using our criterion. It is known that $Z(D) = K((T^n))$. If $1, t, \ldots, t^{n-1}$ is a basis for the field extension L/K, then $\{t^j T^i\}_{0 \leq i,j<n}$ is a basis for D over $Z(D)$. Therefore, the group $G = \langle t^j T^i \rangle_{0 \leq i,j<n}$ is an irreducible subgroup of D^*. On the other hand, one can easily show that $L \cap G \lhd G$ and that $G/L \cap G$ is a cyclic group. Therefore G is an irreducible subgroup of D^* which is abelian-by-cyclic. Thus, by Theorem 3.10, D is a cyclic division algebra. □

The rest of this section is devoted to the observation that one may not be able in general to replace "division algebra" by "central simple algebra" in the statements of some theorems above. First, we observe the following:

THEOREM 3.11. *Let D be an F-central crossed product division algebra, then for any natural number n, $GL_n(D)$ contains an irreducible abelian-by-finite subgroup.*

PROOF. By Lemma 3.1, D^* contains an irreducible abelian-by-finite subgroup G, say. We may view each element of D as a diagonal matrix in $M_n(D)$. Suppose that M is the group of monomial matrices of $GL_n(F)$ and E the group of diagonal matrices of $GL_n(F)$. Set $H = \langle G, M \rangle$. We claim that H as a subgroup of $GL_n(D)$ is irreducible. One can easily show that $F[M] = M_n(F)$. Combining this equality with the fact that $F[G] = D$ we conclude that $F[H] = M_n(D)$. To complete the proof of the claim, it is enough to show that H is abelian-by-finite. It is easily seen that $M/E \simeq S_n$. Suppose that G/A is finite, where A is an abelian normal subgroup of G. Considering the fact that every element of G commutes with every elements of M we obtain $H = GM$ and that AE is an abelian group. Using these facts, one may easily show that H/AE as a homomorphic image of $G/A \times M/E$ is a finite group. Therefore H is an irreducible abelian-by-finite subgroup of $GL_n(D)$, as desired. □

We close the paper with the following remark which shows that in Theorems 3.8, and 3.10, we can not replace "division algebra" by "central simple algebra".

Remark. Suppose that D is a crossed product division algebra and n a natural number. Using the notations of the proof of Theorem 3.10, we have $M/E \simeq S_n$. Therefore, M is an abelian-by-finite group. Hence, H is an irreducible abelian-by-finite subgroup of $GL_n(D)$. Thus, if D is an abelian crossed product (cyclic) division algebra, then $GL_2(D)$ contains an irreducible subgroup which is abelian-by-abelian (abelian-by-cyclic). Furthermore, if F is a field with more than 2 elements by applying to the group M, the group of monomial matrices, we may find an irreducible subgroup of $GL_2(F)$ which is abelian-by-cyclic.

Example 2. We observe that there are some cyclic division algebras D, e.g., the division algebra of real quaternions $\mathbb{H}$, such that $M_n(\mathbb{H})$ is not crossed product for any $n > 1$. In fact it contains no subfield containing $\mathbb{R}$ which is of degree $2n$ over $\mathbb{R}$. But for each natural number n, $GL_n(\mathbb{H})$ contains an irreducible abelian-by-finite subgroup, and it also contains an irreducible abelian-by-cyclic subgroup for $n = 2$ but it is not crossed product. Therefore, Theorems 3.8 and 3.10 are not valid if one replaces division algebras by central simple algebras.

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